MA2108 **Midterm** **18 March 2003**

Time allowed: 1.5 hours

Tutorial Group:(circle one)

Tuesday 9-10 Tuesday 10-11 Wednesday 9-10 Wednesday 10-11

Wednesday 1-2 Wednesday 2-3 Friday 4-6

Matriculation number:____________________

Name:____________________

Signature:____________________

MA2108 Midterm **18 March** **Name:**____________

Problem #	Your Grades
1 (10 points)	
2 (10 points)	
3 (10 points)	
4 (10 points)	
5 (10 points)	
6 (10 points)	
7 (10 points)	
8 (10 points)	
9 (10 points)	
10 (10 points)	
total (100 points)	

MA2108 Midterm **18 March** **Name:**____________

Determine the limit of the following sequences:

1. $\left\{\sin\left(\frac{2+n^2\pi}{6n^2+8}\right)+\ln\left(\frac{3n^3+n^2+1}{4+n^2+3n^3}\right)\right\}$.

Solution.

$$\begin{aligned}&\lim_{n\to\infty}\left(\sin\left(\frac{2+n^2\pi}{6n^2+8}\right)+\ln\left(\frac{3n^3+n^2+1}{4+n^2+3n^3}\right)\right)\\ =&\lim_{n\to\infty}\left(\sin\left(\frac{2/n^2+\pi}{6+8/n^2}\right)+\ln\left(\frac{3+1/n+1/n^3}{4/n^3+1/n+3}\right)\right)\\ =&\sin\frac{\pi}{6}+\ln 1=\frac{1}{2}.\end{aligned}$$

☐

2. $\left\{\frac{2^n+n^{80}80^n+n!}{n!+\ln n}\right\}$.

Solution.

$$\lim_{n\to\infty}\frac{2^n+n^{80}80^n+n!}{n!+\ln n}=\lim_{n\to\infty}\frac{2^n/n!+n^{80}/2^n\cdot 160^n/n!+1}{1+\ln n/n\cdot n/2^n\cdot 2^n/n!}=1$$

☐

3. $\left\{\dfrac{1}{\sqrt{n}(\sqrt{n+2}-\sqrt{n})}\right\}$.

Solution.

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\sqrt{n}(\sqrt{n+2}-\sqrt{n})} &= \lim_{n\to\infty}\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n}(n+2-n)}\\ &= \lim_{n\to\infty}\frac{\sqrt{1+2/n}+1}{2} = 1.\end{aligned}$$

$\square$

4. $\left\{\left(1-\dfrac{2}{3+n^2}\right)^{2n^2+\ln n}\right\}$.

Proof.

$$\begin{aligned}\lim_{n\to\infty}\left(1-\frac{2}{3+n^2}\right)^{2n^2+\ln n} &= \lim_{n\to\infty}\left[\left(1-\frac{2}{3+n^2}\right)^{3+n^2}\right]^{\frac{2n^2+\ln n}{3+n^2}}\\ = \lim_{n\to\infty}\left[\left(1-\frac{2}{3+n^2}\right)^{3+n^2}\right]^{\frac{2+\ln n/n^2}{3/n^2+1}} &= \left(e^{-2}\right)^2 = e^{-4}.\end{aligned}$$

$\square$

Determine convergence or divergence of the following series:

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n + 3n^2 - 2}{2n^2 + n^3 + 1}$.

Solution. Let $a_n = \dfrac{\ln n + 3n^2 - 2}{2n^2 + n^3 + 1}$ and let $b_n = \dfrac{1}{n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n + 3n^2 - 2}{2n^2 + n^3 + 1} \cdot n = \lim_{n\to\infty} \frac{\ln n/n^2 + 3 - 2/n^2}{2/n + 1 + 1/n^3} = 3.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges by limit comparison test. □

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2n}}{9^n \cdot (n!)^2}$.

Solution. Let $a_n = \dfrac{n^{2n}}{9^n \cdot (n!)^2}$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^{2(n+1)} \cdot 9^n \cdot n! \cdot n!}{9^{n+1} \cdot (n+1)! \cdot (n+1)! \cdot n^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{9} \left[\left(1 + \frac{1}{n}\right)^n\right]^2 = \frac{e^2}{9} < 1\end{aligned}$$

and so the series converges by ratio test. □

Determine the absolute convergence, conditional convergence or divergence of the following series:

7. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2+\ln n}{3+\sqrt{n}}$.

Solution. Let $f(x)=\dfrac{2+\ln x}{3+\sqrt{x}}$. Then

$$f'(x)=\frac{1/x\cdot(3+\sqrt{x})-(2+\ln x)\cdot\frac{1}{2\sqrt{x}}}{(3+\sqrt{x})^2}=\frac{6/\sqrt{x}-\ln x}{2\sqrt{x}(3+\sqrt{x})^2}\le 0$$

when $\ln x\ge 6$ or $x\ge e^6$. Thus $b_n=\dfrac{2+\ln n}{3+\sqrt{n}}$ is *eventually* monotone decreasing and positive. Since

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{2+\ln n}{3+\sqrt{n}}=\lim_{n\to\infty}\frac{2/\sqrt{n}+\ln n/\sqrt{n}}{3/\sqrt{n}+1}=0,$$

the series converges by the alternating series test.

