1. Prove the following limits by using $\epsilon - N$ definition

   i) $\lim_{n\to\infty} \frac{2n^2}{3n^2+2} = \frac{2}{3}$.

   ii) $\lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1$.

2. For each of the following statements, either prove that the statement is true or give a counter example to show that the statement is false:

   i) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges. (Hint: Use Theorem 1.4.5)

   ii) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n b_n\}$ diverges. (Hint: Construct a counter-example.)

[From Question 3 onwards, you may assume the limits of the standard sequences.]

3. Evaluate the following limits:

   i) $\lim_{n\to\infty} \frac{n^2+5n^3-1}{3n^3+6n+4}$;

   ii) $\lim_{n\to\infty} \frac{3^n+n^8}{2n^2+7^n}$;

   iii) $\lim_{n\to\infty} \sqrt{\frac{n^4+4n^3+1}{n^3+2n^2}}$.

4. Use the Squeeze theorem to find the following limits:

   i) $\lim_{n\to\infty} \frac{1+|\sin n|}{2n}$;

   ii) $\lim_{n\to\infty} \left(\frac{2n-5}{3n+1}\right)^n$.

5. Do the following sequences tend to $+\infty$ or $-\infty$? Justify your answer.

   i) $\left\{\frac{e^n}{n^{100}}\right\}$;

   ii) $\left\{\frac{n}{\ln \frac{1}{n+2}}\right\}$.

6. Evaluate the following limits (you may assume the limits of the standard sequences and use the Squeeze theorem, etc.)

(a). $\lim_{n\to\infty} \sqrt[3]{\frac{2n^4+n+1}{16n^4+n^2+2}}$;

(b). $\lim_{n\to\infty} \left(3+\ln(\cos\frac{1}{\sqrt{n}})+\frac{n^2}{1.1^n}\right)$;

(c). $\lim_{n\to\infty} \frac{n^4+8^n}{9^n+n+8^n}$;

(d). $\lim_{n\to\infty} \sqrt[n]{\frac{n}{3}}$;

(e). $\lim_{n\to\infty} (\sqrt{3n-2}-\sqrt{3n-3})$;

(f). $\lim_{n\to\infty} \left(\frac{3+(-1)^n}{5}\right)^n$;

(g). $\lim_{n\to\infty} \frac{7^n+\ln n-n!}{n!+n^2}$;

(h). $\lim_{n\to\infty} \frac{n^{100}100^n}{n!}$;

(i). $\lim_{n\to\infty} \frac{(-1)^{n+1}\ln n}{\sqrt{n}}$;

(j). $\lim_{n\to\infty} (3^n+4^n)^{\frac{1}{n}}$;

(k). $\lim_{n\to\infty} n\sin\frac{3}{n}$; (Hint: $\lim_{x\to 0}\frac{\sin x}{x}=1$.)

(l). $\lim_{n\to\infty} \left(1-\frac{1}{3n}\right)^{2n}$;

(m). $\lim_{n\to\infty} \left(1+\frac{1}{n^3}\right)^{n^3+2}$.

7. Let S and T be two bounded sets of real numbers. Show that $S\cup T$ is also a bounded set.

(Hint: Recall that a bounded set means that this set has an upper bound and a lower bound. The assumption says that S has an upper bound and a lower bound, and T has a (possibly different) upper bound and a (possibly different) lower bound. What you need to do is to find an upper bound and a lower bound for the union $S\cup T$, that is, a **common** upper bound and a **common** lower bound for both S and T. You also have to think how to write down your solution in a *logical* way.)

8. i) Show that a sequence $\{a_n\}$ is bounded if and only if $\{|a_n|\}$ is bounded.

(Hint: Try to think: Whence you have an upper bound and a lower bound for $\{a_n\}$, how to give an upper bound and a lower bound for $\{|a_n|\}$, and vice versa.)

ii) Using i) or otherwise, show that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} a_n b_n = 0$.

(Hint: From (i), $\{|b_n|\}$ has an upper bound. Then try $\epsilon - N$ definition.)