LECTURE NOTES ON DIFFERENTIABLE MANIFOLDS

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Contents

1. Tangent Spaces, Vector Fields in $\mathbb{R}^n$ and the Inverse Mapping Theorem

1.1. **Tangent Space to a Level Surface.** Let γ be a curve in $\mathbb{R}^n$: $\gamma\colon t \mapsto (\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t))$. (A curve can be described as a vector-valued function. Converse a vector-valued function gives a curve in $\mathbb{R}^n$.) The *tangent line* at the point $\gamma(t_0)$ is given with the direction

$$\frac{d\gamma}{dt}(t_0) = \left(\frac{d\gamma^1}{dt}(t_0), \ldots, \frac{d\gamma^n}{t}(t_0)\right).$$

(Certainly we need to assume that the derivatives exist. We may talk about *smooth curves*, that is, the curves with all continuous higher derivatives.)

Consider the level surface $f(x^1, x^2, \ldots, x^n) = c$ of a differentiable function f, where x^i refers to i-th coordinate. The *gradient vector* of f at a point $P = (x^1(P), x^2(P), \ldots, x^n(P))$ is

$$\nabla f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}).$$

Given a vector $\vec{u} = (u^1, \ldots, u^n)$, the *directional derivative* is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x^1} u^1 + \cdots + \frac{\partial f}{\partial x^n} u^n.$$

The *tangent space* at the point P on the level surface $f(x^1, \ldots, x^n) = c$ is the $(n-1)$-dimensional (if $\nabla f \neq 0$) space through P normal to the gradient ∇f. In other words, the tangent space is given by the equation

$$\frac{\partial f}{\partial x^1}(P)(x^1 - x^1(P)) + \cdots + \frac{\partial f}{\partial x^n}(P)(x^n - x^n(P)) = 0.$$

From the geometric views, the tangent space *should* consist of all tangents to the smooth curves **on the level surface** through the point P. Assume that γ is a curve through P (when $t = t_0$) that lies in the level surface $f(x^1, \ldots, x^n) = c$, that is

$$f(\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t)) = c.$$

By taking derivatives on both sides,

$$\frac{\partial f}{\partial x^1}(P)(\gamma^1)'(t_0) + \cdots + \frac{\partial f}{\partial x^n}(P)(\gamma^n)'(t_0) = 0$$

and so the tangent line of γ is really normal (orthogonal) to ∇f. When γ *runs over all possible curves* on the level surface through the point P, then we obtain the tangent space at the point P.

Roughly speaking, *a tangent space is a vector space attached to a point in the surface.*

How to obtain the tangent space: *take all tangent lines of smooth curve through this point on the surface.*

1.2. **Tangent Space and Vectors Fields on $\mathbb{R}^n$.**

Now consider the tangent space of $\mathbb{R}^n$. According to the ideas in the previous subsection, first we assume a given point $P \in \mathbb{R}^n$. Then we consider all smooth curves passes through P and then take the tangent lines from the smooth curves. The obtained vector space at the point P is the n-dimensional space. But we can look at in a little detail.

Let γ be a smooth curve through P. We may assume that $\gamma(0) = P$. Let ω be another smooth curve with $\omega(0) = P$. γ is called to be *equivalent* to ω if the directives $\gamma'(0) = \omega'(0)$. The tangent space of $\mathbb{R}^n$ at P, denoted by $T_P(\mathbb{R}^n)$, is then the set of equivalence class of all smooth curves through P.

Let $T(\mathbb{R}^n) = \bigcup\limits_{P \in \mathbb{R}^n} T_P(\mathbb{R}^n)$, called the tangent bundle of $\mathbb{R}^n$. If S is a region of $\mathbb{R}^n$, let $T(S) = \bigcup\limits_{P \in S} T_P(S)$, called the tangent bundle of S.

Note. Each $T_P(\mathbb{R}^n)$ is an n-dimensional vector space, but $T(S)$ is *not* a vector space. In other words, $T(S)$ is obtained by attaching a vector space $T_P(\mathbb{R}^n)$ to each point P in S. Also S is assumed to be a region of $\mathbb{R}^n$, otherwise the tangent space of S (for instance S is a level surface) could be a proper subspace of $T_P(\mathbb{R}^n)$.

If γ is a smooth curve from P to $\mathbb{Q}$ in $\mathbb{R}^n$, then the tangent space $T_P(\mathbb{R}^n)$ moves along γ to $T_Q(\mathbb{R}^n)$. The direction for this moving is given $\gamma'(t)$, which introduces the following important concept.

Definition 1.1. A *vector field* V on a region S of $\mathbb{R}^n$ is a smooth map (also called C^∞-map)

$$V\colon S \to T(S) \quad P \mapsto \vec{v}(P).$$

Let $V\colon P \mapsto \vec{v}(P)$ and $W\colon P \mapsto \vec{w}(P)$ be two vector fields and let $f\colon S \to \mathbb{R}$ be a smooth function. Then $V + W\colon P \mapsto \vec{v}(P) + \vec{w}(P)$ and $fV\colon P \mapsto f(P)\vec{v}(P)$ give (pointwise) addition and scalar multiplication structure on vector fields.

1.3. **Operator Representations of Vector Fields.** Let J be an open interval containing 0 and let $\gamma\colon J \to \mathbb{R}^n$ be a smooth curve with $\gamma(0) = P$. Let $f = f(x^1, \ldots, x^n)$ be a smooth function defined on a neighborhood of P. Assume that the range of γ is contained in the domain of f. By applying the chain rule to the composite $T = f \circ \gamma\colon J \to \mathbb{R}$,

$$D_\gamma(f) := \frac{dT}{dt} = \sum_{i=1}^{n} \frac{d\gamma^i(t)}{dt} \frac{\partial f}{\partial x^i}\bigg|_{x^i=\gamma^i(t)}$$

Proposition 1.2.

$$D_\gamma(af + bg) = aD_\gamma(f) + bD_\gamma(g), \quad \textit{where } a, b \textit{ are constant.}$$

$$D_\gamma(fg) = D_\gamma(f)g + fD_\gamma(g).$$

Let $C^\infty(\mathbb{R}^n)$ denote the set of smooth functions on $\mathbb{R}^n$. An operation D on $C^\infty(\mathbb{R}^n)$ is called a *derivation* if D maps $C^\infty(\mathbb{R}^n)$ to $C^\infty(R^n)$ and satisfies the conditions

$$D(af + bg) = aD(f) + bD(g), \quad \text{where } a, b \text{ are constant.}$$

$$D(fg) = D(f)g + fD(g).$$

Example: For $1 \le i \le n$,

$$\partial_i\colon f \mapsto \frac{\partial f}{\partial x^i}$$

is a derivation.

