LECTURE NOTES ON DIFFERENTIABLE MANIFOLDS

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Contents

1. Tangent Spaces, Vector Fields in $\mathbb{R}^n$ and the Inverse Mapping Theorem

1.1. **Tangent Space to a Level Surface.** Let γ be a curve in $\mathbb{R}^n$: $\gamma\colon t \mapsto (\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t))$. (A curve can be described as a vector-valued function. Converse a vector-valued function gives a curve in $\mathbb{R}^n$.) The *tangent line* at the point $\gamma(t_0)$ is given with the direction

$$\frac{d\gamma}{dt}(t_0) = \left(\frac{d\gamma^1}{dt}(t_0), \ldots, \frac{d\gamma^n}{t}(t_0)\right).$$

(Certainly we need to assume that the derivatives exist. We may talk about *smooth curves*, that is, the curves with all continuous higher derivatives.)

Consider the level surface $f(x^1, x^2, \ldots, x^n) = c$ of a differentiable function f, where x^i refers to i-th coordinate. The *gradient vector* of f at a point $P = (x^1(P), x^2(P), \ldots, x^n(P))$ is

$$\nabla f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}).$$

Given a vector $\vec{u} = (u^1, \ldots, u^n)$, the *directional derivative* is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x^1} u^1 + \cdots + \frac{\partial f}{\partial x^n} u^n.$$

The *tangent space* at the point P on the level surface $f(x^1, \ldots, x^n) = c$ is the $(n-1)$-dimensional (if $\nabla f \neq 0$) space through P normal to the gradient ∇f. In other words, the tangent space is given by the equation

$$\frac{\partial f}{\partial x^1}(P)(x^1 - x^1(P)) + \cdots + \frac{\partial f}{\partial x^n}(P)(x^n - x^n(P)) = 0.$$

From the geometric views, the tangent space *should* consist of all tangents to the smooth curves **on the level surface** through the point P. Assume that γ is a curve through P (when $t = t_0$) that lies in the level surface $f(x^1, \ldots, x^n) = c$, that is

$$f(\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t)) = c.$$

By taking derivatives on both sides,

$$\frac{\partial f}{\partial x^1}(P)(\gamma^1)'(t_0) + \cdots + \frac{\partial f}{\partial x^n}(P)(\gamma^n)'(t_0) = 0$$

and so the tangent line of γ is really normal (orthogonal) to ∇f. When γ *runs over all possible curves* on the level surface through the point P, then we obtain the tangent space at the point P.

Roughly speaking, *a tangent space is a vector space attached to a point in the surface.*

How to obtain the tangent space: *take all tangent lines of smooth curve through this point on the surface.*

1.2. **Tangent Space and Vectors Fields on $\mathbb{R}^n$.** Now consider the tangent space of $\mathbb{R}^n$. According to the ideas in the previous subsection, first we assume a given point $P \in \mathbb{R}^n$. Then we consider all smooth curves passes through P and then take the tangent lines from the smooth curves. The obtained vector space at the point P is the n-dimensional space. But we can look at in a little detail.

Let γ be a smooth curve through P. We may assume that $\gamma(0) = P$. Let ω be another smooth curve with $\omega(0) = P$. γ is called to be *equivalent* to ω if the directives $\gamma'(0) = \omega'(0)$. The tangent space of $\mathbb{R}^n$ at P, denoted by $T_P(\mathbb{R}^n)$, is then the set of equivalence class of all smooth curves through P.

Let $T(\mathbb{R}^n) = \bigcup_{P \in \mathbb{R}^n} T_P(\mathbb{R}^n)$, called the tangent bundle of $\mathbb{R}^n$. If S is a region of $\mathbb{R}^n$, let $T(S) = \bigcup_{P \in S} T_P(S)$, called the tangent bundle of S.

Note. Each $T_P(\mathbb{R}^n)$ is an n-dimensional vector space, but $T(S)$ is *not* a vector space. In other words, $T(S)$ is obtained by attaching a vector space $T_P(\mathbb{R}^n)$ to each point P in S. Also S is assumed to be a region of $\mathbb{R}^n$, otherwise the tangent space of S (for instance S is a level surface) could be a proper subspace of $T_P(\mathbb{R}^n)$.

If γ is a smooth curve from P to $\mathbb{Q}$ in $\mathbb{R}^n$, then the tangent space $T_P(\mathbb{R}^n)$ moves along γ to $T_Q(\mathbb{R}^n)$. The direction for this moving is given $\gamma'(t)$, which introduces the following important concept.

Definition 1.1. A *vector field* V on a region S of $\mathbb{R}^n$ is a smooth map (also called C^∞-map)

$$V\colon S \to T(S) \quad P \mapsto \vec{v}(P).$$

Let $V\colon P \mapsto \vec{v}(P)$ and $W\colon P \mapsto \vec{w}(P)$ be two vector fields and let $f\colon S \to \mathbb{R}$ be a smooth function. Then $V + W\colon P \mapsto \vec{v}(P) + \vec{w}(P)$ and $fV\colon P \mapsto f(P)\vec{v}(P)$ give (pointwise) addition and scalar multiplication structure on vector fields.

1.3. **Operator Representations of Vector Fields.** Let J be an open interval containing 0 and let $\gamma\colon J \to \mathbb{R}^n$ be a smooth curve with $\gamma(0) = P$. Let $f = f(x^1, \ldots, x^n)$ be a smooth function defined on a neighborhood of P. Assume that the range of γ is contained in the domain of f. By applying the chain rule to the composite $T = f \circ \gamma\colon J \to \mathbb{R}$,

$$D_\gamma(f) := \frac{dT}{dt} = \sum_{i=1}^n \frac{d\gamma^i(t)}{dt} \frac{\partial f}{\partial x^i}\bigg|_{x^i = \gamma^i(t)}$$

Proposition 1.2.

$$D_\gamma(af + bg) = aD_\gamma(f) + bD_\gamma(g), \quad \text{where } a, b \text{ are constant.}$$

$$D_\gamma(fg) = D_\gamma(f)g + fD_\gamma(g).$$

Let $C^\infty(\mathbb{R}^n)$ denote the set of smooth functions on $\mathbb{R}^n$. An operation D on $C^\infty(\mathbb{R}^n)$ is called a *derivation* if D maps $C^\infty(\mathbb{R}^n)$ to $C^\infty(R^n)$ and satisfies the conditions

$$D(af + bg) = aD(f) + bD(g), \quad \text{where } a, b \text{ are constant.}$$

$$D(fg) = D(f)g + fD(g).$$

Example: For $1 \leq i \leq n$,

$$\partial_i\colon f \mapsto \frac{\partial f}{\partial x^i}$$

is a derivation.

Proposition 1.3. *Let D be any derivation on $C^\infty(\mathbb{R}^n)$. Given any point P in $\mathbb{R}^n$. Then there exist real numbers $a^1, a^2, \ldots, a^n \in \mathbb{R}$ such that*

$$D(f)(P) = \sum_{i=1}^n a^i \partial_i(f)(P)$$

for any $f \in C^\infty(\mathbb{R}^n)$, where a^i depends on D and P but is independent on f.

Proof. Write x for $(x^1, \dots, x^n)$. Define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(t(x-P)+P)dt.$$

Then

$$\begin{aligned} f(x) - f(P) &= \int_0^1 \frac{d}{dt} f(t(x-P)+P)dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x^i}(t(x-P)+P)\cdot(x^i - x^i(P))dt \\ &= \sum_{i=1}^n (x^i - x^i(P)) \int_0^1 \frac{\partial f}{\partial x^i}(t(x-P)+P)dt = \sum_{i=1}^n (x^i - x^i(P)) g_i(x). \end{aligned}$$

Since D is a derivation, $D(1) = D(1\cdot 1) = D(1)\cdot 1 + 1\cdot D(1)$ and so $D(1) = 0$. It follows that $D(c) = 0$ for any constant c. By applying D to the above equations,

$$\begin{aligned} D(f(x)) = D(f(x) - f(P)) &= \sum_{i=1}^n D(x^i - x^i(P))g_i(x) + (x^i - x^i(P))D(g_i(x)) \\ &= \sum_{i=1}^n D(x^i)g_i(x) + (x^i - x^i(P))D(g_i(x)) \end{aligned}$$

because $D(f(P)) = D(x^i(P)) = 0$. Let $a^i = D(x^i)(P)$ which only depends on D and P. By evaluating at P,

$$D(f)(P) = \sum_{i=1}^n D(x^i)(P)g_i(P) + 0 = \sum_{i=1}^n a^i g_i(P).$$

Since

$$g_i(P) = \int_0^1 \frac{\partial f}{\partial x^i}(t(P-P)+P)dt = \int_0^1 \frac{\partial f}{\partial x^i}(P)dt = \frac{\partial f}{\partial x^i}(P) = \partial_i(f)(P),$$

$$D(f)(P) = \sum_{i=1}^n a^i \partial_i(f)(P),$$

which is the conclusion. □

From this proposition, we can give a new way to looking at vector fields:
Given a vector fields $P \mapsto \vec{v}(P) = (v^1(P), v^2(P), \dots, v^n(P))$, a derivation

$$D_{\vec{v}} = \sum_{i=1}^n v^i(P)\cdot \partial_i$$

on $C^\infty(\mathbb{R}^n)$ is called an *operator representation* of the vector field $P \mapsto \vec{v}(P)$.
Note. The operation $v^i(x)\partial_i$ is given as follows: for any $f \in C^\infty(\mathbb{R}^n)$,

$$D_{\vec{v}}(f)(P) = \sum_{i=1}^n v^i(P)\cdot \partial_i(f)(P)$$

for any P.

From this new view, the tangent spaces $T(\mathbb{R}^n)$ admits a basis $\{\partial_1, \partial_2, \dots, \partial_n\}$.

1.4. **Integral Curves.** Let $V\colon \mathbf{x} \mapsto \vec{v}(\mathbf{x})$ be a (smooth) vector field on an neighborhood U of P. An *integral curve* to V is a smooth curve $\mathbf{s}\colon (-\delta, \epsilon) \to U$, defined for suitable $\delta, \epsilon > 0$, such that

$$\mathbf{s}'(t) = \vec{v}(\mathbf{s}(t))$$

for $-\delta < t < \epsilon$.

Theorem 1.4. *Let $V\colon \mathbf{x} \mapsto \vec{v}(\mathbf{x})$ be a (smooth) vector field on an neighborhood U of P. Then there exists an integral curve to V through P. Any two such curves agree on their common domain.*

Proof. The proof is given by assuming the fundamental existence and uniqueness theorem for systems of first order differential equations.

