Take-home Exam 4

In order to get the full marks for the project of take-home exams, you need to have 50 or more points. Well there are 10 points in this take-home exam.

Question 1 [4 points, 1 for each part]
For each of the following sequence of functions, determine whether is converges pointwise to a function, and find the limiting function if it exists. Justify your answers.

(a) $\left\{\left(1-\frac{x^2}{n}\right)^{nx}\right\}, \quad x \in \mathbb{R}.$

(b) $\{(\cos x)^{2n}\}, \quad x \in \mathbb{R}.$

(c) $\left\{\frac{\sin nx}{\cos nx + nx}\right\}, \quad x \in [1, +\infty).$

(d) $\{f_n(x)\}, \quad f_n(x) = \sum_{k=0}^{n} \frac{x^2}{(1+x^2)^k}, \quad x \in \mathbb{R}.$

Question 2. [6 points, 1 for each part] Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

(a) $F_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0, \frac{1}{2}].$

(b) $F_n(x) = \frac{x^n}{1+x^n}, \quad x \in [0, 1].$

(Hint: Find the limiting function $F(x)$ and check that the limiting function is not continuous but each F_n is, and from this conclude that the sequence of functions does not converge uniformly.)

(c) $F_n(x) = x + \frac{x}{n}\sin nx, \quad x \in [-a, a], \quad a > 0.$

(d) $F_n(x) = x + \frac{x}{n}\sin nx, \quad x \in \mathbb{R}.$

(Hint: Try to find a lower bound of $T_n = \sup_{x\in\mathbb{R}} |F_n(x) - F(x)|$ by taking $x = 2n\pi + \frac{\pi}{2n}$.)

(e) $F_n(x) = \frac{x^n \sin nx}{1+x^n}, \quad x \in [0, \frac{1}{2}].$

(f) $F_n(x) = nx\left(1-x^2\right)^n, \quad x \in [0, 1].$