2002/2003 Semester II MA2108 Advanced Calculus II

Take-home Exam 1

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam already.

Question 1. [*2 points, 1 for each part*]

Prove the following limits by using $\epsilon - N$ definition

i) $\lim_{n\to\infty} \frac{3n+8}{2n+9} = \frac{3}{2}$.

ii) $\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$.

Question 2. [*5 points, 1 for each part*]

For each of the following sequences, either find the limit or show that the limit does not exist.

(a) $\left\{\left(\sqrt{n^2+n} - n\right)\right\}$.

(b) $\left\{(2^n + 3^n)^{\frac{1}{n}}\right\}$.

(c) $\left\{\sqrt[4]{\frac{n! + 2n^5 + \ln n}{n! + 5^n + 3n}}\right\}$.

(d) $\left\{\left(\frac{3n}{3n-1}\right)^{2n+\sqrt{n}}\right\}$.

(e) $\left\{\frac{n^{50} \cdot 50^n \cdot \sin n}{n!}\right\}$.

Question 3. [*3 points, 1 for each part*]

(a) If $\{a_n\}$ is convergent, show that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1+a_n)$ for $n \geq 1$. Assume that $\{a_n\}$ is convergent, find its limit.
(Hint: there will be two possible values for $\lim_{n\to\infty} a_n$. For rejecting one of them, show by induction that $a_n > 0$.)

(c) Find the limit of the sequence

$$\left\{\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}},\ \ldots\right\}.$$

(Hint: This sequence is given recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. You can show by induction that $0 \leq a_n \leq 2$. Then prove that $\{a_n\}$ is monotone increasing. By applying monotone convergence theorem, conclude that $\lim_{n\to\infty} a_n$ exists. There will be two possible values for $\lim_{n\to\infty} a_n$ and you have to decide to reject one of them.)