The requirement for a curve $\mathbf{s}(t) = (s^1(t), \dots, s^n(t))$ to be an integral curve is:

$$\begin{cases} \frac{ds^1(t)}{dt} = v^1(s^1(t), s^2(t), \dots, s^n(t)) \\ \frac{ds^2(t)}{dt} = v^2(s^1(t), s^2(t), \dots, s^n(t)) \\ \cdots\cdots\cdots \\ \frac{ds^n(t)}{dt} = v^n(s^1(t), s^2(t), \dots, s^n(t)) \end{cases}$$

with the initial conditions

$$\mathbf{s}(0) = P \qquad (s^1(0), s^2(0), \dots, s^n(0)) = (x^1(P), x^2(P), \dots, x^n(P))$$

$$\mathbf{s}'(0) = \vec{v}(P) \quad \left(\frac{ds^1}{dt}(0), \dots, \frac{ds^n}{dt}(0)\right) = (v^1(P), \dots, v^n(P)).$$

Thus the statement follows from the fundamental theorem of first order ODE. □

Example 1.5. Let $n = 2$ and let $V\colon P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$, where $v^1(x, y) = x$ and $v^2(x, y) = y$. Given a point $P = (a^1, a^2)$, the equation for the integral curve $\mathbf{s}(t) = (x(t), y(t))$ is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = y(t) \end{cases}$$

with initial conditions $(x(0), y(0)) = (a^1, a^2)$ and $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, a^2)$. Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^t).$$

Example 1.6. Let $n = 2$ and let $V\colon P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$, where $v^1(x, y) = x$ and $v^2(x, y) = -y$. Given a point $P = (a^1, a^2)$, the equation for the integral curve $\mathbf{s}(t) = (x(t), y(t))$ is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = -y(t) \end{cases}$$

with initial conditions $(x(0), y(0)) = (a^1, a^2)$ and $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, -a^2)$. Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^{-t}).$$

1.5. **Implicit- and Inverse-Mapping Theorems.**

Theorem 1.7. *Let D be an open region in $\mathbb{R}^{n+1}$ and let F be a function well-defined on D with continuous partial derivatives. Let $(x_0^1, x_0^2, \ldots, x_0^n, z_0)$ be a point in D where*

$$F(x_0^1, x_0^2, \ldots, x_0^n, z_0) = 0 \qquad \frac{\partial F}{\partial z}(x_0^1, x_0^2, \ldots, x_0^n, z_0) \neq 0.$$

Then there is a neighborhood $N_\epsilon(z_0) \subseteq \mathbb{R}$, a neighborhood $N_\delta(x_0^1, \ldots, x_0^n) \subseteq \mathbb{R}^n$, and a **unique** *function $z = g(x^1, x^2, \ldots, x^n)$ defined for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ with values $z \in N_\epsilon(z_0)$ such that*

1) *$z_0 = g(x_0^1, x_0^2, \ldots, x_0^n)$ and*
$$F(x^1, x^2, \ldots, x^n, g(x^1, \ldots, x^n)) = 0$$
for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.
2) *g has continuous partial derivatives with*
$$\frac{\partial g}{\partial x^i}(x^1, \ldots, x^n) = -\frac{F_{x^i}(x^1, \ldots, x^n, z)}{F_z(x^1, \ldots, x^n, z)}$$
for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ where $z = g(x^1, \ldots, x^n)$.
3) *If F is smooth on D, then $z = g(x^1, \ldots, x^n)$ is smooth on $N_\delta(x_0^1, \ldots, x_0^n)$.*

Proof. **Step 1.** We may assume that $\frac{\partial F}{\partial z}(x_0^1, x_0^2, \ldots, x_0^n, z_0) > 0$. Since F_z is continuous, there exists a neighborhood $N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0)$ in which F_z is continuous and positive. Thus for fixed $(x^1, \ldots, x^n)$, F is strictly increasing on z in this neighborhood. It follows that there exists $c > 0$ such that

$$F(x_0^1, x_0^2, \ldots, x_0^n, z_0 - c) < 0 \qquad F(x_0^1, x_0^2, \ldots, x_0^n, z_0 + c) > 0$$

with

$$(x_0^1, x_0^2, \ldots, x_0^n, z_0 - c), (x_0^1, x_0^2, \ldots, x_0^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0).$$

Step 2. By the continuity of F, there exists a small $\delta > 0$ such that

$$F(x^1, x^2, \ldots, x^n, z_0 - c) < 0 \qquad F(x^1, x^2, \ldots, x^n, z_0 + c) > 0$$

with

$$(x^1, x^2, \ldots, x^n, z_0 - c), (x^1, x^2, \ldots, x^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0)$$

for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.

Step 3. Fixed $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$, F is continuous and strictly increasing on z. There is a **unique** z, $z_0 - c < z < z_0 + c$, such that

$$F(x^1, \ldots, x^n, z) = 0.$$

This defines a function $z = g(x^1, \ldots, x^n)$ for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ with values $z \in (z_0 - c, z_0 + c)$.

Step 4. Prove that $z = g(x^1, \ldots, x^n)$ is continuous. Let $(x_1^1, \ldots, x_1^n) \in N_\delta(x_0^1, \ldots, x_0^n)$. Let $(x_1^1(k), \ldots, x_1^n(k))$ be any sequence in $N_\delta(x_0^1, \ldots, x_0^n)$ converging to $(x_1^1, \ldots, x_1^n)$. Let A be any subsequential limit of $\{z_k = g(x_1^1(k), \ldots, x_1^n(k))\}$, that is $A = \lim_{s\to\infty} z_{k_s}$. Then, by the continuity of F,

$$\begin{aligned} 0 &= \lim_{s\to\infty} F(x_1^1(k_s), \ldots, x_1^n(k_s), z_{k_s}) \\ &= F(\lim_{s\to\infty} x_1^1(k_s), \ldots, \lim_{s\to\infty} x_1^n(k_s), \lim_{s\to\infty} z_{k_s}) \end{aligned}$$

$$= F(x_1^1, \ldots, x_1^n, A).$$

By the unique solution of the equation, $A = g(x_1^1, \ldots, x_1^n)$. Thus $\{z_k\}$ converges $g(x_1^1, \ldots, x_1^n)$ and so g is continuous.

Step 5. Compute the partial derivatives $\frac{\partial z}{\partial x_i}$. Let h be small enough. Let

$$z + k = g(x^1, \ldots, x^{i-1}, x^i + h, x^{i+1}, \ldots, x^n),$$

that is

$$F(x^1, \ldots, x^i + h, \ldots, x^n, z + k) = 0$$

with $z_0 - c < z + k < z_0 + c$. Then

$$\begin{aligned} 0 &= F(x^1, \ldots, x^i + h, \ldots, x^n, z + k) - F(x^1, \ldots, x^n, z) \\ &= F_{x^i}(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})h + F_z(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})k \end{aligned}$$

by the mean value theorem (Consider the function

$$\phi(t) = F(x^1, \ldots, x^i + th, \ldots, x^n, z + tk)$$

for $0 \leq t \leq 1$. Then $\phi(1) - \phi(0) = \phi'(\xi)(1-0)$.), where $\tilde{x}^i$ is between x^i and $x^i + h$, and $\tilde{z}$ is between z and $z + k$. Now

$$\begin{aligned} \frac{\partial g}{\partial x_i} &= \lim_{h \to 0} \frac{g(x^1, \ldots, x^{i-1}, x^i + h, x^{i+1}, \ldots, x^n) - z}{h} = \lim_{h \to 0} \frac{k}{h} \\ &= -\lim_{h \to 0} \frac{F_{x^i}(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})}{F_z(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})}. = -\frac{F_{x_i}}{F_z}, \end{aligned}$$

where $\tilde{z} \to z$ as $h \to 0$ because g is continuous (and so $k \to 0$ as $h \to 0$).

Step 6. Since F_z is not zero in this small neighborhood, g_{x_i} is continuous for each i. If F is smooth, then all higher derivatives of g are continuous and so g is also smooth. □

Theorem 1.8 (Implicit Function Theorem)**.** *Let D be an open region in $\mathbb{R}^{m+n}$ and let $F_1, F_2, \ldots, F_n$ be functions well-defined on D with continuous partial derivatives. Let $(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n)$ be a point in D where*

$$\begin{cases} F_1(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n) = 0 \\ F_2(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n) = 0 \\ \cdots\cdots\cdots\cdots \\ F_n(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n) = 0 \end{cases}$$

and the Jacobian

$$J = \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u^1, u^2, \ldots, u^n)} = \det\left(\frac{\partial F_i}{\partial u^j}\right) \neq 0$$

at the point $(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n)$. Then there are neighborhoods $N_\delta(x_0^1, \ldots, x_0^m)$, $N_{\epsilon_1}(u_0^1)$, $N_{\epsilon_2}(u_0^2)$, …, $N_{\epsilon_n}(u_0^n)$, and **unique** *functions*

$$\begin{cases} u^1 = g_1(x^1, x^2, \ldots, x^m) \\ u^2 = g_2(x^1, x^2, \ldots, x^m) \\ \cdots\cdots\cdots\cdots \\ u^n = g_n(x^1, x^2, \ldots, x^m) \end{cases}$$

defined for $(x^1, \ldots, x^m) \in N_\delta(x_0^1, \ldots, x_0^m)$ with values $u^1 \in N_{\epsilon_1}(u_0^1), \ldots, u^n \in N_{\epsilon_n}(u_0^n)$ such that

1) *$u_0^i = g_i(x_0^1, x_0^2, \ldots, x_0^m)$ and*

$$F_i(x^1, x^2, \ldots, x^n, g_i(x^1, \ldots, x^m)) = 0$$

for all $1 \le i \le n$ and all $(x^1, \ldots, x^m) \in N_\delta(x_0^1, \ldots, x_0^m)$.

2) *Each g_i has continuous partial derivatives with*

$$\frac{\partial g_i}{\partial x^j}(x^1, \ldots, x^m) = -\frac{1}{J} \cdot \frac{\partial(F_1, \ldots, F_n)}{\partial(u^1, u^2, \ldots, u^{j-1}, x^j, u^{j+1}, \ldots, u^n)}$$

for all $(x^1, \ldots, x^m) \in N_\delta(x_0^1, \ldots, x_0^m)$ where $u^i = g_i(x^1, \ldots, x^m)$.

3) *If each F_i is smooth on D, then each $u^i = g_i(x^1, \ldots, x^m)$ is smooth on $N_\delta(x_0^1, \ldots, x_0^m)$.*

Sketch of Proof. The proof is given by induction on n. Assume that the statement holds for $n-1$ with $n > 1$. (We already prove that the statement holds for $n = 1$.) Since the matrix

$$\left(\frac{\partial F_i}{\partial u^j}\right)$$

is invertible at the point $P = (x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n)$ (because the determinant is not zero), we may assume that

$$\frac{\partial F_n}{\partial u^n}(P) \neq 0.$$

(The entries in the last column can not be all 0 and so, if $\frac{\partial F_i}{\partial u^n}(P) \neq 0$, we can interchange F_i and F_n.)

