1. Examples of Manifolds

1.1. Open Stiefel Manifolds and Grassmann Manifolds. The *open Stiefel manifold* is the space of k-tuples of linearly independent vectors in $\mathbb{R}^n$:

$$\tilde{V}_{k,n} = \{(\vec{v}_1, \ldots, \vec{v}_k)^T \mid \vec{v}_i \in \mathbb{R}^n,\ \{\vec{v}_1, \ldots, \vec{v}_k\} \text{ linearly independent}\},$$

where $\tilde{V}_{k,n}$ is considered as the subspace of $k \times n$ matrixes $M(k,n) \cong \mathbb{R}^{kn}$. Since $\tilde{V}_{k,n}$ is an open subset of $M(k,n) = \mathbb{R}^{kn}$, $\tilde{V}_{k,n}$ is an open submanifold of $\mathbb{R}^{kn}$.

The *Grassmann manifold* $G_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$, that is, all k-planes through the origin. Let

$$\pi \colon \tilde{V}_{k,n} \to G_{k,n}$$

be the quotient by sending k-tuples of linearly independent vectors to the k-planes spanned by k vectors. The topology in $G_{k,n}$ is given by quotient topology of π, namely, U is an open set of $G_{k,n}$ if and only if $\pi^{-1}(U)$ is open in $\tilde{V}_{k,n}$.

For $(\vec{v}_1, \ldots, \vec{v}_k)^T \in \tilde{V}_{k,n}$, write $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle$ for the k-plane spanned by $\vec{v}_1, \ldots, \vec{v}_k$. Observe that two k-tuples $(\vec{v}_1, \ldots, \vec{v}_k)^T$ and $(\vec{w}_1, \ldots, \vec{w}_k)^T$ spanned the same k-plane if and only if each of them is basis for the common plane, if and only if there is nonsingular $k \times k$ matrix P such that

$$P(\vec{v}_1, \ldots, \vec{v}_k)^T = (\vec{w}_1, \ldots, \vec{w}_k)^T.$$

This gives the identification rule for the Grassmann manifold $G_{k,n}$. Let $\mathrm{GL}_k(\mathbb{R})$ be the space of general linear groups on $\mathbb{R}^k$, that is, $\mathrm{GL}_k(\mathbb{R})$ consists of $k \times k$ nonsingular matrixes, which is an open subset of $M(k,k) = \mathbb{R}^{k^2}$. Then $G_{k,n}$ is the quotient of $\tilde{V}_{k,n}$ by the action of $\mathrm{GL}_k(\mathbb{R})$.

First we prove that $G_{k,n}$ is Hausdorff. If $k = n$, then $G_{n,n}$ is only one point. So we assume that $k < n$. Given an k-plane X and $\vec{w} \in \mathbb{R}^n$, let $\rho_{\vec{w}}$ be the square of the Euclidian distance from $\vec{w}$ to X. Let $\{e_1, \ldots, e_k\}$ be the orthogonal basis for X, then

$$\rho_{\vec{w}}(X) = \vec{w} \cdot \vec{w} - \sum_{j=1}^{k} (\vec{w} \cdot e_j)^2.$$

Fixing any $\vec{w} \in \mathbb{R}^n$, we obtain the continuous map

$$\rho_{\vec{w}} \colon G_{k,n} \longrightarrow \mathbb{R}$$

because $\rho_{\vec{w}} \circ \pi \colon \tilde{V}_{k,n} \to \mathbb{R}$ is continuous and $G_{k,n}$ given by the quotient topology. (Here we use the property of quotient topology that any function f from the quotient space $G_{k,n}$ to any space is continuous if and only if $f \circ \pi$ from $\tilde{V}_{k,n}$ to that space is continuous.) Given any two distinct points X and Y in $G_{k,n}$, we can choose a $\vec{w}$ such that $\rho_{\vec{w}}(X) \neq \rho_{\vec{w}}(Y)$. Let V_1 and V_2 be disjoint open subsets of $\mathbb{R}$ such that $\rho_{\vec{w}}(X) \in V_1$ and $\rho_{\vec{w}}(Y) \in V_2$. Then $\rho_{\vec{w}}^{-1}(V_1)$ and $\rho_{\vec{w}}^{-1}(V_2)$ are two open subset of $G_{k,n}$ that separate X and Y, and so $G_{k,n}$ is Hausdorff.

Now we check that $G_{k,n}$ is manifold of dimension $k(n-k)$ by showing that, for any X in $G_{k,n}$, there is an open neighborhood U_X of α such that $U_X \cong \mathbb{R}^{k(n-k)}$.

