Question 1 (a). Let \( a_n = \frac{n^2 - 1}{2n^2 + n} \). Then \( \lim_{n \to \infty} a_n = \frac{1}{2} \neq 0 \) and so the series \( \sum_{n=1}^{\infty} \frac{n^2 - 1}{2n^2 + n} \) is divergent by the divergence test.

Question 1 (b). Let \( a_n = \sin \frac{n\pi}{2} \). Then \( \{a_n\} = \{1, 0, -1, 0, 1, 0, -1, 0, \ldots\} \) and so \( \lim_{n \to \infty} a_n \) does not exist. Thus the series \( \sum_{n=1}^{\infty} \sin \frac{n\pi}{2} \) is divergent by the divergence test.

Question 1 (c). Let \( a_n = \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) and let \( b_n = \frac{1}{n} \). Then
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n + n^3 + 4}{n^2 + 1 + \ln n} = \lim_{n \to \infty} \frac{1 + \frac{n^3 + 4}{n^2 + 1 + \ln n}}{1 + \frac{1}{n^2} + \frac{\ln n}{n^2}} = 0 + 1 + 0 = 1.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent, so is \( \sum_{n=1}^{\infty} \frac{n^2 + 1 + \ln n}{n + n^3 + 4} \) by the limit comparison test.

Question 1 (d). Observe that
\[
3 + \sin \frac{n\pi}{2} \leq \frac{4}{n^2}.
\]
Since \( \sum_{n=1}^{\infty} \frac{4}{n^2} \) is convergent by the \( p \)-series, the positive series \( \sum_{n=1}^{\infty} \frac{3 + \sin \frac{n\pi}{2}}{n^2} \) is convergent by the comparison test.

Question 1 (e). Observe that
\[
\frac{2^n + 3}{3^{n+1} - n} \leq \frac{2^n + 2^n}{3^{n+1}} = \frac{2^{n+1}}{3^{n+1}} = \left(\frac{2}{3}\right)^{n+1}
\]
for \( n \geq 2 \). Since \( \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+1} \) is convergent by the geometric series, the positive series \( \sum_{n=1}^{\infty} \frac{2^n + 3}{3^{n+1} - n} \) is convergent by the comparison test.

Question 1 (f). Let \( a_n = \frac{2}{n^{1+\frac{1}{n}}} \) and let \( b_n = \frac{1}{n} \). Observe that
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{1}{n} \cdot \frac{n^{1+\frac{1}{n}}}{2} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}.
\]
Since \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent by the harmonic series, the positive series \( \sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}} \) is divergent by the limit comparison test.
Question 1 (g). Observe that
\[
\frac{4 + (-1)^n}{2n} \geq \frac{3}{2n}.
\]
Since \(\sum_{n=1}^{\infty} \frac{3}{2n} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n}\) is divergent by the harmonic series, the positive series \(\sum_{n=1}^{\infty} \frac{4 + (-1)^n}{2n}\) is divergent by the comparison test. \(\square\)

Question 1 (h). Observe that
\[
\frac{1}{n(1 + \ln n)^p} = \frac{(1 + \ln n)^{-p}}{n} \geq \frac{1}{n}
\]
for \(p \leq 0\). Since \(\sum_{n=1}^{\infty} \frac{1}{n}\) is divergent by the harmonic series, the positive series \(\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)^p}\) is divergent for \(p \leq 0\) by the comparison test. \(\square\)

Question 1 (i). Observe that
\[
\frac{n}{n^2 + 1} \geq \frac{n}{2n^2 + n^2} = \frac{n}{2n^2} = \frac{1}{2n}.
\]
Since \(\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}\) is divergent by the harmonic series, the positive series \(\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}\) is divergent by the comparison test. \(\square\)

Question 2(a). Since \(\sum_{n=1}^{\infty} a_n\) is convergent, we have \(\lim_{n \to \infty} a_n = 0\) and so there exists a positive integer \(N\) such that \(a_n = |a_n| = |a_n - 0| < 1\) for \(n > N\). It follows
\[
a_n^2 = a_n \cdot a_n \leq 1 \cdot a_n = a_n
\]
for \(n > N\). By the comparison test, the positive series \(\sum_{n=1}^{\infty} a_n^2\) is convergent. \(\square\)

Another Solution of Question 2 (a). Since \(\sum_{n=1}^{\infty} a_n\) is convergent, we have \(\lim_{n \to \infty} a_n = 0\).
Let \(b_n = a_n^2\). Then
\[
\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} a_n = 0,
\]
that is, \(b_n << a_n\). By limit comparison test, \(\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n^2\) is convergent. \(\square\)
Question 2 (b). Let \( a_n = \frac{1}{n^2} \). Then \( \sum_{n=1}^{\infty} a_n \) is convergent but \( \sum_{n=1}^{\infty} \sqrt{a_n} \) is divergent by the p-series. \( \square \)

Question 3 (a). Let \( f(x) = \frac{1}{x(1 + \ln x)} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since
\[
\int_1^\infty f(x) \, dx = \int_1^\infty \frac{1}{x(1 + \ln x)} \, dx = \int_0^{\infty} \frac{1}{1 + y} \, dy = \ln(1 + y) \bigg|_0^\infty = +\infty.
\]
is divergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is divergent by the integral test. \( \square \)

Question 3 (b). Let \( f(x) = \frac{1}{x[1 + (\ln x)^2]} \). Then \( f(x) \) is a positive monotone decreasing function over \([1, +\infty)\). Since
\[
\int_1^\infty f(x) \, dx = \int_1^\infty \frac{1}{x[1 + (\ln x)^2]} \, dx = \int_0^{\infty} \frac{1}{1 + y^2} \, dy = \arctan y \bigg|_0^\infty = \frac{\pi}{2}
\]
is convergent, the series \( \sum_{n=1}^{\infty} \frac{1}{n(1 + \ln n)} \) is convergent by the integral test. \( \square \)