From the previous theorem, there is a solution

$$u^n = g_n(x^1, \ldots, x^m, u^1, \ldots, u^{n-1})$$

to the last equation. Consider

$$\begin{cases} G_1 = F_1(x^1, \ldots, x^m, u^1, \ldots, u^{n-1}, g_n) \\ G_2 = F_2(x^1, \ldots, x^m, u^1, \ldots, u^{n-1}, g_n) \\ \cdots\cdots\cdots\cdots \\ G_{n-1} = F_{n-1}(x^1, \ldots, x^m, u^1, \ldots, u^{n-1}, g_n). \end{cases}$$

Then

$$\frac{\partial G_i}{\partial u^j} = \frac{\partial F_i}{\partial u^j} + \frac{\partial F_i}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j}$$

for $1 \le i, j \le n-1$, where

$$\frac{\partial F_n}{\partial u^j} + \frac{\partial F_n}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j} = 0.$$

Let

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial g_n}{\partial u^1} & \frac{\partial g_n}{\partial u^2} & \frac{\partial g_n}{\partial u^3} & \cdots & \frac{\partial g_n}{\partial u^{n-1}} & 1 \end{pmatrix}$$

Then

$$\left(\frac{\partial F_i}{\partial u^j}\right) \cdot B = \begin{pmatrix} \left(\frac{\partial G_i}{\partial u^j}\right)_{n-1,n-1} & * \\ 0 & \frac{\partial F_n}{\partial u^n} \end{pmatrix}.$$

By taking the determinant,

$$J = \frac{\partial(F_1,\dots,F_n)}{\partial(u^1,\dots,u^n)} = \frac{\partial F_n}{\partial u^n}\cdot\frac{\partial(G_1,\dots,G_{n-1})}{\partial(u^1,\dots,u^{n-1})}.$$

Thus $\frac{\partial(G_1,\dots,G_{n-1})}{\partial(u^1,\dots,u^{n-1})} \neq 0$ at P and, by induction, there are solutions

$$u^i = g_i(x^1,\dots,x^m)$$

for $1 \le i \le n-1$. □

Theorem 1.9 (Inverse Mapping Theorem). *Let D be an open region in $\mathbb{R}^n$. Let*

$$\begin{cases} x^1 = f_1(u^1,\dots,u^n) \\ x^2 = f_2(u^1,\dots,u^n) \\ \quad\cdots\cdots \\ x^n = f_n(u^1,\dots,u^n) \end{cases}$$

be functions defined on D with continuous partial derivatives. Let $(u^1_0,\dots,u^n_0) \in D$ satisfy $x^i_0 = f_i(u^1_0,\dots,u^n_0)$ and the Jacobian

$$\frac{\partial(x^1,\dots,x^n)}{\partial(u^1,\dots,u^n)} \neq 0 \qquad \text{at} \quad (u^1_0,\dots,u^n_0).$$

Then there are neighborhood $N_\delta(x^1_0,\dots,x^n_0)$ and $N_\epsilon(u^1_0,\dots,u^n_0)$ such that

$$\begin{cases} u^1 = f_1^{-1}(x^1,\dots,x^n) \\ u^2 = f_2^{-1}(x^1,\dots,x^n) \\ \quad\cdots\cdots \\ u^n = f_n^{-1}(x^1,\dots,x^n) \end{cases}$$

is well-defined and has continuous partial derivatives on $N_\delta(x^1_0,\dots,x^n_0)$ with values in $N_\epsilon(u^1_0,\dots,u^n_0)$. Moreover if each f_i is smooth, then each f_i^{-1} is smooth.

Proof. Let $F_i = f_i(u^1,\dots,u^n) - x_i$. The assertion follows from the Implicit Function Theorem. □

2. Topological and Differentiable Manifolds, Diffeomorphisms, Immersions, Submersions and Submanifolds

2.1. **Topological Spaces.**

Definition 2.1. Let X be a set. A *topology* $\mathcal{U}$ for X is a collection of subsets of X satisfying

i) $\emptyset$ and X are in $\mathcal{U}$;
ii) the intersection of two members of $\mathcal{U}$ is in $\mathcal{U}$;
iii) the union of any number of members of $\mathcal{U}$ is in $\mathcal{U}$.

The set X with $\mathcal{U}$ is called a *topological space*. The members $U \in \mathcal{U}$ are called the *open sets*.

Let X be a topological space. A subset $N \subseteq X$ with $x \in N$ is called a *neighborhood* of x if there is an open set U with $x \in U \subseteq N$. For example, if X is a metric space, then the closed ball $D_\epsilon(x)$ and the open ball $B_\epsilon(x)$ are neighborhoods of x. A subset C is said to *closed* if $X \setminus C$ is open.

Definition 2.2. A function $f\colon X \to Y$ between two topological spaces is said to be *continuous* if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X.

A continuous function from a topological space to a topological space is often simply called a *map*. A *space* means a *Hausdorff space*, that is, a topological spaces where any two points has disjoint neighborhoods.

Definition 2.3. Let X and Y be topological spaces. We say that X and Y are *homeomorphic* if there exist continuous functions $f\colon X \to Y, g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We write $X \cong Y$ and say that f and g are *homeomorphisms* between X and Y.

By the definition, a function $f\colon X \to Y$ is a homeomorphism if and only if

i) f is a bijective;
ii) f is continuous and
iii) f^{-1} is also continuous.

Equivalently f is a homeomorphism if and only if 1) f is a bijective, 2) f is continuous and 3) f is an open map, that is f sends open sets to open sets. Thus a homeomorphism between X and Y is a bijective between the points and the open sets of X and Y.

A very general question in topology is how to classify topological spaces under homeomorphisms. For example, we know (from complex analysis and others) that any simple closed loop is homeomorphic to the unit circle S^1. Roughly speaking topological classification of curves is known. The topological classification of (two-dimensional) surfaces is known as well. However the topological classification of 3-dimensional manifolds (we will learn manifolds later.) is quite open.

The famous Poicaré conjecture is related to this problem, which states that any simply connected 3-dimensional (topological) manifold is homeomorphic to the 3-sphere S^3. A space X is called *simply connected* if (1) X is path-connected (that is, given any two points, there is a continuous path joining them) and (2) the fundamental group $\pi_1(X)$ is trivial (roughly speaking, any loop can be deformed to be the constant loop in X). The *manifolds* are the objects that we are going to discuss in this course.

2.2. **Topological Manifolds.** A Hausdorff space M is called a *(topological) n-manifold* if each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. Roughly speaking, an n-manifold is *locally* $\mathbb{R}^n$. Sometimes M is denoted as M^n for mentioning the dimension of M.

(**Note.** If you are not familiar with topological spaces, you just think that M is a subspace of $\mathbb{R}^N$ for a large N.)

For example, $\mathbb{R}^n$ and the n-sphere S^n is an n-manifold. A 2-dimensional manifold is called a *surface*. The objects traditionally called 'surfaces in 3-space' can be made into manifolds in a standard way. The compact surfaces have been classified as spheres or projective planes with various numbers of handles attached.

By the definition of manifold, the closed n-disk D^n is not an n-manifold because it has the 'boundary' S^{n-1}. D^n is an example of 'manifolds with boundary'. We give the definition of manifold with boundary as follows.

A Hausdorff space M is called an *n-manifold with boundary* ($n \geq 1$) if each point in M has a neighborhood homeomorphic to an open set in the half space

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

Manifold is one of models that we can do calculus 'locally'. By means of calculus, we need local coordinate systems. Let $x \in M$. By the definition, there is a an open neighborhood $U(x)$ of x and a homeomorphism ϕ_x from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The collection $\{(U(x), \phi_x) | x \in M\}$ has the property that 1) $\{U(x) | x \in M\}$ is an open cover and 2) ϕ_x is a homeomorphism from $U(x)$

onto an open set in $\mathbb{R}^n_+$. The subspace $\phi_x(U_x)$ in $\mathbb{R}^n_+$ plays a role as a local coordinate system. The collection $\{(U(x), \phi_x)|x \in M\}$ is somewhat too large and we may like less local coordinate systems. This can be done as follows.

Let M be a space. A *chart* of M is a pair (U, ϕ) such that 1) U is an open set in M and 2) ϕ is a homeomorphism from U onto an open set in $\mathbb{R}^n_+$. The map

$$\phi\colon U \to \mathbb{R}^n_+$$

can be given by n coordinate functions $\phi_1, \ldots, \phi_n$. If P denotes a point of U, these functions are often written as

$$x^1(P), x^2(P), \ldots, x^n(P)$$

or simply $x^1, x^2, \ldots, x^n$. They are called *local coordinates* on the manifold.

An *atlas* for M means a collection of charts $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ such that $\{U_\alpha|\alpha \in J\}$ is an open cover of M.

Proposition 2.4. *A Hausdorff space M is a manifold (with boundary) if and only if M has an atlas.*

Proof. Suppose that M is a manifold. Then the collection $\{(U(x), \phi_x)|x \in M\}$ is an atlas. Conversely suppose that M has an atlas. For any $x \in M$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x that is homeomorphic to an open set in $\mathbb{R}^n_+$. Thus M is a manifold. □

We define a subset ∂M as follows: $x \in \partial M$ if there is a chart (U_α, ϕ_α) such that $x \in U_\alpha$ and $\phi_\alpha(x) \in \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 0\}$. ∂M is called the boundary of M. For example the boundary of D^n is S^{n-1}.

Proposition 2.5. *Let M be a n-manifold with boundary. Then ∂M is an $(n-1)$-manifold without boundary.*

Proof. Let $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ be an atlas for M. Let $J' \subseteq J$ be the set of indices such that $U_\alpha \cap \partial M \neq \emptyset$ if $\alpha \in J'$. Then Clearly

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M}|\alpha \in J'\}$$

can be made into an atlas for ∂M. □

Note. The key point here is that if U is open in $\mathbb{R}^n_+$, then $U \cap \mathbb{R}^{n-1}$ is also open because: Since U is open in $\mathbb{R}^n_+$, there is an open subset V of $\mathbb{R}^n$ such that $U = V \cap \mathbb{R}^n_+$. Now if $x \in U \cap \mathbb{R}^{n-1}$, there is an open disk $E_\epsilon(x) \subseteq V$ and so

$$E_\epsilon(x) \cap \mathbb{R}^{n-1} \subseteq V \cap \mathbb{R}^{n-1} = U \cap \mathbb{R}^{n-1}$$

is an open $(n-1)$-dimensional ϵ-disk in $\mathbb{R}^{n-1}$ centered at x.

2.3. **Differentiable Manifolds.**

Definition 2.6. A Hausdorff space M is called a *differential manifold of class C^k (with boundary)* if there is an atlas of M

$$\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$$

such that

For any $\alpha, \beta \in J$, the composites

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$

is differentiable of class C^k.

The atlas $\{(U_\alpha, \phi_\alpha | \alpha \in J\}$ is called a *differential atlas of class C^k* on M.

(**Note.** Assume that M is a subspace of $\mathbb{R}^N$ with $N >> 0$. If M has an atlas $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ such that each $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n_+$ is differentiable of class C^k, then M is a differentiable manifold of class C^k. This is the definition of differentiable (smooth) manifolds in [6] as in the beginning they already assume that M is a subspace of $\mathbb{R}^N$ with N large. In our definition (the usual definition of differentiable manifolds using charts), we only assume that M is a (Hausdorff) topological space and so ϕ_α is *only an identification* of an abstract U_α with an open subset of $\mathbb{R}^n_+$. In this case we can not talk differentiability of ϕ_α unless U_α is regarded as a subspace of a (large dimensional) Euclidian space.)

Two differential atlases of class C^k $\{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ and $\{(V_\beta, \psi_\beta) | \beta \in J\}$ are called *equivalent* if

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\} \cup \{(V_\beta, \psi_\beta) | \beta \in J\}$$

is again a differential atlas of class C^k (this is an equivalence relation). A *differential structure of class C^k* on M is an equivalence class of differential atlases of class C^k on M. Thus a differential manifold of class C^k means a manifold with a differential structure of class C^k. A *smooth* manifold means a differential manifold of class C^∞.