Let $X \in G_{k,n}$ be spanned by $(\vec{v}_1, \ldots, \vec{v}_k)^T$. There exists a nonsingular $n \times n$ matrix Q such that

$$(\vec{v}_1, \ldots, \vec{v}_k)^T = (I_k, 0)Q,$$

where I_k is the unit $k \times k$-matrix. Fixing Q, define

$$X_\alpha = \{(P_k, B_{k,n-k})Q \mid \det(P_k) \neq 0, B_{k,n-k} \in M(k, n-k)\} \subseteq \tilde{V}_{k,n}.$$

Then E_X is an open subset of $\tilde{V}_{k,n}$. Let $U_X = \pi(E_X) \subseteq G_{k,n}$. Since

$$\pi^{-1}(U_X) = E_X$$

is open in $\tilde{V}_{k,n}$, U_X is open in $G_{k,n}$ with $X \in U_X$. From the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_k(\mathbb{R}) \times M(k, n-k) & \xrightarrow[\cong]{(P, A) \mapsto (P, PA)Q} & E_X \\ \Big\downarrow \text{proj.} & & \Big\downarrow \pi \\ M(k, n-k) & \xrightarrow[\phi_X^{-1}]{A \mapsto \langle (I_k, A)Q \rangle} & U_X, \end{array}$$

U_X is homeomorphic to $M(k, n-k) = \mathbb{R}^{k(n-k)}$ and so $G_{k,n}$ is a (topological) manifold.

For checking that $G_{k,n}$ is a smooth manifold, let X and $Y \in G_{k,n}$ be spanned by $(\vec{v}_1, \ldots, \vec{v}_k)^T$ and $(\vec{w}_1, \ldots, \vec{w}_k)^T$, respectively. There exists nonsingular $n \times n$ matrixes Q and $\tilde{Q}$ such that

$$(\vec{v}_1, \ldots, \vec{v}_k)^T = (I_k, 0)Q, \quad (\vec{w}_1, \ldots, \vec{w}_k)^T = (I_k, 0)\tilde{Q}.$$

Consider the maps:

$$M(k, n-k) \xrightarrow[\phi_X^{-1}]{} U_X \qquad A \mapsto \langle (I_k, A)Q \rangle$$

$$M(k, n-k) \xrightarrow[\phi_Y^{-1}]{} U_Y \qquad A \mapsto \langle (I_k, A)\tilde{Q} \rangle.$$

If $Z \in U_X \cap U_Y$, then

$$Z = \langle (I_k, A_Z)Q \rangle = \langle (I_k, B_Z)\tilde{Q} \rangle$$

for unique $A, B \in M(k, n-k)$. It follows that there is a nonsingular $k \times k$ matrix P such that

$$(I_k, B_Z)\tilde{Q} = P(I_k, A_Z)Q \Leftrightarrow (I_k, B_Z) = P(I_k, A_Z)Q\tilde{Q}^{-1}.$$

Let

$$T = Q\tilde{Q}^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

Then

$$(I_k, B_Z) = (P, PA_Z)T = (PT_{11} + PA_ZT_{21}, PT_{12} + PA_ZT_{22})$$

$$\begin{cases} I_k = P(T_{11} + A_ZT_{21}) \\ B_Z = P(T_{12} + A_ZT_{22}). \end{cases}$$

It follows that

$Z \in U_X \cap U_Y$ if and only if $\det(T_{11} + A_Z T_{21}) \neq 0$ (that is, $T_{11} + A_Z T_{21}$ is invertible).

From the above, the composite

$$\phi_X(U_X \cap U_Y) \xrightarrow{\phi_X^{-1}} U_X \cap U_Y \xrightarrow{\phi_Y} M(k,n)$$

is given by

$$A \mapsto (T_{11} + AT_{21})^{-1}(T_{12} + AT_{22}),$$

which is smooth. Thus $G_{k,n}$ is a smooth manifold.

As a special case, $G_{1,n}$ is the space of lines (through the origin) of $\mathbb{R}^n$, which is also called *projective space* denoted by $\mathbb{R}\mathrm{P}^{n-1}$. From the above, $\mathbb{R}\mathrm{P}^{n-1}$ is a manifold of dimension $n-1$.

1.2. **Stiefel Manifold.** The *Stiefel manifold*, denoted by $V_{k,n}$, is defined to be the set of k orthogonal unit vectors in $\mathbb{R}^n$ with topology given as a subspace of $\tilde{V}_{k,n} \subseteq M(k,n)$. Thus

$$V_{k,n} = \{A \in M(k,n) \mid A \cdot A^T = I_k\}.$$

We prove that $V_{k,n}$ is a smooth submanifold of $M(k,n)$ by using Pre-image Theorem.

