

1. Find the limit inferior and limit superior of the following sequences

a) $\left\{ \frac{2 - (-1)^n n}{4n + 2} \right\}$.

b) $\left\{ \left(0.9 + \sin \frac{n\pi}{2} \right)^n \right\}$.

c) $\sqrt[n]{\frac{(n!)^2}{(2n)!}}$. (Hint: Use Exercise 8.2 in the lecture notes.)

2. Let $\{a_n\}$ be a bounded sequence of real numbers. Show that

$$\limsup_{n \rightarrow \infty} \sqrt{|a_n|} = \sqrt{\limsup_{n \rightarrow \infty} |a_n|}.$$

(Hint: Let $b_n = \sup\{|a_n|, |a_{n+1}|, |a_{n+2}|, \dots\}$ and let

$$B_n = \sup\{\sqrt{|a_n|}, \sqrt{|a_{n+1}|}, \sqrt{|a_{n+2}|}, \dots\}.$$

Recall from the definition that $\overline{\lim}_{n \rightarrow \infty} \sqrt{|a_n|} = \lim_{n \rightarrow \infty} B_n$ and $\overline{\lim}_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} b_n$.

Prove that $B_n \leq \sqrt{b_n}$ and $\sqrt{b_n} \leq B_n$.)

3. Let $\{a_n\}$ and $\{b_n\}$ be Cauchy sequences. Show that $\{a_n + b_n\}$ and $\{a_n b_n\}$ are also Cauchy sequences.

4. For each of the following series, calculate the n -th partial sum S_n , and determine whether the series is convergent or divergent.

i) $\sum_{n=1}^{\infty} \ln \frac{n+2}{n+3}$.

ii) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.