Note: A general manifold is also called *topological manifold.* Kervaire and Milnor [4] have shown that the topological sphere S^7 has 28 distinct oriented smooth structures.

Definition 2.7. let M and N be smooth manifolds (with boundary) of dimensions m and n respectively. A map $f \colon M \to N$ is called *smooth* if for some smooth atlases $\{(U_\alpha, \phi_\alpha | \alpha \in I\}$ for M and $\{(V_\beta, \psi_\beta)\beta \in J\}$ for N the functions

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}|_{\phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha)} \colon \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n_+$$

are of class C^∞.

Proposition 2.8. *If $f \colon M \to N$ is smooth with respect to atlases*

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\}, \quad \{(V_\beta, \phi_\beta | \beta \in J\}$$

for M, N then it is smooth with respect to equivalent atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \quad \{(V'_\gamma, \eta_\gamma | \beta \in J'\}$$

Proof. Since f is smooth with respect with the atlases

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\}, \ \{(V_\beta, \phi_\beta | \beta \in J\},$$

f is smooth with respect to the smooth atlases

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\} \cup \{(U'_\delta, \theta_\delta | \alpha \in I'\}, \quad \{(V_\beta, \phi_\beta | \beta \in J\} \cup \{(V'_\gamma, \eta_\gamma | \beta \in J'\}$$

by look at the local coordinate systems. Thus f is smooth with respect to the atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \ \{(V'_\gamma, \eta_\gamma | \beta \in J'\}.$$

□

Thus the definition of smooth maps between two smooth manifolds is independent of choice of atlas.

Definition 2.9. A smooth map $f\colon M \to N$ is called a *diffeomorphism* if f is one-to-one and onto, and if the inverse $f^{-1}\colon N \to M$ is also smooth.

Definition 2.10. Let M be a smooth n-manifold, possibly with boundary. A subset X is called a *properly embedded submanifold* of dimension $k \leq n$ if X is a closed in M and, for each $P \in X$, there exists a chart (U, ϕ) about P in M such that

$$\phi(U \cap X) = \phi(U) \cap \mathbb{R}^k_+,$$

where $\mathbb{R}^k_+ \subseteq \mathbb{R}^n_+$ is the standard inclusion.

Note. In the above definition, the collection $\{(U \cap X, \phi|_{U\cap X})\}$ is an atlas for making X to a smooth k-manifold with boundary $\partial X = X \cap \partial M$.

If $\partial M = \emptyset$, by dropping the requirement that X is a closed subset but keeping the requirement on local charts, X is called simply a *submanifold* of M.

2.4. **Tangent Space.** Let S be an open region of $\mathbb{R}^n$. Recall that, for $P \in S$, the tangent space $T_P(S)$ is just the n-dimensional vector space by putting the origin at P. Let T be an open region of $\mathbb{R}^m$ and let $f = (f_1, \ldots, f_m)\colon S \to T$ be a smooth map. Then f induces a *linear transformation*

$$Tf\colon T_P(S) \to T_{f(P)}(T)$$

given by

$$Tf(\vec{v}) = \left(\frac{\partial f_i}{\partial x^j}\right)_{m\times n} \cdot \begin{pmatrix} v^1 \\ v^2 \\ \cdots \\ v^n \end{pmatrix}_{n\times 1} = \begin{pmatrix} v^1\partial_1(f_1) + v^2\partial_2(f_1) + \cdots + v^n\partial_n(f_1) \\ v^1\partial_1(f_2) + v^2\partial_2(f_2) + \cdots + v^n\partial_n(f_2) \\ \cdots\cdots\cdots\cdots \\ v^1\partial_1(f_m) + v^2\partial_2(f_m) + \cdots + v^n\partial_n(f_m) \end{pmatrix},$$

namely Tf is obtained by taking directional derivatives of $(f_1, \ldots, f_m)$ along vector $\vec{v}$ for any $\vec{v} \in T_P(S)$.

Now we are going to define the *tangent space* to a (differentiable) manifold M at a point P as follows:

First we consider the set

$$\mathcal{T}_P = \{(U, \phi, \vec{v}) \mid P \in U, \quad (U, \phi) \text{ is a chart } \quad \vec{v} \in T(\phi(P))(\phi(U))\}.$$

The point is that there are possibly many charts around P. Each chart creates an n-dimension vector space. So we need to define an *equivalence relation* in $\mathcal{T}_P$ such that, $\mathcal{T}_P$ modulo these relations is only one copy of n-dimensional vector space which is also independent on the choice of charts.

Let $(U, \phi, \vec{v})$ and $(V, \psi, \vec{w})$ be two elements in $\mathcal{T}_P$. That is (U, ϕ) and (V, ψ) are two charts with $P \in U$ and $P \in V$. By the definition,

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

is diffeomorphism and so it induces an isomorphism of vector spaces

$$T(\psi \circ \phi^{-1})\colon T_{\phi(P)}(\phi(U \cap V)) \longrightarrow T_{\psi(P)}(\psi(U \cap V)).$$

Now $(U, \phi, \vec{v})$ is called equivalent to $(V, \psi, \vec{w})$, denoted by $(U, \phi, \vec{v}) \sim (V, \psi, \vec{w})$, if

$$T(\psi \circ \phi^{-1})(\vec{v}) = \vec{w}.$$

Define $T_P(M)$ to be the quotient

$$T_P(M) = \mathcal{T}_P/\sim .$$

Exercise 2.1. Let M be a differentiable n-manifold and let P be any point in M. Prove that $T_P(M)$ is an n-dimensional vector space. [Hint: Fixed a chart (U, ϕ) and defined

$$a(U, \phi, \vec{v}) + b(U, \phi, \vec{w}) := (U, \phi, a\vec{v} + b\vec{w}).$$

Now given any $(V, \psi, \vec{x}), (\tilde{V}, \tilde{\psi}, \vec{y}) \in \mathcal{T}_P$, consider the map

$$\phi \circ \psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V) \qquad \phi \circ \tilde{\psi}^{-1} \colon \tilde{\psi}(U \cap \tilde{V}) \to \phi(U \cap \tilde{V})$$

and define

$$a(V, \psi, \vec{x}) + b(\tilde{V}, \tilde{\psi}, \vec{y}) = (U, \phi, aT(\phi \circ \psi^{-1})(\vec{x}) + bT(\phi \circ \tilde{\psi}^{-1})(\vec{y})).$$

Then prove that this operation gives a well-defined vector space structure on T_P, that is, independent on the equivalence relation.]

The tangent space $T_P(M)$, as a vector space, can be described as follows: given any chart (U, ϕ) with $P \in U$, there is a unique isomorphism

$$T_\phi \colon T_P(M) \to T_{\phi(P)}(\phi(U)).$$

by choosing $(U, \phi, \vec{v})$ as representatives for its equivalence class. If (V, ψ) is another chart with $P \in V$, then there is a commutative diagram

$$\begin{array}{ccc}
T_P(M) & \xrightarrow[\cong]{T_\phi} & T_{\phi(P)}(\phi(U \cap V)) \\
\Vert & & \downarrow{\scriptstyle T(\psi \circ \phi^{-1})} \\
T_P(M) & \xrightarrow[\cong]{T_\psi} & T_{\psi(P)}(\psi(U \cap V)),
\end{array} \tag{1}$$

where $T(\psi \circ \phi^{-1})$ is the linear isomorphism induced by the Jacobian matrix of the differentiable map $\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$.

Exercise 2.2. Let $f \colon M \to N$ be a smooth map, where M and N need not to have the same dimension. Prove that there is a unique linear transformation

$$Tf \colon T_P(M) \longrightarrow T_{f(P)}(N)$$

such that the diagram

$$\begin{array}{ccc}
T_P(M) & \xrightarrow[\cong]{T_\phi} & T_{\phi(P)}(\phi(U)) \\
\downarrow{\scriptstyle Tf} & & \downarrow{\scriptstyle T(\psi \circ f \circ \phi^{-1})} \\
T_{f(P)}(N) & \xrightarrow[\cong]{T_\psi} & T_{\psi(f(P))}(\psi(V))
\end{array}$$

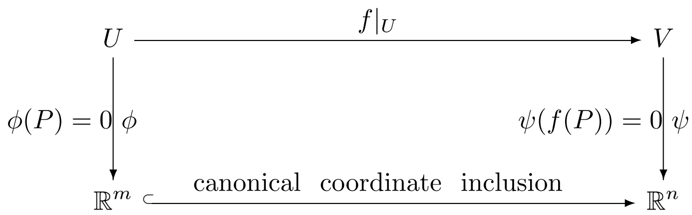
commutes for any chart (U, ϕ) with $P \in U$ and any chart (V, ψ) with $f(P) \in V$. [First fix a choice of (U, ϕ) with $P \in U$ and (V, ψ) with $f(P) \in V$, the linear transformation Tf is uniquely defined by the above diagram. Then use Diagram (1) to check that Tf is independent on choices of charts.

2.5. **Immersions.** A smooth map $f\colon M \to N$ is called *immersion* at P if the linear transformation

$$Tf\colon T_P(M) \to T_{f(P)}(M)$$

is injective.

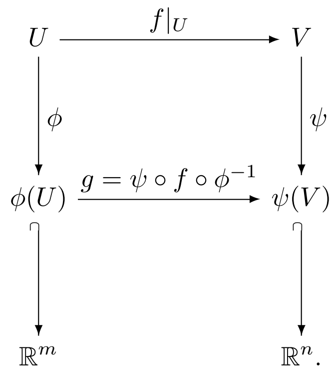
Theorem 2.11 (Local Immersion Theorem)**.** *Suppose that $f\colon M^m \to N^n$ is immersion at P. Then there exist charts (U, ϕ) about P and (V, ψ) about $f(P)$ such that the diagram*



commutes.

Proof. We may assume that $\phi(P) = 0$ and $\psi(f(P)) = 0$. (Otherwise replacing ϕ and ψ by $\phi - \phi(P)$ and $\psi - \psi(f(P))$, respectively.)

Consider the commutative diagram



By the assumption,

$$Tg\colon T_0(\phi(U)) \longrightarrow T_0(\psi(V))$$

is an injective linear transformation and so

$$\operatorname{rank}(Tg) = m$$

at the origin. The matric for Tg is

$$\begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \frac{\partial g^1}{\partial x^2} & \cdots & \frac{\partial g^1}{\partial x^m} \\ \frac{\partial g^2}{\partial x^1} & \frac{\partial g^2}{\partial x^2} & \cdots & \frac{\partial g^2}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g^m}{\partial x^1} & \frac{\partial g^m}{\partial x^2} & \cdots & \frac{\partial g^m}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g^n}{\partial x^1} & \frac{\partial g^n}{\partial x^2} & \cdots & \frac{\partial g^n}{\partial x^m} \end{pmatrix}. \tag{2}$$

By changing basis of $\mathbb{R}^n$ (corresponding to change the rows), we may assume that the first m rows forms an invertible matrix $A_{m\times m}$ at the origin.