Let $S(k)$ be the space of symmetric matrixes. Then $S(k) \cong \mathbb{R}^{\frac{(k+1)k}{2}}$ is a smooth manifold of dimension. Consider the map

$$f\colon M(k,n) \to S(k) \qquad A \mapsto AA^T.$$

For any $A \in M(k,n)$, $Tf_A\colon T_A(M(k,n)) \to T_{f(A)}(S(k))$ is given by setting $Tf_A(B)$ is the directional derivative along B for any $B \in T_A(M(k,n))$, that is,

$$\begin{aligned} Tf_A(B) &= \lim_{s\to 0} \frac{f(A+sB) - f(A)}{s} \\ &= \lim_{s\to 0} \frac{(A+sB)(A+sB)^T - AA^T}{s} \\ &= \lim_{s\to 0} \frac{AA^T + sAB^T + sBA^T + s^2BB^T - AA^T}{s} = AB^T + BA^T. \end{aligned}$$

We check that $Tf_A\colon T_A(M(k,n)) \to T_{f(A)}(S(k))$ is surjective for any $A \in f^{-1}(I_k)$.

By the identification of $M(k,n)$ and $S(k)$ with Euclidian spaces, $T_A(M(k,n)) = M(k,n)$ and $T_{f(A)}(S(k)) = S(k))$. Let $A \in f^{-1}(I_k)$ and let $C \in T_{f(A)}(S(k))$. Define

$$B = \frac{1}{2}CA \in T_A(M(k,n)).$$

Then

$$Tf_A(B) = AB^T + BA^T = \frac{1}{2}AA^TC^T + \frac{1}{2}CAA^T \overset{AA^T=I_k}{=\!=\!=} \frac{1}{2}C^T + \frac{1}{2}C \overset{C=C^T}{=\!=\!=} C.$$

Thus $Tf\colon T_A(M(k,n)) \to T_{f(A)}(S(k))$ is onto and so I_k is a regular value of f. Thus, by Pre-image Theorem, $V_{k,n} = f^{-1}(I_k)$ is a smooth submanifold of $M(k,n)$ of dimension

$$kn - \frac{(k+1)k}{2} = \frac{k(2n-k-1)}{2}.$$

Special Cases: When $k = n$, then $V_{n,n} = O(n)$ the orthogonal group. From the above, $O(n)$ is a (smooth) manifold of dimension $\frac{n(n-1)}{2}$. (**Note.** $O(n)$ is a Lie group, namely, a smooth manifold plus a topological group such that the multiplication and inverse are smooth.)

When $k = 1$, then $V_{1,n} = S^{n-1}$ which is manifold of dimension $n - 1$.

When $k = n - 1$, then $V_{n-1,n}$ is a manifold of dimension $\frac{(n-1)n}{2}$. One can check that

$$V_{n-1,n} \cong SO(n)$$

the subgroup of $O(n)$ with determinant 1. In general case, $V_{k,n} = O(n)/O(n - k)$.

As a space, $V_{k,n}$ is compact. This follows from that $V_{k,n}$ is a closed subspace of the k-fold Cartesian product of S^{n-1} because $V_{k,n}$ is given by k unit vectors $(\vec{v}_1, \ldots, \vec{v}_k)^T$ in $\mathbb{R}^n$ that are solutions to $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$, and the fact that any closed subspace of compact Hausdorff space is compact. The composite

$$V_{k,n} \hookrightarrow \tilde{V}_{k,n} \xrightarrow{\pi} G_{k,n}$$

is onto and so the Grassmann manifold $G_{k,n}$ is also compact. Moreover the above composite is a smooth map because π is smooth and $V_{k,n}$ is a submanifold. This gives the diagram

$$\begin{array}{ccccc} V_{k,n} & \xrightarrow{\text{submanifold}} & M(k,n) & \xrightarrow[A \mapsto AA^T]{\text{submersion at } I_k} & S(k) \\ \Big\downarrow{\scriptstyle \text{smooth}} & & & & \\ G_{k,n} & & & & \end{array}$$

Note. By the construction, $G_{k,n}$ is the quotient of $V_{k,n}$ by the action of $O(k)$. This gives identifications:

$$G_{k,n} = V_{k,n}/O(k) = O(n)/(O(k) \times O(n - k)).$$