Proposition 1.3. *Let D be any derivation on $C^\infty(\mathbb{R}^n)$. Given any point P in $\mathbb{R}^n$. Then there exist real numbers $a^1, a^2, \ldots, a^n \in \mathbb{R}$ such that*

$$D(f)(P) = \sum_{i=1}^{n} a^i \partial_i(f)(P)$$

for any $f \in C^\infty(\mathbb{R}^n)$, where a^i depends on D and P but is independent on f.

Proof. Write x for $(x^1, \ldots, x^n)$. Define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(t(x - P) + P)dt.$$

Then

$$f(x) - f(P) = \int_0^1 \frac{d}{dt} f(t(x - P) + P)dt$$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x^i}(t(x-P)+P)\cdot(x^i - x^i(P))dt$$

$$= \sum_{i=1}^n (x^i - x^i(P)) \int_0^1 \frac{\partial f}{\partial x^i}(t(x-P)+P)dt = \sum_{i=1}^n (x^i - x^i(P)) g_i(x).$$

Since D is a derivation, $D(1) = D(1\cdot 1) = D(1)\cdot 1 + 1\cdot D(1)$ and so $D(1) = 0$. It follows that $D(c) = 0$ for any constant c. By applying D to the above equations,

$$D(f(x)) = D(f(x) - f(P)) = \sum_{i=1}^n D(x^i - x^i(P)) g_i(x) + (x^i - x^i(P)) D(g_i(x))$$

$$= \sum_{i=1}^n D(x^i) g_i(x) + (x^i - x^i(P)) D(g_i(x))$$

because $D(f(P)) = D(x^i(P)) = 0$. Let $a^i = D(x^i)(P)$ which only depends on D and P. By evaluating at P,

$$D(f)(P) = \sum_{i=1}^n D(x^i)(P) g_i(P) + 0 = \sum_{i=1}^n a^i g_i(P).$$

Since

$$g_i(P) = \int_0^1 \frac{\partial f}{\partial x^i}(t(P-P)+P)dt = \int_0^1 \frac{\partial f}{\partial x^i}(P)dt = \frac{\partial f}{\partial x^i}(P) = \partial_i(f)(P),$$

$$D(f)(P) = \sum_{i=1}^n a^i \partial_i(f)(P),$$

which is the conclusion. □

From this proposition, we can give a new way to looking at vector fields:
Given a vector fields $P \mapsto \vec{v}(P) = (v^1(P), v^2(P), \ldots, v^n(P))$, a derivation

$$D_{\vec{v}} = \sum_{i=1}^n v^i(P)\cdot \partial_i$$

on $C^\infty(\mathbb{R}^n)$ is called an *operator representation* of the vector field $P \mapsto \vec{v}(P)$.
Note. The operation $v^i(x)\partial_i$ is given as follows: for any $f \in C^\infty(\mathbb{R}^n)$,

$$D_{\vec{v}}(f)(P) = \sum_{i=1}^n v^i(P)\cdot \partial_i(f)(P)$$

for any P.

From this new view, the tangent spaces $T(\mathbb{R}^n)$ admits a basis $\{\partial_1, \partial_2, \ldots, \partial_n\}$.

1.4. **Integral Curves.** Let $V\colon \mathbf{x} \mapsto \vec{v}(\mathbf{x})$ be a (smooth) vector field on an neighborhood U of P. An *integral curve* to V is a smooth curve $\mathbf{s}\colon (-\delta, \epsilon) \to U$, defined for suitable $\delta, \epsilon > 0$, such that

$$\mathbf{s}'(t) = \vec{v}(\mathbf{s}(t))$$

for $-\delta < t < \epsilon$.

Theorem 1.4. *Let $V\colon \mathbf{x} \mapsto \vec{v}(\mathbf{x})$ be a (smooth) vector field on an neighborhood U of P. Then there exists an integral curve to V through P. Any two such curves agree on their common domain.*

Proof. The proof is given by assuming the fundamental existence and uniqueness theorem for systems of first order differential equations.

The requirement for a curve $\mathbf{s}(t) = (s^1(t), \dots, s^n(t))$ to be an integral curve is:

$$\begin{cases} \frac{ds^1(t)}{dt} = v^1(s^1(t), s^2(t), \dots, s^n(t)) \\ \frac{ds^2(t)}{dt} = v^2(s^1(t), s^2(t), \dots, s^n(t)) \\ \cdots\cdots\cdots \\ \frac{ds^n(t)}{dt} = v^n(s^1(t), s^2(t), \dots, s^n(t)) \end{cases}$$

with the initial conditions

$$\mathbf{s}(0) = P \qquad (s^1(0), s^2(0), \dots, s^n(0)) = (x^1(P), x^2(P), \dots, x^n(P))$$

$$\mathbf{s}'(0) = \vec{v}(P) \quad \left(\frac{ds^1}{dt}(0), \dots, \frac{ds^n}{dt}(0)\right) = (v^1(P), \dots, v^n(P)).$$

Thus the statement follows from the fundamental theorem of first order ODE. ☐

Example 1.5. Let $n = 2$ and let $V\colon P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$, where $v^1(x, y) = x$ and $v^2(x, y) = y$. Given a point $P = (a^1, a^2)$, the equation for the integral curve $\mathbf{s}(t) = (x(t), y(t))$ is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = y(t) \end{cases}$$

with initial conditions $(x(0), y(0)) = (a^1, a^2)$ and $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, a^2)$. Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^t).$$

Example 1.6. Let $n = 2$ and let $V\colon P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$, where $v^1(x, y) = x$ and $v^2(x, y) = -y$. Given a point $P = (a^1, a^2)$, the equation for the integral curve $\mathbf{s}(t) = (x(t), y(t))$ is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = -y(t) \end{cases}$$

with initial conditions $(x(0), y(0)) = (a^1, a^2)$ and $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, -a^2)$. Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^{-t}).$$

1.5. Implicit- and Inverse-Mapping Theorems.

Theorem 1.7. *Let D be an open region in $\mathbb{R}^{n+1}$ and let F be a function well-defined on D with continuous partial derivatives. Let $(x_0^1, x_0^2, \ldots, x_0^n, z_0)$ be a point in D where*

$$F(x_0^1, x_0^2, \ldots, x_0^n, z_0) = 0 \qquad \frac{\partial F}{\partial z}(x_0^1, x_0^2, \ldots, x_0^n, z_0) \neq 0.$$