Define a function

$$h = (h^1, h^2, \ldots, h^n)\colon \phi(U)\times\mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$$

by setting

$$h^i(x^1,\ldots,x^m,x^{m+1},\ldots,x^n) = g^i(x^1,\ldots,x^m)$$

for $1\leq i\leq m$ and

$$h^i(x^1,\ldots,x^m,x^{m+1},\ldots,x^n) = g^i(x^1,\ldots,x^m) + x^i$$

for $m+1\leq i\leq n$. Then Jacobian matrix of h is

$$\begin{pmatrix} A_{m\times m} & 0_{m\times(n-m)} \\ B_{(n-m)\times m} & I_{n-m} \end{pmatrix},$$

where B is taken from $(m+1)$-st row to n-th row in the matrix (2). Thus the Jacobian of h is not zero at the origin. By the Inverse Mapping Theorem, h is an diffeomorphism in a small neighborhood of the origin. It follows that there exist open neighborhoods $\tilde{U}\subseteq U$ of P and $\tilde{V}\subseteq V$ of $f(P)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\quad f|_{\tilde{U}} \quad} & \tilde{V} \\
\Big\downarrow \phi|_{\tilde{U}} & & \Big\downarrow \psi|_{\tilde{V}} \\
\phi(\tilde{U}) & \xrightarrow{\quad g = \psi \circ f \circ \phi^{-1} \quad} & \psi(\tilde{V}) \\
\Big\| & & \cong \Big\downarrow h^{-1} \\
\phi(\tilde{U}) \times 0 & \hookrightarrow & \phi(\tilde{U}) \times U_2 \\
\Big\downarrow & & \Big\downarrow \\
\mathbb{R}^m = \mathbb{R}^m \times 0 & \hookrightarrow & \mathbb{R}^n,
\end{array}
$$

where U_2 is a small neighborhood of the origin in $\mathbb{R}^{n-m}$. □

Theorem 2.12. *Let $f\colon M \to N$ be a smooth map. Suppose that*

1) *f is immersion at every point $P \in M$,*
2) *f is one-to-one and*
3) *$f\colon M \to f(M)$ is a homeomorphism.*

Then $f(M)$ is a smooth submanifold of M and $f\colon M \to f(M)$ is a diffeomorphism.

Note. In Condition 3, we need that if U is an open subset of M, then there is an open subset V of N such that $V \cap f(M) = f(U)$.

Proof. For any point P in M, we can choose the charts as in Theorem 2.11. By Condition 3, $f(U)$ is an open subset of $f(M)$. The charts $\{(f(U), \psi|_{f(U)})\}$ gives an atlas for $f(M)$ such that $f(M)$ is a submanifold of M. Now $f\colon M \to f(M)$ is a diffeomorphism because it is locally diffeomorphism and the inverse exists. □

Condition 3 is important in this theorem, namely an injective immersion need not give a diffeomorphism with its image. (Construct an example for this.) An injective immersion satisfying condition 3 is called an *embedding*.

2.6. **Submersions.** A smooth map $f\colon M \to N$ is called *submersion* at P if the linear transformation

$$Tf\colon T_P(M) \to T_{f(P)}(M)$$

is surjective.

Theorem 2.13 (Local Submersion Theorem)**.** *Suppose that $f\colon M^m \to N^n$ is submersion at P. Then there exist charts (U,ϕ) about P and (V,ψ) about $f(P)$ such that the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad f|_U \quad} & V \\ {\scriptstyle \phi(P)=0}\Big\downarrow{\scriptstyle \phi} & & {\scriptstyle \psi(f(P))=0}\Big\downarrow{\scriptstyle \psi} \\ \mathbb{R}^m & \xrightarrow{\text{canonical coordinate proj.}} & \mathbb{R}^n \end{array}$$

commutes.

For a smooth map of manifolds $f\colon M \to N$, a point $Q \in N$ is called *regular* if $Tf\colon T_P(M) \to T_Q(N)$ is surjective for every $P \in f^{-1}(Q)$, the pre-image of Q.

Theorem 2.14 (Pre-image Theorem)**.** *Let $f\colon M \to N$ be a smooth map and let $Q \in N$ such that $f^{-1}(Q)$ is not empty. Suppose that Q is regular. Then $f^{-1}(Q)$ is a submanifold of M with* $\dim f^{-1}(Q) = \dim M - \dim N$*.*

Proof. From the above theorem, for any $P \in f^{-1}(Q)$,

$$\phi|_{f^{-1}(Q)}\colon f^{-1}(Q) \cap U \longrightarrow \mathbb{R}^{m-n}$$

gives a chart about P. □

Let Z be a submanifold of N. A smooth map $f\colon M \to N$ is said to be *transversal* to Z if

$$\mathrm{Im}(Tf\colon T_P(M) \to T_{f(P)}(N)) + T_{f(P)}(Z) = T_{f(P)}(N)$$

for every $x \in f^{-1}(Z)$.

Theorem 2.15. *If a smooth map $f\colon M \to N$ is transversal to a submanifold $Z \subseteq N$, then $f^{-1}(Z)$ is a submanifold of M. Moreover the codimension of $f^{-1}(Z)$ in M equals to the codimension of Z in N.*

Proof. Given $P \in f^{-1}(Z)$, since Z is a submanifold, there is a chart (V,ψ) of N about $f(P)$ such that $V = V_1 \times V_2$ with $V_1 = V \cap Z$ and $(V_1, \psi|_{V_1})$ is a chart of Z about $f(P)$. By the assumption, the composite

$$f^{-1}(V) \xrightarrow{f|_{f^{-1}(V)}} V \xrightarrow{\text{proj.}} V_2$$

is submersion. By the Pre-image Theorem, $f^{-1}(V) \cap f^{-1}(Z)$ is a submanifold of the open subset $f^{-1}(V)$ of M and so there is a chart about P such that Z is a submanifold of M.

With respect to the assertion about the codimensions,

$$\mathrm{codim}(f^{-1}(Z)) = \dim V_2 = \mathrm{codim}(Z).$$

□

Consider the special case that both M and Z are submanifolds of N. Then the transversal condition is

$$T_P(M) + T_P(Z) = T_P(N)$$

for any $P \in M \cap Z$.

Corollary 2.16. *The intersection of two transversal submanifolds of N is again a submanifold. Moreover*

$$\operatorname{codim}(M \cap Z) = \operatorname{codim}(M) + \operatorname{codim}(Z)$$

in N.

3. Examples of Manifolds

3.1. **Open Stiefel Manifolds and Grassmann Manifolds.** The *open Stiefel manifold* is the space of k-tuples of linearly independent vectors in $\mathbb{R}^n$:

$$\tilde{V}_{k,n} = \{(\vec{v}_1, \ldots, \vec{v}_k)^T \mid \vec{v}_i \in \mathbb{R}^n,\ \{\vec{v}_1, \ldots, \vec{v}_k\} \text{ linearly independent}\},$$

where $\tilde{V}_{k,n}$ is considered as the subspace of $k \times n$ matrixes $M(k,n) \cong \mathbb{R}^{kn}$. Since $\tilde{V}_{k,n}$ is an open subset of $M(k,n) = \mathbb{R}^{kn}$, $\tilde{V}_{k,n}$ is an open submanifold of $\mathbb{R}^{kn}$.

The *Grassmann manifold* $G_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$, that is, all k-planes through the origin. Let

$$\pi\colon \tilde{V}_{k,n} \to G_{k,n}$$

be the quotient by sending k-tuples of linearly independent vectors to the k-planes spanned by k vectors. The topology in $G_{k,n}$ is given by quotient topology of π, namely, U is an open set of $G_{k,n}$ if and only if $\pi^{-1}(U)$ is open in $\tilde{V}_{k,n}$.

For $(\vec{v}_1, \ldots, \vec{v}_k)^T \in \tilde{V}_{k,n}$, write $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle$ for the k-plane spanned by $\vec{v}_1, \ldots, \vec{v}_k$. Observe that two k-tuples $(\vec{v}_1, \ldots, \vec{v}_k)^T$ and $(\vec{w}_1, \ldots, \vec{w}_k)^T$ spanned the same k-plane if and only if each of them is basis for the common plane, if and only if there is nonsingular $k \times k$ matrix P such that

$$P(\vec{v}_1, \ldots, \vec{v}_k)^T = (\vec{w}_1, \ldots, \vec{w}_k)^T.$$

This gives the identification rule for the Grassmann manifold $G_{k,n}$. Let $\mathrm{GL}_k(\mathbb{R})$ be the space of general linear groups on $\mathbb{R}^k$, that is, $\mathrm{GL}_k(\mathbb{R})$ consists of $k \times k$ nonsingular matrixes, which is an open subset of $M(k,k) = \mathbb{R}^{k^2}$. Then $G_{k,n}$ is the quotient of $\tilde{V}_{k,n}$ by the action of $\mathrm{GL}_k(\mathbb{R})$.

First we prove that $G_{k,n}$ is Hausdorff. If $k = n$, then $G_{n,n}$ is only one point. So we assume that $k < n$. Given an k-plane X and $\vec{w} \in \mathbb{R}^n$, let $\rho_{\vec{w}}$ be the square of the Euclidian distance from $\vec{w}$ to X. Let $\{e_1, \ldots, e_k\}$ be the orthogonal basis for X, then

$$\rho_{\vec{w}}(X) = \vec{w} \cdot \vec{w} - \sum_{j=1}^{k} (\vec{w} \cdot e_j)^2.$$

Fixing any $\vec{w} \in \mathbb{R}^n$, we obtain the continuous map

$$\rho_{\vec{w}}\colon G_{k,n} \longrightarrow \mathbb{R}$$

because $\rho_{\vec{w}} \circ \pi\colon \tilde{V}_{k,n} \to \mathbb{R}$ is continuous and $G_{k,n}$ given by the quotient topology. (Here we use the property of quotient topology that any function f from the quotient space $G_{k,n}$ to any space is continuous if and only if $f \circ \pi$ from $\tilde{V}_{k,n}$ to that space is continuous.) Given any two distinct points X and Y in $G_{k,n}$, we can choose a $\vec{w}$ such that $\rho_{\vec{w}}(X) \neq \rho_{\vec{w}}(Y)$. Let V_1 and V_2 be disjoint open subsets of $\mathbb{R}$ such that $\rho_{\vec{w}}(X) \in V_1$ and $\rho_{\vec{w}}(Y) \in V_2$. Then $\rho_{\vec{w}}^{-1}(V_1)$ and $\rho_{\vec{w}}^{-1}(V_2)$ are two open subset of $G_{k,n}$ that separate X and Y, and so $G_{k,n}$ is Hausdorff.

Now we check that $G_{k,n}$ is manifold of dimension $k(n-k)$ by showing that, for any X in $G_{k,n}$, there is an open neighborhood U_X of α such that $U_X \cong \mathbb{R}^{k(n-k)}$.

