2002/2003 Semester II MA2108 Advanced Calculus II

Take-home Exam 2

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.

Question 1[2 points, 1 for each part]

Let a_1 and b_1 be positive numbers with $a_1 > b_1$. Let $a_2 = \dfrac{a_1 + b_1}{2}$ be their arithmetic mean and let $b_2 = \sqrt{a_1 b_1}$ be their geometric mean. Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}.$$

(a) Show by mathematical induction that $a_n > a_{n+1} > b_{n+1} > b_n$.

(b) Prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(**Note.** Gauss called this the common value of these limits the *arithmetic-geometric mean* of the numbers $a = a_1$ and $b = b_1$.)

Question 2. [3 points, 1 for each part]

Find limit inferior and limit superior of each of the following sequences.

(a) $\left\{ \dfrac{n + (-1)^n n^2}{n^2 + 1} \right\}$.

(b) $\{[1.5 + (-1)^n]^n\}$.

(c) $\left\{ \left(1 + \dfrac{1}{n}\right) \left(1 + \sin \dfrac{n\pi}{8}\right)^{\frac{1}{n}} \right\}$

Question 3 [5 points, 1 for each part]

Determine the convergence or divergence of each of the following series. Justify your answers.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 2k - 1}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(2 + \ln n)}$.

(c) $\displaystyle\sum_{n=1}^{\infty} 6^n \left(1 - \frac{2}{n+1}\right)^{n^2}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$.

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k+1} - \sqrt{k}}{k}$.