Since

$$\frac{2+\ln n}{3+\sqrt{n}}\ge\frac{2}{\sqrt{n}+\sqrt{n}}=\frac{1}{\sqrt{n}}$$

for $n\ge 9$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges by p-series, the series $\displaystyle\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{2+\ln n}{3+\sqrt{n}}\right|$ diverges by comparison test.

In conclusion, the series is conditionally convergent. ☐

8. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(1+\cos n\right)^n\left(1-\frac{1}{n}\right)^{n^2}$.

Solution. Since

$$\varlimsup_{n\to\infty}\sqrt[n]{\left|(-1)^{n+1}\left(1+\cos n\right)^n\left(1-\frac{1}{n}\right)^{n^2}\right|}$$

$$=\varlimsup_{n\to\infty}\left(1+\cos n\right)\cdot\left(1-\frac{1}{n}\right)^n\le\varlimsup_{n\to\infty}2\left(1-\frac{1}{n}\right)^n=\frac{2}{e}<1,$$

the series is absolutely convergent by the root test for general series. ☐

9. Find limit inferior and limit superior of the sequences:

$$\left\{\left(1+\frac{1}{n}\right)\left(\frac{\sqrt{3}}{2}+\sin\frac{n\pi}{3}\right)^{\frac{1}{n}}\right\}$$

Solution. Let $a_n = \left(1+\frac{1}{n}\right)\left(\frac{\sqrt{3}}{2}+\sin\frac{n\pi}{3}\right)^{\frac{1}{n}}$. Then

$$0 \le a_n \le \left(1+\frac{1}{n}\right)\cdot(\sqrt{3})^{\frac{1}{n}}$$

because $-\frac{\sqrt{3}}{2} \le \sin\frac{n\pi}{3} \le \frac{\sqrt{3}}{2}$ for $n = 1, 2, 3, \ldots$, and so

$$0 = \lim_{n\to\infty} 0 = \varliminf_{n\to\infty} 0 \le \varliminf_{n\to\infty} a_n$$

$$\le \varlimsup_{n\to\infty} a_n \le \varlimsup_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot(\sqrt{3})^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot(\sqrt{3})^{\frac{1}{n}} = 1$$

Let $\frac{n\pi}{3} = 2k\pi + \frac{4\pi}{3}$, that is, $n = 6k+4$ for $k = 1, 2, \ldots$. Then

$$\lim_{k\to\infty} a_{6k+4} = \lim_{k\to\infty}\left(1+\frac{1}{6k+4}\right)\left(\frac{\sqrt{3}}{2}+\sin\left(2k\pi+\frac{4\pi}{3}\right)\right)^{\frac{1}{6k+4}}$$

$$\lim_{k\to\infty} 0 = 0$$

and so $\varliminf_{n\to\infty} a_n \le 0$. Thus

$$\varliminf_{n\to\infty} a_n = 0.$$

Let $\frac{n\pi}{3} = k\pi$, that is, $n = 3k$, for $k = 1, 2, 3 \ldots$. Then

$$\lim_{k\to\infty} a_{3k} = \lim_{k\to\infty}\left(1+\frac{1}{3k}\right)\left(\frac{\sqrt{3}}{2}+\sin(k\pi)\right)^{\frac{1}{3k}}$$

$$= \lim_{k\to\infty}\left(1+\frac{1}{3k}\right)\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{3k}} = 1$$

and so $\varlimsup_{n\to\infty} a_n \ge 1$. Thus

$$\varlimsup_{n\to\infty} a_n = 1.$$

□

10. Let $\sum_{n=1}^{\infty} a_n$ be a **convergent positive** series. Does the series $\sum_{n=1}^{\infty} \sin(a_n)$ converge? Justify your answer.

Solution. Yes. Let $b_n = \sin(a_n)$. Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$. Let $\epsilon = \frac{\pi}{2}$. There exists N such that

$$|a_n| < \frac{\pi}{2}$$

for $n > N$. Since $a_n \geq 0$, $0 \leq a_n < \frac{\pi}{2}$ for $n > N$ and so

$$b_n = \sin(a_n) \geq 0$$

for $n > N$. In other words, $\sum_{n=1}^{\infty} b_n$ is eventually positive.

Since

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\sin(a_n)}{a_n} \overset{x=a_n}{=\!=\!=} \lim_{x\to 0} \frac{\sin x}{x} = 1,$$

the series $\sum_{n=1}^{\infty} b_n$ converges by limit comparison test. ☐