Let $X \in G_{k,n}$ be spanned by $(\vec{v}_1, \dots, \vec{v}_k)^T$. There exists a nonsingular $n \times n$ matrix Q such that

$$(\vec{v}_1, \dots, \vec{v}_k)^T = (I_k, 0)Q,$$

where I_k is the unit $k \times k$-matrix. Fixing Q, define

$$X_\alpha = \{(P_k, B_{k,n-k})Q \mid \det(P_k) \neq 0, B_{k,n-k} \in M(k, n-k)\} \subseteq \tilde{V}_{k,n}.$$

Then E_X is an open subset of $\tilde{V}_{k,n}$. Let $U_X = \pi(E_X) \subseteq G_{k,n}$. Since

$$\pi^{-1}(U_X) = E_X$$

is open in $\tilde{V}_{k,n}$, U_X is open in $G_{k,n}$ with $X \in U_X$. From the commutative diagram

$$\begin{array}{ccc}
\mathrm{GL}_k(\mathbb{R}) \times M(k, n-k) & \xrightarrow[\cong]{(P, A) \mapsto (P, PA)Q} & E_X \\
\Big\downarrow \text{proj.} & & \Big\downarrow \pi \\
M(k, n-k) & \xrightarrow[\phi_X^{-1}]{A \mapsto \langle (I_k, A)Q \rangle} & U_X,
\end{array}$$

U_X is homeomorphic to $M(k, n-k) = \mathbb{R}^{k(n-k)}$ and so $G_{k,n}$ is a (topological) manifold.

For checking that $G_{k,n}$ is a smooth manifold, let X and $Y \in G_{k,n}$ be spanned by $(\vec{v}_1, \dots, \vec{v}_k)^T$ and $(\vec{w}_1, \dots, \vec{w}_k)^T$, respectively. There exists nonsingular $n \times n$ matrixes Q and $\tilde{Q}$ such that

$$(\vec{v}_1, \dots, \vec{v}_k)^T = (I_k, 0)Q, \quad (\vec{w}_1, \dots, \vec{w}_k)^T = (I_k, 0)\tilde{Q}.$$

Consider the maps:

$$M(k, n-k) \xrightarrow[\phi_X^{-1}]{} U_X \qquad A \mapsto \langle (I_k, A)Q \rangle$$

$$M(k, n-k) \xrightarrow[\phi_Y^{-1}]{} U_Y \qquad A \mapsto \langle (I_k, A)\tilde{Q} \rangle.$$

If $Z \in U_X \cap U_Y$, then

$$Z = \langle (I_k, A_Z)Q \rangle = \langle (I_k, B_Z)\tilde{Q} \rangle$$

for unique $A, B \in M(k, n-k)$. It follows that there is a nonsingular $k \times k$ matrix P such that

$$(I_k, B_Z)\tilde{Q} = P(I_k, A_Z)Q \Leftrightarrow (I_k, B_Z) = P(I_k, A_Z)Q\tilde{Q}^{-1}.$$

Let

$$T = Q\tilde{Q}^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ & \\ T_{21} & T_{22} \end{pmatrix}.$$

Then

$$(I_k, B_Z) = (P, PA_Z)T = (PT_{11} + PA_ZT_{21}, PT_{12} + PA_ZT_{22})$$

$$\begin{cases} I_k = P(T_{11} + A_ZT_{21}) \\ \\ B_Z = P(T_{12} + A_ZT_{22}). \end{cases}$$

It follows that

$$Z \in U_X \cap U_Y \text{ if and only if } \det(T_{11} + A_ZT_{21}) \neq 0 \text{ (that is, } T_{11} + A_ZT_{21} \text{ is invertible).}$$

From the above, the composite

$$\phi_X(U_X \cap U_Y) \xrightarrow{\phi_X^{-1}} U_X \cap U_Y \xrightarrow{\phi_Y} M(k,n)$$

is given by

$$A \mapsto (T_{11} + AT_{21})^{-1} (T_{12} + AT_{22}),$$

which is smooth. Thus $G_{k,n}$ is a smooth manifold.

As a special case, $G_{1,n}$ is the space of lines (through the origin) of $\mathbb{R}^n$, which is also called *projective space* denoted by $\mathbb{R}\mathrm{P}^{n-1}$. From the above, $\mathbb{R}\mathrm{P}^{n-1}$ is a manifold of dimension $n-1$.

3.2. **Stiefel Manifold.** The *Stiefel manifold*, denoted by $V_{k,n}$, is defined to be the set of k orthogonal unit vectors in $\mathbb{R}^n$ with topology given as a subspace of $\tilde{V}_{k,n} \subseteq M(k,n)$. Thus

$$V_{k,n} = \{A \in M(k,n) \mid A \cdot A^T = I_k\}.$$

We prove that $V_{k,n}$ is a smooth submanifold of $M(k,n)$ by using Pre-image Theorem.

Let $S(k)$ be the space of symmetric matrixes. Then $S(k) \cong \mathbb{R}^{\frac{(k+1)k}{2}}$ is a smooth manifold of dimension. Consider the map

$$f\colon M(k,n) \to S(k) \qquad A \mapsto AA^T.$$

For any $A \in M(k,n)$, $Tf_A\colon T_A(M(k,n)) \to T_{f(A)}(S(k))$ is given by setting $Tf_A(B)$ is the directional derivative along B for any $B \in T_A(M(k,n))$, that is,

$$\begin{aligned} Tf_A(B) &= \lim_{s\to 0} \frac{f(A+sB) - f(A)}{s} \\ &= \lim_{s\to 0} \frac{(A+sB)(A+sB)^T - AA^T}{s} \\ &= \lim_{s\to 0} \frac{AA^T + sAB^T + sBA^T + s^2BB^T - AA^T}{s} = AB^T + BA^T. \end{aligned}$$

We check that $Tf_A\colon T_A(M(k,n)) \to T_{f(A)}(S(k))$ is surjective for any $A \in f^{-1}(I_k)$.

By the identification of $M(k,n)$ and $S(k)$ with Euclidian spaces, $T_A(M(k,n)) = M(k,n)$ and $T_{f(A)}(S(k)) = S(k))$. Let $A \in f^{-1}(I_k)$ and let $C \in T_{f(A)}(S(k))$. Define

$$B = \frac{1}{2} CA \in T_A(M(k,n)).$$

Then

$$Tf_A(B) = AB^T + BA^T = \frac{1}{2}AA^TC^T + \frac{1}{2}CAA^T \xlongequal{AA^T = I_k} \frac{1}{2}C^T + \frac{1}{2}C \xlongequal{C=C^T} C.$$

Thus $Tf\colon T_A(M(k,n)) \to T_{f(A)}(S(k))$ is onto and so I_k is a regular value of f. Thus, by Pre-image Theorem, $V_{k,n} = f^{-1}(I_k)$ is a smooth submanifold of $M(k,n)$ of dimension

$$kn - \frac{(k+1)k}{2} = \frac{k(2n-k-1)}{2}.$$

Special Cases: When $k = n$, then $V_{n,n} = O(n)$ the orthogonal group. From the above, $O(n)$ is a (smooth) manifold of dimension $\frac{n(n-1)}{2}$. (**Note.** $O(n)$ is a Lie group, namely, a smooth manifold plus a topological group such that the multiplication and inverse are smooth.)

When $k = 1$, then $V_{1,n} = S^{n-1}$ which is manifold of dimension $n-1$.

When $k = n-1$, then $V_{n-1,n}$ is a manifold of dimension $\frac{(n-1)n}{2}$. One can check that

$$V_{n-1,n} \cong SO(n)$$

the subgroup of $O(n)$ with determinant 1. In general case, $V_{k,n} = O(n)/O(n-k)$.

As a space, $V_{k,n}$ is compact. This follows from that $V_{k,n}$ is a closed subspace of the k-fold Cartesian product of S^{n-1} because $V_{k,n}$ is given by k unit vectors $(\vec{v}_1, \ldots, \vec{v}_k)^T$ in $\mathbb{R}^n$ that are solutions to $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$, and the fact that any closed subspace of compact Hausdorff space is compact. The composite

$$V_{k,n} \hookrightarrow \tilde{V}_{k,n} \xrightarrow{\pi} G_{k,n}$$

is onto and so the Grassmann manifold $G_{k,n}$ is also compact. Moreover the above composite is a smooth map because π is smooth and $V_{k,n}$ is a submanifold. This gives the diagram

$$\begin{array}{ccccc}
V_{k,n} & \xhookrightarrow{\text{submanifold}} & M(k,n) & \underset{A \mapsto AA^T}{\xrightarrow{\text{submersion at } I_k}} & S(k) \\
\Big\downarrow{\scriptstyle \text{smooth}} & & & & \\
G_{k,n} & & & &
\end{array}$$

Note. By the construction, $G_{k,n}$ is the quotient of $V_{k,n}$ by the action of $O(k)$. This gives identifications:

$$G_{k,n} = V_{k,n}/O(k) = O(n)/(O(k) \times O(n-k)).$$

4. Fibre Bundles and Vector Bundles

4.1. **Fibre Bundles.** A *bundle* means a triple (E, p, B), where $p\colon E \to B$ is a (continuous) map. The space B is called the *base space*, the space E is called the *total space*, and the map p is called the *projection* of the bundle. For each $b \in B$, $p^{-1}(b)$ is called the *fibre* of the bundle over $b \in B$.

Intuitively, a bundle can be thought as a union of fibres $f^{-1}(b)$ for $b \in B$ parameterized by B and *glued together* by the topology of the space E. Usually a Greek letter ($\xi, \eta, \zeta, \lambda$, etc) is used to denote a bundle; then $E(\xi)$ denotes the total space of ξ, and $B(\xi)$ denotes the base space of ξ.

A *morphism* of bundles $(\phi, \bar{\phi})\colon \xi \to \xi'$ is a pair of (continuous) maps $\phi\colon E(\xi) \to E(\xi')$ and $\bar{\phi}\colon B(\xi) \to B(\xi')$ such that the diagram

$$\begin{array}{ccc}
E(\xi) & \xrightarrow{\phi} & E(\xi') \\
\Big\downarrow{\scriptstyle p(\xi)} & & \Big\downarrow{\scriptstyle p(\xi')} \\
B(\xi) & \xrightarrow{\bar{\phi}} & B(\xi')
\end{array}$$

commutes.

The trivial bundle is the projection of the Cartesian product:

$$p\colon B \times F \to B, \qquad (x, y) \mapsto x.$$

Roughly speaking, a *fibre bundle* $p\colon E \to B$ is a "locally trivial" bundle with a "fixed fibre" F. More precisely, for any $x \in B$, there exists an open neighborhood U of x such that $p^{-1}(U)$ is a trivial bundle, in other words, there is a homeomorphism $\phi_U\colon p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi_x} & p^{-1}(U) \\ \Big\downarrow \pi_1 & & \Big\downarrow p \\ U & = & U \end{array}$$

commutes, that is, $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$.

Similar to manifolds, we can use "chart" to describe fibre bundles. A *chart* (U, ϕ) for a bundle $p\colon E \to B$ is (1) an open set U of B and (2) a homeomorphism $\phi\colon U \times F \to p^{-1}(U)$ such that $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$. An *atlas* is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ is an open covering of B.

Proposition 4.1. *A bundle $p\colon E \to B$ is a fibre bundle with fibre F if and only if it has an atlas.*

Proof. Suppose that $p\colon E \to B$ is a fibre bundle. Then the collection $\{(U(x), \phi_x) | x \in B\}$ is an atlas.