Then there is a neighborhood $N_\epsilon(z_0) \subseteq \mathbb{R}$, a neighborhood $N_\delta(x_0^1, \ldots, x_0^n) \subseteq \mathbb{R}^n$, and a **unique** *function $z = g(x^1, x^2, \ldots, x^n)$ defined for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ with values $z \in N_\epsilon(z_0)$ such that*

1) *$z_0 = g(x_0^1, x_0^2, \ldots, x_0^n)$ and*
$$F(x^1, x^2, \ldots, x^n, g(x^1, \ldots, x^n)) = 0$$
for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.
2) *g has continuous partial derivatives with*
$$\frac{\partial g}{\partial x^i}(x^1, \ldots, x^n) = -\frac{F_{x^i}(x^1, \ldots, x^n, z)}{F_z(x^1, \ldots, x^n, z)}$$
for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ where $z = g(x^1, \ldots, x^n)$.
3) *If F is smooth on D, then $z = g(x^1, \ldots, x^n)$ is smooth on $N_\delta(x_0^1, \ldots, x_0^n)$.*

Proof. **Step 1.** We may assume that $\frac{\partial F}{\partial z}(x_0^1, x_0^2, \ldots, x_0^n, z_0) > 0$. Since F_z is continuous, there exists a neighborhood $N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0)$ in which F_z is continuous and positive. Thus for fixed $(x^1, \ldots, x^n)$, F is strictly increasing on z in this neighborhood. It follows that there exists $c > 0$ such that

$$F(x_0^1, x_0^2, \ldots, x_0^n, z_0 - c) < 0 \qquad F(x_0^1, x_0^2, \ldots, x_0^n, z_0 + c) > 0$$

with

$$(x_0^1, x_0^2, \ldots, x_0^n, z_0 - c), (x_0^1, x_0^2, \ldots, x_0^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0).$$

Step 2. By the continuity of F, there exists a small $\delta > 0$ such that

$$F(x^1, x^2, \ldots, x^n, z_0 - c) < 0 \qquad F(x^1, x^2, \ldots, x^n, z_0 + c) > 0$$

with

$$(x^1, x^2, \ldots, x^n, z_0 - c), (x^1, x^2, \ldots, x^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0)$$

for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.

Step 3. Fixed $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$, F is continuous and strictly increasing on z. There is a **unique** z, $z_0 - c < z < z_0 + c$, such that

$$F(x^1, \ldots, x^n, z) = 0.$$

This defines a function $z = g(x^1, \ldots, x^n)$ for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ with values $z \in (z_0 - c, z_0 + c)$.

Step 4. Prove that $z = g(x^1, \ldots, x^n)$ is continuous. Let $(x_1^1, \ldots, x_1^n) \in N_\delta(x_0^1, \ldots, x_0^n)$. Let $(x_1^1(k), \ldots, x_1^n(k))$ be any sequence in $N_\delta(x_0^1, \ldots, x_0^n)$ converging to $(x_1^1, \ldots, x_1^n)$. Let A be any subsequential limit of $\{z_k = g(x_1^1(k), \ldots, x_1^n(k))\}$, that is $A = \lim_{s\to\infty} z_{k_s}$. Then, by the continuity of F,

$$\begin{aligned} 0 &= \lim_{s\to\infty} F(x_1^1(k_s), \ldots, x_1^n(k_s), z_{k_s}) \\ &= F(\lim_{s\to\infty} x_1^1(k_s), \ldots, \lim_{s\to\infty} x_1^n(k_s), \lim_{s\to\infty} z_{k_s}) \end{aligned}$$

$$= F(x_1^1,\dots,x_1^n,A).$$

By the unique solution of the equation, $A = g(x_1^1,\dots,x_1^n)$. Thus $\{z_k\}$ converges $g(x_1^1,\dots,x_1^n)$ and so g is continuous.

Step 5. Compute the partial derivatives $\frac{\partial z}{\partial x_i}$. Let h be small enough. Let

$$z+k = g(x^1,\dots,x^{i-1},x^i+h,x^{i+1},\dots,x^n),$$

that is

$$F(x^1,\dots,x^i+h,\dots,x^n,z+k)=0$$

with $z_0-c<z+k<z_0+c$. Then

$$\begin{aligned}0 &= F(x^1,\dots,x^i+h,\dots,x^n,z+k)-F(x^1,\dots,x^n,z)\\ &= F_{x^i}(x^1,\dots,\tilde{x}^i,\dots,x^n,\tilde{z})h+F_z(x^1,\dots,\tilde{x}^i,\dots,x^n,\tilde{z})k\end{aligned}$$

by the mean value theorem (Consider the function

$$\phi(t)=F(x^1,\dots,x^i+th,\dots,x^n,z+tk)$$

for $0\le t\le 1$. Then $\phi(1)-\phi(0)=\phi'(\xi)(1-0)$.), where $\tilde{x}^i$ is between x^i and x^i+h, and $\tilde{z}$ is between z and $z+k$. Now

$$\begin{aligned}\frac{\partial g}{\partial x_i} &= \lim_{h\to 0}\frac{g(x^1,\dots,x^{i-1},x^i+h,x^{i+1},\dots,x^n)-z}{h}=\lim_{h\to 0}\frac{k}{h}\\ &= -\lim_{h\to 0}\frac{F_{x^i}(x^1,\dots,\tilde{x}^i,\dots,x^n,\tilde{z})}{F_z(x^1,\dots,\tilde{x}^i,\dots,x^n,\tilde{z})}.=-\frac{F_{x_i}}{F_z},\end{aligned}$$

where $\tilde{z}\to z$ as $h\to 0$ because g is continuous (and so $k\to 0$ as $h\to 0$).

Step 6. Since F_z is not zero in this small neighborhood, g_{x_i} is continuous for each i. If F is smooth, then all higher derivatives of g are continuous and so g is also smooth. □

