NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2002-2003

MA2108 ADVANCED CALCULUS II

April 2003 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions in **Section A**. Section A carries a total of 60 marks.

3. Answer no more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [16 marks]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{5 + \ln\left(\cos\dfrac{\ln n}{\sqrt{n}}\right) + \dfrac{n^3}{1.1^n}\right\}$.

(b) $\left\{\dfrac{8^n \cdot n^{100} + \ln n - n!}{n! + n^2}\right\}$.

(c) $\left\{\left(\dfrac{n}{n+1}\right)^{2n+\ln n}\right\}$.

(d) $\left\{\dfrac{\sqrt[n]{n!}}{n}\right\}$.

Question 2 [16 marks]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^n}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{1.1}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n! \cdot 3^n}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$.

Question 3 [10 marks]

Find the radius of convergence of each of the following power series. Justify your answer.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^2}(3x+1)^k.$

(b) $\displaystyle\sum_{k=1}^{\infty} \left(1-\frac{1}{k}\right)^{k^2} x^{k^2}.$

Question 4 [18 marks]

(a) Determine whether the following sequence of functions converge uniformly on the indicated intervals. Justify your answers.

$$F_n(x) = \frac{x^n \cos nx}{1+x^n}, \quad x \in [0, \tfrac{2}{3}].$$

(b) Determine whether the following series of functions converge uniformly on the indicated intervals. Justify your answers.

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k^2+x^2}, \quad x \in [0, \infty).$

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k+3x} \quad x \in [0, \infty).$

SECTION B

Answer not more than **TWO (2)** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

(a) Evaluate $\lim_{n\to\infty} \int_0^1 \frac{n + \cos(nx^2)}{x^2 + n}\, dx$. Justify your answer.

(b) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(2x+1)^n.$$

Justify your answer.

(c) Suppose that $f(x) = \sum_{k=0}^{\infty} c_k x^k$ and $e^{f(x)} = \sum_{k=0}^{\infty} d_k x^k$. has positive radius of convergence. Show that, for each $n \geq 1$,

$$d_n = \frac{1}{n} \sum_{k=1}^{n} k c_k d_{n-k}.$$

Question 6 [20 marks]

(a) Consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n} \cdot x}{e^{n^2 x}}.$$

Is $f(x)$ continuous on $[0, +\infty)$? Justify your answer.

(b) Using any applicable method, find the Taylor series of the function $f(x) = \ln\left(\frac{1+2x}{1-x}\right)$ at $x_0 = 0$, and specify the interval on which the series converges to the function.

(c) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Prove that

$$\underline{\lim}_{n\to\infty}\, a_n + \underline{\lim}_{n\to\infty}\, b_n \leq \underline{\lim}_{n\to\infty}\, (a_n + b_n).$$

Question 7 [20 marks]

(a) Let $f(x) = x^2 \cdot \sqrt[3]{1+x^9}$. Find $f^{(28)}(0)$.

(b) Determine the absolute convergence, conditional convergence or divergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n)}{n(\ln n + 1)\left[\ln(\ln n + 1)\right]^2}$. Justify your answers.

(c) Does the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^x}$ converge uniformly on $(1, +\infty)$? Justify your answer.