Conversely suppose that $p\colon E \to B$ has an atlas. For any $x \in B$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x with the property that $p|_{p^{-1}(U)}\colon p^{-1}(U_\alpha) \to U_\alpha$ is a trivial bundle. Thus $p\colon E \to B$ is a fibre bundle. □

Let ξ be a fibre bundle with fibre F and an atlas $\{(U_\alpha, \phi_\alpha)\}$. The composite

$$\phi_\alpha^{-1} \circ \phi_\beta\colon (U_\alpha \cap U_\beta) \times F \xrightarrow{\phi_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} (U_\alpha \cap U_\beta) \times F$$

has the property that

$$\phi_\alpha^{-1} \circ \phi_\beta(x, y) = (x, g_{\alpha\beta}(x, y))$$

for any $x \in U_\alpha \cap U_\beta$ and $y \in F$. Consider the continuous map $g_{\alpha\beta}\colon U_{\alpha\beta} \times F \to F$. Fixing any x, $g_{\alpha\beta}(x, -)\colon F \to F$, $y \mapsto g_{\alpha\beta}(x, y)$ is a homeomorphism with inverse given by $g_{\beta\alpha}(x, -)$. This gives a *transition function*

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F, F),$$

where $\mathrm{Homeo}(F, F)$ is the group of all homeomorphisms from F to F.

Exercise 4.1. *Prove that the transition functions $\{g_{\alpha\beta}\}$ satisfy the following equation*

$$g_{\alpha\beta}(x) \circ g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{3}$$

By choosing $\alpha = \beta = \gamma$, $g_{\alpha\alpha}(x) \circ g_{\alpha\alpha}(x) = g_{\alpha\alpha}(x)$ and so

$$g_{\alpha\alpha}(x) = x \qquad x \in U_\alpha \tag{4}$$

.

By choosing $\alpha = \gamma$, $g_{\alpha\beta}(x) \circ g_{\beta\alpha}(x) = g_{\alpha\alpha}(x) = x$ and so

$$g_{\beta\alpha}(x) = g_{\alpha\beta}(x)^{-1} \qquad x \in U_\alpha \cap U_\beta. \tag{5}$$

We need to introduce a *topology* on $\mathrm{Homeo}(F, F)$ such that the transition functions $g_{\alpha\beta}$ are continuous. The topology on $\mathrm{Homeo}(F, F)$ is given by *compact-open topology* briefly reviewed as follows:

Let X and Y be topological spaces. Let $\mathrm{Map}(X, Y)$ denote the set of all continuous maps from X to Y. Given any compact set K of X and any open set U of Y, let

$$W_{K,U} = \{f \in \mathrm{Map}(X, Y) \mid f(K) \subseteq U\}.$$

Then the *compact-open topology* is generated by $W_{K,U}$, that is, an open set in $\mathrm{Map}(X,Y)$ is an arbitrary union of a finite intersection of subsets with the form $W_{K,U}$.

$\mathrm{Map}(F,F)$ be the set of all continuous maps from F to F with compact open topology. Then $\mathrm{Homeo}(F,F)$ is a subset of $\mathrm{Map}(F,F)$ with subspace topology.

Proposition 4.2. *If* $\mathrm{Homeo}(F,F)$ *has the compact-open topology, then the transition functions* $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to \mathrm{Homeo}(F,F)$ *are continuous.*

Proof. Given $W_{K,U}$, we show that $g_{\alpha\beta}^{-1}(W_{K,U})$ is open in $U_\alpha\cap U_\beta$. Let $x_0\in U_\alpha\cap U_\beta$ such that $g_{\alpha\beta}(x_0)\in W_{(K,U)}$. We need to show that there is a neighborhood V is x_0 such that $g_{\alpha\beta}(V)\subseteq W_{K,U}$, or $g_{\alpha\beta}(V\times K)\subseteq U$. Since U is open and $g_{\alpha\beta}\colon (U_\alpha\cap U_\beta)\times F\to F$ is continuous, $g^{-1}(U)$ is an open set of $(U_\alpha\cap U_\beta)\times F$ with $x_0\times K\subseteq g_{\alpha\beta}^{-1}(U)$. For each $y\in K$, there exist open neighborhoods $V(y)$ of x and $N(y)$ of y such that $V(x)\times N(y)\subseteq g_{\alpha\beta}^{-1}(U)$. Since $\{N(y)\mid y\in K\}$ is an open cover of the compact set K, there is a finite cover $\{N(y_1),\ldots,N(y_n)\}$ of K. Let $V=\bigcap_{j=1}^{n}V(y_j)$. Then $V\times K\subseteq g_{\alpha\beta}^{-1}(U)$ and so $g_{\alpha\beta}(V)\subseteq W_{K,U}$. □

Proposition 4.3. *If F regular and locally compact, then the composition and evaluation maps*

$$\mathrm{Homeo}(F,F)\times\mathrm{Homeo}(F,F)\longrightarrow \mathrm{Homeo}(F,F)\qquad (g,f)\mapsto f\circ g$$

$$\mathrm{Homeo}(F,F)\times F\longrightarrow F\qquad (f,y)\mapsto f(y)$$

are continuous.

Proof. Suppose that $f\circ g\in W_{K,U}$. Then $f(g(K))\subseteq U$, or $g(K)\subseteq f^{-1}(U)$, and the latter is open. Since F is regular and locally compact, there is an open set V such that

$$g(K)\subseteq V\subseteq \bar V\subseteq f^{-1}(U)$$

and the closure $\bar V$ is compact. If $g'\in W_{K,V}$ and $f'\in W_{\bar V,U}$, then $f'\circ g'\in W_{K,U}$. Thus $W_{K,V}$ and $W_{\bar V,U}$ are neighborhoods of g and f whose composition product lies in $W_{K,U}$. This implies that $\mathrm{Homeo}(F,F)\times\mathrm{Homeo}(F,F)\to\mathrm{Homeo}(F,F)$ is continuous.

Let U be an open set of F and let $f_0(y_0)\in U$ or $y_0\in f_0^{-1}(U)$. Since F is regular an locally compact, there is a neighborhood V of y_0 such that $\bar V$ is compact and $y_0\in V\subseteq\bar V\subseteq g_0^{-1}(U)$. If $g\in W_{\bar V,U}$ and $y\in V$, then $g(y)\in U$ and so the evaluation map $\mathrm{Homeo}(F,F)\times F\to F$ is continuous. □

Proposition 4.4. *If F is compact Hausdorff, then the inverse map*

$$\mathrm{Homeo}(F,F)\longrightarrow\mathrm{Homeo}(F,F)\qquad f\mapsto f^{-1}$$

is continuous.

Proof. Suppose that $g_0^{-1}\in W_{K,U}$. Then $g_0^{-1}(K)\subseteq U$ or $K\subseteq g_0(U)$. It follows that

$$F\smallsetminus K\supseteq F\smallsetminus g_0(U)=g_0(F\smallsetminus U)$$

because g_0 is a homeomorphism. Note that $F\smallsetminus U$ is compact, $F\smallsetminus K$ is open and $g_0\in W_{F\smallsetminus U,F\smallsetminus K}$. If $g\in W_{F\smallsetminus U,F\smallsetminus K}$, then, from the above arguments, $g^{-1}\in W_{K,U}$ and hence the result. □

Note. If F is regular and locally compact, then $\mathrm{Homeo}(F,F)$ is a topological monoid, namely compact-open topology only fails in the continuity of g^{-1}. A modification on compact-open topology eliminates this defect [1].

4.2. **G-Spaces and Principal G-Bundles.** Let G be a topological group and let X be a space. A right G-action on X means a(continuous) map $\mu\colon X\times G\to X, (x,g)\mapsto x\cdot g$ such that $x\cdot 1=x$ and $(x\cdot g)\cdot h = x\cdot(gh)$. In this case, we call X a (right) G-space. Let X and Y be (right) G-spaces. A continuous map $f\colon X\to Y$ is called a G-map if $f(x\cdot g)=f(x)\cdot g$ for any $x\in X$ and $g\in G$. Let X/G be the set of G-orbits xG, $x\in X$, with quotient topology.

Proposition 4.5. *Let X be a G-space.*

1) *For fixing any $g\in G$, the map $x\mapsto x\cdot g$ is a homeomorphism.*
2) *The projection $\pi\colon X\to X/G$ is an open map.*

Proof. (1). The inverse is given by $x\mapsto x\cdot g^{-1}$.

(2) If U is an open set of X,

$$\pi^{-1}(\pi(U))=\bigcup_{g\in G} U\cdot g$$

is open because it is a union of open sets, and so $\pi(U)$ is open by quotient topology. Thus π is an open map. □

We are going to find some conditions such that $\pi\colon X\to X/G$ has canonical fibre bundle structure with fibre G. Given any point $\bar{x}\in X/G$, choose $x\in X$ such that $\pi(x)=\bar{x}$. Then

$$\pi^{-1}(\bar{x})=\{x\cdot g\mid g\in G\}=G/H_x,$$

where $H_x=\{g\in G\mid x\cdot g=x\}$.

For having constant fibre G, we need to assume that the G-action on X is free, namely

$$x\cdot g=x \quad\Longrightarrow\quad g=1$$

for any $x\in X$. This is equivalent to the property that

$$x\cdot g=x\cdot h \quad\Longrightarrow\quad g=h$$

for any $x\in X$. In this case we call X a *free G-space.*

Since a fibre bundle is locally trivial (locally Cartesian product), there is always a local cross-section from the base space to the total space. Our second condition is that the projection $\pi\colon X\to X/G$ has local cross-sections. More precisely, for any $\bar{x}\in X/G$, there is an open neighborhood $U(\bar{x})$ with a continuous map $s_{\bar{x}}\colon U(\bar{x})\to X$ such that $\pi\circ s_{\bar{x}}=\mathrm{id}_{U(\bar{x})}$.

(**Note.** For every point $\bar{x}$, we can always choose a pre-image of π, the local cross-section means the pre-images can be chosen "continuously" in a neighborhood. This property depends on the topology structure of X and X/G.)

Assume that X is a (right) free G-space with local cross-sections to $\pi\colon X\to X/G$. Let $\bar{x}$ be any point in X/G. Let $U(\bar{x})$ be a neighborhood of $\bar{x}$ with a (continuous) crosse-section $s_{\bar{x}}\colon U(\bar{x})\to X$. Define

$$\phi_{\bar{x}}\colon U(\bar{x})\times G\longrightarrow \pi^{-1}(U(\bar{x}))\qquad (\bar{y},g)\longrightarrow s_{\bar{x}}(\bar{y})\cdot g$$

for any $y\in U(\bar{x})$.

Exercise 4.2. Let X be a (right) free G-space with local cross-sections to $\pi\colon X\to X/G$. Then the continuous map $\phi_{\bar{x}}\colon U(\bar{x})\times G\to\pi^{-1}(U(\bar{x}))$ is one-to-one and onto. □

We need to find the third condition such that $\phi_{\bar{x}}$ is a homeomorphism. Let

$$X^*=\{(x,x\cdot g)\mid x\in X, g\in G\}\subseteq X\times X.$$

A function

$$\tau\colon X^* \longrightarrow G$$

such that

$$x \cdot \tau(x, x') = x' \qquad \text{for all } (x, x') \in X$$

is called a *translation function.* (**Note.** If X is a free G-space, then translation function is unique because, for any $(x, x') \in X^*$, there is a unique $g \in G$ such that $x' = x \cdot g$, and so, by definition, $\tau(x, x') = g$.)