Theorem 1.8 (Implicit Function Theorem). *Let D be an open region in $\mathbb{R}^{2n}$ and let $F_1,F_2,\dots,F_n$ be functions well-defined on D with continuous partial derivatives. Let $(x_0^1,x_0^2,\dots,x_0^n,u_0^1,u_0^2,\dots,u_0^n)$ be a point in D where*

$$\begin{cases} F_1(x_0^1,x_0^2,\dots,x_0^n,u_0^1,u_0^2,\dots,u_0^n)=0\\ F_2(x_0^1,x_0^2,\dots,x_0^n,u_0^1,u_0^2,\dots,u_0^n)=0\\ \qquad\cdots\cdots\cdots\cdots\\ F_n(x_0^1,x_0^2,\dots,x_0^n,u_0^1,u_0^2,\dots,u_0^n)=0\end{cases}$$

and the Jacobian

$$J=\frac{\partial(F_1,F_2,\dots,F_n)}{\partial(u^1,u^2,\dots,u^n)}=\det\left(\frac{\partial F_i}{\partial u^j}\right)\neq 0$$

at the point $(x_0^1,x_0^2,\dots,x_0^n,u_0^1,u_0^2,\dots,u_0^n)$. Then there are neighborhoods $N_\delta(x_0^1,\dots,x_0^n)$, $N_{\epsilon_1}(u_0^1)$, $N_{\epsilon_2}(u_0^2)$, …, $N_{\epsilon_n}(u_0^n)$, and **unique** *functions*

$$\begin{cases} u^1=g_1(x^1,x^2,\dots,x^n)\\ u^2=g_2(x^1,x^2,\dots,x^n)\\ \qquad\cdots\cdots\cdots\cdots\\ u^n=g_n(x^1,x^2,\dots,x^n)\end{cases}$$

defined for $(x^1,\dots,x^n)\in N_\delta(x_0^1,\dots,x_0^n)$ with values $u^1\in N_{\epsilon_1}(u_0^1),\dots,u^n\in N_{\epsilon_n}(u_0^n)$ such that

1) *$u_0^i = g_i(x_0^1, x_0^2, \ldots, x_0^n)$ and*

$$F_i(x^1, x^2, \ldots, x^n, g_i(x^1, \ldots, x^n)) = 0$$

for all $1 \leq i \leq n$ and all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.

2) *Each g_i has continuous partial derivatives with*

$$\frac{\partial g_i}{\partial x^j}(x^1, \ldots, x^n) = -\frac{1}{J} \cdot \frac{\partial(F_1, \ldots, F_n)}{\partial(u^1, u^2, \ldots, u^{j-1}, x^j, u^{j+1}, \ldots, u^n)}$$

for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ where $u^i = g_i(x^1, \ldots, x^n)$.

3) *If each F_i is smooth on D, then each $u^i = g_i(x^1, \ldots, x^n)$ is smooth on $N_\delta(x_0^1, \ldots, x_0^n)$.*

Sketch of Proof. The proof is given by induction on n. Assume that the statement holds for $n-1$ with $n > 1$. (We already prove that the statement holds for $n = 1$.) Since the matrix

$$\left(\frac{\partial F_i}{\partial u^j}\right)$$

is invertible at the point $P = (x_0^1, x_0^2, \ldots, x_0^n, u_0^1, u_0^2, \ldots, u_0^n)$ (because the determinant is not zero), we may assume that

$$\frac{\partial F_n}{\partial u^n}(P) \neq 0.$$

(The entries in the last column can not be all 0 and so, if $\frac{\partial F_i}{\partial u^n}(P) \neq 0$, we can interchange F_i and F_n.)

From the previous theorem, there is a solution

$$u^n = g_n(x^1, \ldots, x^n, u^1, \ldots, u^{n-1})$$

to the last equation. Consider

$$\begin{cases} G_1 = F_1(x^1, \ldots, x^n, u^1, \ldots, u^{n-1}, g_n) \\ G_2 = F_2(x^1, \ldots, x^n, u^1, \ldots, u^{n-1}, g_n) \\ \cdots\cdots\cdots\cdots \\ G_{n-1} = F_{n-1}(x^1, \ldots, x^n, u^1, \ldots, u^{n-1}, g_n). \end{cases}$$

Then

$$\frac{\partial G_i}{\partial u^j} = \frac{\partial F_i}{\partial u^j} + \frac{\partial F_i}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j}$$

for $1 \leq i, j \leq n-1$, where

$$\frac{\partial F_n}{\partial u^j} + \frac{\partial F_n}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j} = 0.$$

Let

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial g_n}{\partial u^1} & \frac{\partial g_n}{\partial u^2} & \frac{\partial g_n}{\partial u^3} & \cdots & \frac{\partial g_n}{\partial u^{n-1}} & 1 \end{pmatrix}$$

Then

$$\left(\frac{\partial F_i}{\partial u^j}\right) \cdot B = \begin{pmatrix} \left(\frac{\partial G_i}{\partial u^j}\right)_{n-1,n-1} & * \\ 0 & \frac{\partial F_n}{\partial u^n} \end{pmatrix}.$$

By taking the determinant,

$$J = \frac{\partial(F_1,\ldots,F_n)}{\partial(u^1,\ldots,u^n)} = \frac{\partial F_n}{\partial u^n}\cdot\frac{\partial(G_1,\ldots,G_{n-1})}{\partial(u^1,\ldots,u^{n-1})}.$$

Thus $\frac{\partial(G_1,\ldots,G_{n-1})}{\partial(u^1,\ldots,u^{n-1})} \neq 0$ at P and, by induction, there are solutions

$$u^i = g_i(x^1,\ldots,x^n)$$

for $1\le i\le n-1$. □

Theorem 1.9 (Inverse Mapping Theorem)**.** *Let D be an open region in $\mathbb{R}^n$. Let*

$$\begin{cases} x^1 = f_1(u^1,\ldots,u^n)\\ x^2 = f_2(u^1,\ldots,u^n)\\ \quad\cdots\cdots\\ x^n = f_n(u^1,\ldots,u^n)\end{cases}$$

be functions defined on D with continuous partial derivatives. Let $(u_0^1,\ldots,u_0^n)\in D$ satisfy $x_0^i = f_i(u_0^1,\ldots,u_0^n)$ and the Jacobian

$$\frac{\partial(x^1,\ldots,x^n)}{\partial(u^1,\ldots,u^n)} \neq 0 \qquad \text{at} \quad (u_0^1,\ldots,u_0^n).$$

Then there are neighborhood $N_\delta(x_0^1,\ldots,x_0^n)$ and $N_\epsilon(u_0^1,\ldots,u_0^n)$ such that

$$\begin{cases} u^1 = f_1^{-1}(x^1,\ldots,x^n)\\ u^2 = f_2^{-1}(x^1,\ldots,x^n)\\ \quad\cdots\cdots\\ u^n = f_n^{-1}(x^1,\ldots,x^n)\end{cases}$$

is well-defined and has continuous partial derivatives on $N_\delta(x_0^1,\ldots,x_0^n)$ with values in $N_\epsilon(u_0^1,\ldots,u_0^n)$. Moreover if each f_i is smooth, then each f_i^{-1} is smooth.

Proof. Let $F_i = f_i(u^1,\ldots,u^n) - x_i$. The assertion follows from the Implicit Function Theorem. □

2. Topological and Differentiable Manifolds, Diffeomorphisms, Immersions, Submersions and Submanifolds

2.1. **Topological Spaces.**

Definition 2.1. Let X be a set. A *topology* $\mathcal{U}$ for X is a collection of subsets of X satisfying

i) $\emptyset$ and X are in $\mathcal{U}$;
ii) the intersection of two members of $\mathcal{U}$ is in $\mathcal{U}$;
iii) the union of any number of members of $\mathcal{U}$ is in $\mathcal{U}$.

The set X with $\mathcal{U}$ is called a *topological space*. The members $U\in\mathcal{U}$ are called the *open sets*.

Let X be a topological space. A subset $N\subseteq X$ with $x\in N$ is called a *neighborhood* of x if there is an open set U with $x\in U\subseteq N$. For example, if X is a metric space, then the closed ball $D_\epsilon(x)$ and the open ball $B_\epsilon(x)$ are neighborhoods of x. A subset C is said to *closed* if $X\setminus C$ is open.

Definition 2.2. A function $f\colon X\to Y$ between two topological spaces is said to be *continuous* if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X.

A continuous function from a topological space to a topological space is often simply called a *map*. A *space* means a *Hausdorff space*, that is, a topological spaces where any two points has disjoint neighborhoods.

Definition 2.3. Let X and Y be topological spaces. We say that X and Y are *homeomorphic* if there exist continuous functions $f\colon X \to Y, g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We write $X \cong Y$ and say that f and g are *homeomorphisms* between X and Y.

By the definition, a function $f\colon X \to Y$ is a homeomorphism if and only if

- i) f is a bijective;
- ii) f is continuous and
- iii) f^{-1} is also continuous.

Equivalently f is a homeomorphism if and only if 1) f is a bijective, 2) f is continuous and 3) f is an open map, that is f sends open sets to open sets. Thus a homeomorphism between X and Y is a bijective between the points and the open sets of X and Y.

A very general question in topology is how to classify topological spaces under homeomorphisms. For example, we know (from complex analysis and others) that any simple closed loop is homeomorphic to the unit circle S^1. Roughly speaking topological classification of curves is known. The topological classification of (two-dimensional) surfaces is known as well. However the topological classification of 3-dimensional manifolds (we will learn manifolds later.) is quite open.

The famous Poicaré conjecture is related to this problem, which states that any simply connected 3-dimensional (topological) manifold is homeomorphic to the 3-sphere S^3. A space X is called *simply connected* if (1) X is path-connected (that is, given any two points, there is a continuous path joining them) and (2) the fundamental group $\pi_1(X)$ is trivial (roughly speaking, any loop can be deformed to be the constant loop in X). The *manifolds* are the objects that we are going to discuss in this course.

2.2. **Topological Manifolds.** A Hausdorff space M is called a *(topological) n-manifold* if each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. Roughly speaking, an n-manifold is *locally* $\mathbb{R}^n$. Sometimes M is denoted as M^n for mentioning the dimension of M.

(**Note.** If you are not familiar with topological spaces, you just think that M is a subspace of $\mathbb{R}^N$ for a large N.)

For example, $\mathbb{R}^n$ and the n-sphere S^n is an n-manifold. A 2-dimensional manifold is called a *surface*. The objects traditionally called 'surfaces in 3-space' can be made into manifolds in a standard way. The compact surfaces have been classified as spheres or projective planes with various numbers of handles attached.

By the definition of manifold, the closed n-disk D^n is not an n-manifold because it has the 'boundary' S^{n-1}. D^n is an example of 'manifolds with boundary'. We give the definition of manifold with boundary as follows.

A Hausdorff space M is called an *n-manifold with boundary* ($n \geq 1$) if each point in M has a neighborhood homeomorphic to an open set in the half space

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

Manifold is one of models that we can do calculus 'locally'. By means of calculus, we need local coordinate systems. Let $x \in M$. By the definition, there is a an open neighborhood $U(x)$ of x and a homeomorphism ϕ_x from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The collection $\{(U(x), \phi_x) | x \in M\}$ has the property that 1) $\{U(x) | x \in M\}$ is an open cover and 2) ϕ_x is a homeomorphism from $U(x)$

onto an open set in $\mathbb{R}^n_+$. The subspace $\phi_x(U_x)$ in $\mathbb{R}^n_+$ plays a role as a local coordinate system. The collection $\{(U(x), \phi_x)|x \in M\}$ is somewhat too large and we may like less local coordinate systems. This can be done as follows.

Let M be a space. A *chart* of M is a pair (U, ϕ) such that 1) U is an open set in M and 2) ϕ is a homeomorphism from U onto an open set in $\mathbb{R}^n_+$. The map

$$\phi\colon U \to \mathbb{R}^n_+$$

can be given by n coordinate functions $\phi_1, \ldots, \phi_n$. If P denotes a point of U, these functions are often written as

$$x^1(P), x^2(P), \ldots, x^n(P)$$

or simply $x^1, x^2, \ldots, x^n$. They are called *local coordinates* on the manifold.

An *atlas* for M means a collection of charts $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ such that $\{U_\alpha|\alpha \in J\}$ is an open cover of M.

Proposition 2.4. *A Hausdorff space M is a manifold (with boundary) if and only if M has an atlas.*

Proof. Suppose that M is a manifold. Then the collection $\{(U(x), \phi_x)|x \in M\}$ is an atlas. Conversely suppose that M has an atlas. For any $x \in M$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x that is homeomorphic to an open set in $\mathbb{R}^n_+$. Thus M is a manifold. □

We define a subset ∂M as follows: $x \in \partial M$ if there is a chart (U_α, ϕ_α) such that $x \in U_\alpha$ and $\phi_\alpha(x) \in \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 0\}$. ∂M is called the boundary of M. For example the boundary of D^n is S^{n-1}.

Proposition 2.5. *Let M be a n-manifold with boundary. Then ∂M is an $(n-1)$-manifold without boundary.*

Proof. Let $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ be an atlas for M. Let $J' \subseteq J$ be the set of indices such that $U_\alpha \cap \partial M \neq \emptyset$ if $\alpha \in J'$. Then Clearly

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M}|\alpha \in J'\}$$

can be made into an atlas for ∂M. □

Note. The key point here is that if U is open in $\mathbb{R}^n_+$, then $U \cap \mathbb{R}^{n-1}$ is also open because: Since U is open in $\mathbb{R}^n_+$, there is an open subset V of $\mathbb{R}^n$ such that $U = V \cap \mathbb{R}^n_+$. Now if $x \in U \cap \mathbb{R}^{n-1}$, there is an open disk $E_\epsilon(x) \subseteq V$ and so

$$E_\epsilon(x) \cap \mathbb{R}^{n-1} \subseteq V \cap \mathbb{R}^{n-1} = U \cap \mathbb{R}^{n-1}$$

is an open $(n-1)$-dimensional ϵ-disk in $\mathbb{R}^{n-1}$ centered at x.

2.3. **Differentiable Manifolds.**

Definition 2.6. A Hausdorff space M is called a *differential manifold of class C^k (with boundary)* if there is an atlas of M

$$\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$$

such that

For any $\alpha, \beta \in J$, the composites

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$

is differentiable of class C^k.

The atlas $\{(U_\alpha, \phi_\alpha | \alpha \in J\}$ is called a *differential atlas of class C^k* on M.

(**Note.** Assume that M is a subspace of $\mathbb{R}^N$ with $N >> 0$. If M has an atlas $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ such that each $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n_+$ is differentiable of class C^k, then M is a differentiable manifold of class C^k. This is the definition of differentiable (smooth) manifolds in [3] as in the beginning they already assume that M is a subspace of $\mathbb{R}^N$ with N large. In our definition (the usual definition of differentiable manifolds using charts), we only assume that M is a (Hausdorff) topological space and so ϕ_α is *only an identification* of an abstract U_α with an open subset of $\mathbb{R}^n_+$. In this case we can not talk differentiability of ϕ_α unless U_α is regarded as a subspace of a (large dimensional) Euclidian space.)

Two differential atlases of class C^k $\{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ and $\{(V_\beta, \psi_\beta) | \beta \in J\}$ are called *equivalent* if

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\} \cup \{(V_\beta, \psi_\beta) | \beta \in J\}$$

is again a differential atlas of class C^k (this is an equivalence relation). A *differential structure of class C^k* on M is an equivalence class of differential atlases of class C^k on M. Thus a differential manifold of class C^k means a manifold with a differential structure of class C^k. A *smooth* manifold means a differential manifold of class C^∞.

Note: A general manifold is also called *topological manifold.* Kervaire and Milnor [1] have shown that the topological sphere S^7 has 28 distinct oriented smooth structures.

Definition 2.7. let M and N be smooth manifolds (with boundary) of dimensions m and n respectively. A map $f \colon M \to N$ is called *smooth* if for some smooth atlases $\{(U_\alpha, \phi_\alpha | \alpha \in I\}$ for M and $\{(V_\beta, \psi_\beta) \beta \in J\}$ for N the functions

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}|_{\phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha)} \colon \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n_+$$

are of class C^∞.

Proposition 2.8. *If $f \colon M \to N$ is smooth with respect to atlases*

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\}, \quad \{(V_\beta, \phi_\beta | \beta \in J\}$$

for M, N then it is smooth with respect to equivalent atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \quad \{(V'_\gamma, \eta_\gamma | \beta \in J'\}$$

Proof. Since f is smooth with respect with the atlases

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\}, \ \{(V_\beta, \phi_\beta | \beta \in J\},$$

f is smooth with respect to the smooth atlases

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\} \cup \{(U'_\delta, \theta_\delta | \alpha \in I'\}, \quad \{(V_\beta, \phi_\beta | \beta \in J\} \cup \{(V'_\gamma, \eta_\gamma | \beta \in J'\}$$

by look at the local coordinate systems. Thus f is smooth with respect to the atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \ \{(V'_\gamma, \eta_\gamma | \beta \in J'\}.$$

□

Thus the definition of smooth maps between two smooth manifolds is independent of choice of atlas.

Definition 2.9. A smooth map $f\colon M \to N$ is called a *diffeomorphism* if f is one-to-one and onto, and if the inverse $f^{-1}\colon N \to M$ is also smooth.

Definition 2.10. Let M be a smooth n-manifold, possibly with boundary. A subset X is called a *properly embedded submanifold* of dimension $k \leq n$ if X is a closed in M and, for each $P \in X$, there exists a chart (U, ϕ) about P in M such that

$$\phi(U \cap X) = \phi(U) \cap \mathbb{R}^k_+,$$

where $\mathbb{R}^k_+ \subseteq \mathbb{R}^n_+$ is the standard inclusion.

Note. In the above definition, the collection $\{(U \cap X, \phi|_{U\cap X})\}$ is an atlas for making X to a smooth k-manifold with boundary $\partial X = X \cap \partial M$.

If $\partial M = \emptyset$, by dropping the requirement that X is a closed subset but keeping the requirement on local charts, X is called simply a *submanifold* of M.

2.4. **Tangent Space.** Let S be an open region of $\mathbb{R}^n$. Recall that, for $P \in S$, the tangent space $T_P(S)$ is just the n-dimensional vector space by putting the origin at P. Let T be an open region of $\mathbb{R}^m$ and let $f = (f_1, \ldots, f_m)\colon S \to T$ be a smooth map. Then f induces a *linear transformation*

$$Tf\colon T_P(S) \to T_{f(P)}(T)$$

given by

$$Tf(\vec{v}) = \left(\frac{\partial f_i}{\partial x^j}\right)_{m\times n} \cdot \begin{pmatrix} v^1 \\ v^2 \\ \cdots \\ v^n \end{pmatrix}_{n\times 1} = \begin{pmatrix} v^1\partial_1(f_1) + v^2\partial_2(f_1) + \cdots + v^n\partial_n(f_1) \\ v^1\partial_1(f_2) + v^2\partial_2(f_2) + \cdots + v^n\partial_n(f_2) \\ \cdots\cdots\cdots\cdots \\ v^1\partial_1(f_m) + v^2\partial_2(f_m) + \cdots + v^n\partial_n(f_m) \end{pmatrix},$$

namely Tf is obtained by taking directional derivatives of $(f_1, \ldots, f_m)$ along vector $\vec{v}$ for any $\vec{v} \in T_P(S)$.

Now we are going to define the *tangent space* to a (differentiable) manifold M at a point P as follows:

First we consider the set

$$\mathcal{T}_P = \{(U, \phi, \vec{v}) \mid P \in U, \quad (U, \phi) \text{ is a chart } \quad \vec{v} \in T(\phi(P))(\phi(U))\}.$$

The point is that there are possibly many charts around P. Each chart creates an n-dimension vector space. So we need to define an *equivalence relation* in $\mathcal{T}_P$ such that, $\mathcal{T}_P$ modulo these relations is only one copy of n-dimensional vector space which is also independent on the choice of charts.

Let $(U, \phi, \vec{v})$ and $(V, \psi, \vec{w})$ be two elements in $\mathcal{T}_P$. That is (U, ϕ) and (V, ψ) are two charts with $P \in U$ and $P \in V$. By the definition,

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

is diffeomorphism and so it induces an isomorphism of vector spaces

$$T(\psi \circ \phi^{-1})\colon T_{\phi(P)}(\phi(U \cap V)) \longrightarrow T_{\psi(P)}(\psi(U \cap V)).$$

Now $(U, \phi, \vec{v})$ is called equivalent to $(V, \psi, \vec{w})$, denoted by $(U, \phi, \vec{v}) \sim (V, \psi, \vec{w})$, if

$$T(\psi \circ \phi^{-1})(\vec{v}) = \vec{w}.$$

Define $T_P(M)$ to be the quotient

$$T_P(M) = \mathcal{T}_P/\sim .$$

Exercise 2.1. Let M be a differentiable n-manifold and let P be any point in M. Prove that $T_P(M)$ is an n-dimensional vector space. [Hint: Fixed a chart (U,ϕ) and defined

$$a(U,\phi,\vec{v}) + b(U,\phi,\vec{w}) := (U,\phi,a\vec{v}+b\vec{w}).$$

Now given any $(V,\psi,\vec{x}),(\tilde{V},\tilde{\psi},\vec{y}) \in \mathcal{T}_P$, consider the map

$$\phi\circ\psi^{-1}\colon \psi(U\cap V)\to\phi(U\cap V) \qquad \phi\circ\tilde{\psi}^{-1}\colon \tilde{\psi}(U\cap\tilde{V})\to\phi(U\cap\tilde{V})$$

and define

$$a(V,\psi,\vec{x}) + b(\tilde{V},\tilde{\psi},\vec{y}) = (U,\phi,aT(\phi\circ\psi^{-1})(\vec{x}) + bT(\phi\circ\tilde{\psi}^{-1})(\vec{y})).$$

Then prove that this operation gives a well-defined vector space structure on T_P, that is, independent on the equivalence relation.]

The tangent space $T_P(M)$, as a vector space, can be described as follows: given any chart (U,ϕ) with $P\in U$, there is a unique isomorphism

$$T_\phi\colon T_P(M)\to T_{\phi(P)}(\phi(U)).$$

by choosing $(U,\phi,\vec{v})$ as representatives for its equivalence class. If (V,ψ) is another chart with $P\in V$, then there is a commutative diagram

(1)
$$\begin{tikzcd}
T_P(M) \arrow[r, "T_\phi", "\cong"'] \arrow[d, equal] & T_{\phi(P)}(\phi(U\cap V)) \arrow[d, "T(\psi\circ\phi^{-1})"] \\
T_P(M) \arrow[r, "T_\psi", "\cong"'] & T_{\psi(P)}(\psi(U\cap V)),
\end{tikzcd}$$

where $T(\psi\circ\phi^{-1})$ is the linear isomorphism induced by the Jacobian matrix of the differentiable map $\psi\circ\phi^{-1}\colon \phi(U\cap V)\to\psi(U\cap V)$.

Exercise 2.2. Let $f\colon M\to N$ be a smooth map, where M and N need not to have the same dimension. Prove that there is a unique linear transformation

$$Tf\colon T_P(M)\longrightarrow T_{f(P)}(N)$$

such that the diagram

$$\begin{tikzcd}
T_P(M) \arrow[r, "T_\phi", "\cong"'] \arrow[d, "Tf"] & T_{\phi(P)}(\phi(U)) \arrow[d, "T(\psi\circ f\circ\phi^{-1})"] \\
T_{f(P)}(N) \arrow[r, "T_\psi", "\cong"'] & T_{\psi(f(P))}(\psi(V))
\end{tikzcd}$$

commutes for any chart (U,ϕ) with $P\in U$ and any chart (V,ψ) with $f(P)\in V$. [First fix a choice of (U,ϕ) with $P\in U$ and (V,ψ) with $f(P)\in V$, the linear transformation Tf is uniquely defined by the above diagram. Then use Diagram (1) to check that Tf is independent on choices of charts.

2.5. **Immersions.** A smooth map $f\colon M \to N$ is called *immersion* at P if the linear transformation

$$Tf\colon T_P(M) \to T_{f(P)}(M)$$

is injective.

Theorem 2.11 (Local Immersion Theorem)**.** *Suppose that $f\colon M^m \to N^n$ is immersion at P. Then there exist charts (U, ϕ) about P and (V, ψ) about $f(P)$ such that the diagram*

$$\begin{tikzcd}
U \arrow[r, "f|_U"] \arrow[d, "\phi(P)=0"', "\phi"] & V \arrow[d, "\psi(f(P))=0"', "\psi"] \\
\mathbb{R}^m \arrow[r, hook, "\text{canonical coordinate inclusion}"] & \mathbb{R}^n
\end{tikzcd}$$

commutes.

Proof. We may assume that $\phi(P) = 0$ and $\psi(f(P)) = 0$. (Otherwise replacing ϕ and ψ by $\phi - \phi(P)$ and $\psi - \psi(f(P))$, respectively.)

Consider the commutative diagram

$$\begin{tikzcd}
U \arrow[r, "f|_U"] \arrow[d, "\phi"] & V \arrow[d, "\psi"] \\
\phi(U) \arrow[r, "g = \psi \circ f \circ \phi^{-1}"] \arrow[d, hook] & \psi(V) \arrow[d, hook] \\
\mathbb{R}^m & \mathbb{R}^n.
\end{tikzcd}$$

By the assumption,

$$Tg\colon T_0(\phi(U)) \longrightarrow T_0(\psi(V))$$

is an injective linear transformation and so

$$\operatorname{rank}(Tg) = m$$

at the origin. The matric for Tg is

$$\begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \frac{\partial g^1}{\partial x^2} & \cdots & \frac{\partial g^1}{\partial x^m} \\ \frac{\partial g^2}{\partial x^1} & \frac{\partial g^2}{\partial x^2} & \cdots & \frac{\partial g^2}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g^m}{\partial x^1} & \frac{\partial g^m}{\partial x^2} & \cdots & \frac{\partial g^m}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g^n}{\partial x^1} & \frac{\partial g^n}{\partial x^2} & \cdots & \frac{\partial g^n}{\partial x^m} \end{pmatrix}. \tag{2}$$

By changing basis of $\mathbb{R}^n$ (corresponding to change the rows), we may assume that the first m rows forms an invertible matrix $A_{m\times m}$ at the origin.

Define a function

$$h=(h^1,h^2,\ldots,h^n)\colon \phi(U)\times\mathbb{R}^{n-m}\longrightarrow\mathbb{R}^n$$

by setting

$$h^i(x^1,\ldots,x^m,x^{m+1},\ldots,x^n)=g^i(x^1,\ldots,x^m)$$

for $1\leq i\leq m$ and

$$h^i(x^1,\ldots,x^m,x^{m+1},\ldots,x^n)=g^i(x^1,\ldots,x^m)+x^i$$

for $m+1\leq i\leq n$. Then Jacobian matrix of h is

$$\begin{pmatrix} A_{m\times m} & 0_{m\times(n-m)} \\ B_{(n-m)\times m} & I_{n-m} \end{pmatrix},$$

where B is taken from $(m+1)$-st row to n-th row in the matrix (2). Thus the Jacobian of h is not zero at the origin. By the Inverse Mapping Theorem, h is an diffeomorphism in a small neighborhood of the origin. It follows that there exist open neighborhoods $\tilde{U}\subseteq U$ of P and $\tilde{V}\subseteq V$ of $f(P)$ such that the following diagram commutes

$$\begin{tikzcd}
\tilde{U} \arrow[r, "f|_{\tilde{U}}"] \arrow[d, "\phi|_{\tilde{U}}"] & \tilde{V} \arrow[d, "\psi|_{\tilde{V}}"] \\
\phi(\tilde{U}) \arrow[r, "g = \psi \circ f \circ \phi^{-1}"] \arrow[d, equal] & \psi(\tilde{V}) \arrow[d, "h^{-1}", "\cong"'] \\
\phi(\tilde{U}) \times 0 \arrow[r, hook] \arrow[d, hook] & \phi(\tilde{U}) \times U_2 \arrow[d, hook] \\
\mathbb{R}^m = \mathbb{R}^m \times 0 \arrow[r, hook] & \mathbb{R}^n,
\end{tikzcd}$$

where U_2 is a small neighborhood of the origin in $\mathbb{R}^{n-m}$. ☐

Theorem 2.12. *Let $f\colon M \to N$ be a smooth map. Suppose that*

1) *f is immersion at every point $P \in M$,*
2) *f is one-to-one and*
3) *$f\colon M \to f(M)$ is a homeomorphism.*

Then $f(M)$ is a smooth submanifold of M and $f\colon M \to f(M)$ is a diffeomorphism.

Note. In Condition 3, we need that if U is an open subset of M, then there is an open subset V of N such that $V \cap f(M) = f(U)$.

Proof. For any point P in M, we can choose the charts as in Theorem 2.11. By Condition 3, $f(U)$ is an open subset of $f(M)$. The charts $\{(f(U), \psi|_{f(U)})\}$ gives an atlas for $f(M)$ such that $f(M)$ is a submanifold of M. Now $f\colon M \to f(M)$ is a diffeomorphism because it is locally diffeomorphism and the inverse exists. ☐

Condition 3 is important in this theorem, namely an injective immersion need not give a diffeomorphism with its image. (Construct an example for this.) An injective immersion satisfying condition 3 is called an *embedding*.

2.6. **Submersions.** A smooth map $f\colon M \to N$ is called *submersion* at P if the linear transformation

$$Tf\colon T_P(M) \to T_{f(P)}(M)$$

is surjective.

Theorem 2.13 (Local Submersion Theorem)**.** *Suppose that $f\colon M^m \to N^n$ is submersion at P. Then there exist charts (U,ϕ) about P and (V,ψ) about $f(P)$ such that the diagram*

$$\begin{tikzcd}
U \arrow[r, "f|_U"] \arrow[d, "\phi(P)=0"', "\phi"] & V \arrow[d, "\psi(f(P))=0"', "\psi"] \\
\mathbb{R}^m \arrow[r, hook, "\text{canonical coordinate proj.}"] & \mathbb{R}^n
\end{tikzcd}$$

commutes.

For a smooth map of manifolds $f\colon M \to N$, a point $Q \in N$ is called *regular* if $Tf\colon T_P(M) \to T_Q(N)$ is surjective for every $P \in f^{-1}(Q)$, the pre-image of Q.

Theorem 2.14 (Pre-image Theorem)**.** *Let $f\colon M \to N$ be a smooth map and let $Q \in N$ such that $f^{-1}(Q)$ is not empty. Suppose that Q is regular. Then $f^{-1}(Q)$ is a submanifold of M with* $\dim f^{-1}(Q) = \dim M - \dim N$.

Proof. From the above theorem, for any $P \in f^{-1}(Q)$,

$$\phi|_{f^{-1}(Q)}\colon f^{-1}(Q) \cap U \longrightarrow \mathbb{R}^{m-n}$$

gives a chart about P. ☐

Let Z be a submanifold of N. A smooth map $f\colon M \to N$ is said to be *transversal* to Z if

$$\mathrm{Im}(Tf\colon T_P(M) \to T_{f(P)}(N)) + T_{f(P)}(Z) = T_{f(P)}(N)$$

for every $x \in f^{-1}(Z)$.

Theorem 2.15. *If a smooth map $f\colon M \to N$ is transversal to a submanifold $Z \subseteq N$, then $f^{-1}(Z)$ is a submanifold of M. Moreover the codimension of $f^{-1}(Z)$ in M equals to the codimension of Z in N.*

Proof. Given $P \in f^{-1}(Z)$, since Z is a submanifold, there is a chart (V,ψ) of N about $f(P)$ such that $V = V_1 \times V_2$ with $V_1 = V \cap Z$ and $(V_1, \psi|_{V_1})$ is a chart of Z about $f(P)$. By the assumption, the composite

$$f^{-1}(V) \xrightarrow{f|_{f^{-1}(V)}} V \xrightarrow{\text{proj.}} V_2$$

is submersion. By the Pre-image Theorem, $f^{-1}(V) \cap f^{-1}(Z)$ is a submanifold of the open subset $f^{-1}(V)$ of M and so there is a chart about P such that Z is a submanifold of M.

With respect to the assertion about the codimensions,

$$\mathrm{codim}(f^{-1}(Z)) = \dim V_2 = \mathrm{codim}(Z).$$

☐

Consider the special case that both M and Z are submanifolds of N. Then the transversal condition is

$$T_P(M) + T_P(Z) = T_P(N)$$

for any $P \in M \cap Z$.

Corollary 2.16. *The intersection of two transversal submanifolds of N is again a submanifold. Moreover*

$$\operatorname{codim}(M \cap Z) = \operatorname{codim}(M) + \operatorname{codim}(Z)$$

in N.

3. Fibre Bundles and Vector Bundles

4. Tangent Bundles and Vector Fields

5. Cotangent Bundles and Tensor Fields

6. Orientation of Manifolds

7. Tensor Algebras and Exterior Algebras

8. DeRham Cohomology

9. Integration on Manifolds

10. Stokes' Theorem

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