Proposition 4.6. *Let X be a (right) free G-space with local cross-sections to $\pi\colon X \to X/G$. Then the following statements are equivalent each other:*

1) *The translation function $\tau\colon X^* \to G$ is continuous.*
2) *For any $\bar{x} \in X/G$, the map $\phi_{\bar{x}}\colon U(\bar{x}) \times G \to \pi^{-1}(U(\bar{x}))$ is a homeomorphism.*
3) *There is an atlas $\{(U_\alpha, \phi_\alpha\}$ of X/G such that the homeomorphisms*

$$\phi_\alpha\colon U_\alpha \times G \longrightarrow \pi^{-1}(U_\alpha)$$

satisfy the condition $\phi_\alpha(\bar{y}, gh) = \phi_\alpha(\bar{y}, g) \cdot h$, that is ϕ_α is a homeomorphism of G-spaces.

Proof. (1) $\Longrightarrow$ (2). Consider the (continuous) map

$$\theta\colon \pi^{-1}(U(\bar{x})) \longrightarrow U(\bar{x}) \times G \qquad z \mapsto (\pi(z), \tau(s_{\bar{x}}(\pi(z)), z)).$$

Then

$$\theta \circ \phi_{\bar{x}}(\bar{y}, g) = \theta(s_{\bar{x}}(\bar{y}) \cdot g) = (\bar{y}, \tau(s_{\bar{x}}(\bar{y}), s_{\bar{x}}(\bar{y}) \cdot g)) = (\bar{y}, g),$$

$$\phi_{\bar{x}} \circ \theta(z) = \phi_{\bar{x}}(\pi(z), \tau(s_{\bar{x}}(\pi(z)), z)) = s_{\bar{x}}(\pi(z)) \cdot \tau(s_{\bar{x}}(\pi(z)), z) = z.$$

Thus $\phi_{\bar{x}}$ is a homeomorphism.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Note that the translation function is unique for free G-spaces. It suffices to show that the restriction

$$\tau(X)\colon X^* \cap (\pi^{-1}(U_\alpha) \times \pi^{-1}(U_\alpha)) = \left(\pi^{-1}(U_\alpha)\right)^* \longrightarrow G$$

is continuous. Consider the commutative diagram

$$\begin{array}{ccc}
(U_\alpha \times G)^* & \xrightarrow[\cong]{\phi_\alpha^*} & \left(\pi^{-1}(U_\alpha)\right)^* \\
\Big\downarrow{\scriptstyle \tau(U_\alpha \times G)} & & \Big\downarrow{\scriptstyle \tau(X)} \\
G & = & G.
\end{array}$$

Since

$$\tau(U_\alpha \times G)((\bar{y}, g), (\bar{y}, h)) = g^{-1}h$$

is continuous, the translation function restricted to $\left(\pi^{-1}(U_\alpha)\right)^*$

$$\tau(X) = \tau(U_\alpha \times G) \circ ((\phi_\alpha)^*)^{-1}$$

is continuous for each α and so $\tau(X)$ is continuous. $\square$

Now we give the definition. A *principal G-bundle* is a free G-space X such that

$$\pi\colon X \to X/G$$

has local cross-sections and one of the (equivalent) conditions in Proposition 4.6 holds.

Example. Let Γ be a topological group and let G be a closed subgroup. Then the action of G on Γ given by $(a, g) \mapsto ag$ for $a \in \Gamma$ and $g \in G$ is free. Then translation function is given by $\tau(a, b) = a^{-1}b$, which is continuous. Thus $\Gamma \to \Gamma/G$ is principal G-bundle if and only if it has local cross-sections.

4.3. **The Associated Principal G-Bundles of Fibre Bundles.** We come back to look at fibre bundles ξ given by $p\colon E \to B$ with fibre F. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas and let

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F, F)$$

be the transition functions. A topological group G is called a *group of the bundle* ξ if

1) There is a group homomorphism
$$\theta\colon G \longrightarrow \mathrm{Homeo}(F, F).$$
2) There exists an atlas of ξ such that the transition functions $g_{\alpha\beta}$ lift to G via θ, that is, there is commutative diagram
$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & \mathrm{Homeo}(F, F) \\ \Vert & & \uparrow{\scriptstyle \theta} \\ U_\alpha \cap U_\beta & \xrightarrow{\quad g_{\alpha\beta} \quad} & G, \end{array}$$
(where we use the same notation $g_{\alpha\beta}$.)
3) The transition functions
$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$$
are continuous.
4) The G-action on F via θ is continuous, that is, the composite
$$G \times F \xrightarrow{\theta \times \mathrm{id}_F} \mathrm{Homeo}(F, F) \times F \xrightarrow{\text{evaluation}} F$$
is continuous.

We write $\bar{\xi} = \{(U_\alpha, g_{\alpha\beta})\}$ for the set of transition functions to the atlas $\{(U_\alpha, \phi_\alpha)\}$.

Note. In Steenrod's definition [11, p.7], θ is assume to be a monomorphism (equivalently, the G-action on F is effective, that is, if $y \cdot g = y$ for all $y \in F$, then $g = 1$.).

We are going to construct a principal G-bundle $\pi\colon E^G \to B$. Then prove that the total space $E = F \times_G E^G$ and $p\colon E \to B$ can be obtained canonically from $\pi\colon E^G \to B$. In other words, all fibre bundles can obtained through principal G-bundles through this way. Also the topological group G plays an important role for fibre bundles. Namely, by choosing different topological groups G, we may get different properties for the fibre bundle ξ. For instance, if we can choose G to be trivial (that is, $g_{\alpha\beta}$ lifts to the trivial group), then fibre bundle is trivial. We will see that the bundle group G for n-dimensional vector bundles can be chosen as the general linear group $\mathrm{GL}_n(\mathbb{R})$. The vector bundle is *orientable* if and only if the transition functions can left to the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ matrices whose determinant is positive. If $n = 2m$, then $\mathrm{GL}_m(\mathbb{C}) \subseteq \mathrm{GL}_{2m}(\mathbb{R})$. The vector bundle admits (almost) complex structure if and only if the transition functions can left to $\mathrm{GL}_m(\mathbb{C})$. (For manifolds, one can consider the structure on the tangent bundles. For instance, an oriented manifold means its tangent bundle is oriented.)

Proposition 4.7. *If $\bar{\xi}$ is the set of transition functions for the space B and topological group G, then there is a principal G-bundle ξ^G given by*

$$\pi\colon E^G \longrightarrow B$$

and an atlas $\{(U_\alpha, \phi_\alpha)\}$ such that $\bar{\xi}$ is the set of transition functions to this atlas.

Proof. The proof is given by construction. Let

$$\bar{E} = \bigcup_\alpha U_\alpha \times G \times \alpha,$$

that is $\bar{E}$ is the disjoint union of $U_\alpha \times G$. Now define a relation on $\bar{E}$ by

$$(b, g, \alpha) \sim (b', g', \beta) \Longleftrightarrow b = b', g = g_{\alpha\beta}(b)g'.$$

This is an equivalence relation by Equations (3)-(5). Let $E^G = \bar{E}/\sim$ with quotient topology and let $\{b, g, \alpha\}$ for the class of (b, g, α) in E^G. Define $\pi\colon E^G \to B$ by

$$\pi\{b, g, \alpha\} = b,$$

then π is clearly well-defined (and so continuous). The right G-action on E^G is defined by

$$\{b, g, \alpha\} \cdot h = \{b, gh, \alpha\}.$$

This is well-defined (and so continuous) because if $(b', g', \beta) \sim (b, g, \alpha)$, then

$$(b', g'h, \beta) = (b, (g_{\alpha\beta}(b)g)h, \beta) = (b, g_{\alpha\beta}(b)(gh), \beta) \sim (b, gh, \alpha).$$

Define $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$ by setting

$$\phi_\alpha(b, g) = \{b, g, \alpha\},$$

then ϕ_α is continuous and satisfies $\pi \circ \phi_\alpha(b, g) = b$ and

$$\phi_\alpha(b, g) = \{b, 1 \cdot g, \alpha\} = \{b, 1, \alpha\} \cdot g$$

for $b \in U_\alpha$ and $g \in G$. The map ϕ_α is a homeomorphism because, for fixing α, the map

$$\coprod_\beta (U_\alpha \cap U_\beta) \times G \times \beta \longrightarrow U_\alpha \times G \qquad (b, g', \beta) \mapsto (b, g_{\alpha\beta}(b)g')$$

induces a map $\pi^{-1}(U_\alpha) \to U_\alpha \times G$ which the inverse of ϕ_α. Moreover,

$$\phi_\alpha(b, g_{\alpha\beta}(b)g) = \{b, g_{\alpha\beta}(b)g, \alpha\} = \{b, g, \beta\} = \theta_\beta(b, g)$$

for $b \in U_\alpha \cap U_\beta$ and $g \in G$. Thus the $\{(U_\alpha, g_{\alpha\beta})\}$ is the set of transition function to the atlas $\{(U_\alpha, \phi_\alpha)\}$. □

Let X be a right G-space and let Y be a left G-space. The product over G is defined by

$$X \times_G Y = X \times Y/(xg, y) \sim (x, gy)$$

with quotient topology. Note that the composite

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\pi_X} & X & \xrightarrow{\pi} & X/G \\ (x, y) & \mapsto & x & \mapsto & \bar{x} \end{array}$$

factors through $X \times_G Y$. Let $p\colon X \times_G Y \to X/G$ be the resulting map. For any $\bar{x} \in X/G$, choose $x \in \pi^{-1}(\bar{x}) \subseteq X$, then

$$p^{-1}(\bar{x}) = \pi^{-1}(\bar{x}) \times_G Y = x \times Y/H_x,$$

where $H_x = \{g \in G \mid xg = x\}$. Thus if X is a free right G-space, then the projection $p\colon X \times_G Y \to X/G$ has the constant fibre Y.

Proposition 4.8. *Let $\pi\colon X \to X/G$ be a (right) principal G-bundle and let Y be any left G-space. Then*

$$p\colon X \times_G Y \longrightarrow X/G$$

is a fibre bundle with fibre Y.

Proof. Consider a chart (U_α, ϕ_α) for $\pi\colon X \to X/G$. Since the homeomorphism $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$ is a G-map, there is a commutative diagram

$$\begin{array}{ccccccc} U_\alpha \times Y & = & (U_\alpha \times G) \times_G Y & \xrightarrow[\cong]{\phi_\alpha} & \pi^{-1}(U_\alpha) \times_G Y & = & p^{-1}(U_\alpha) \\ \downarrow{\scriptstyle \pi_{U_\alpha}} & & \downarrow{\scriptstyle \pi_{U_\alpha}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ U_\alpha & = & U_\alpha & = & U_\alpha & = & U_\alpha \end{array}$$

and hence the result. □

Let ξ be a (right) principal G-bundle given by $\pi\colon X \to X/G$. Let Y be any left G-space. Then fibre bundle

$$p\colon X \times_G Y \longrightarrow X/G$$

is called *induced fibre bundle* of ξ, denoted by $\xi[Y]$.

Now let $p\colon E \to B$ is a fibre bundle with fibre F and bundle group G. Observe that the action of $\mathrm{Homeo}(F, F)$ on F is a left action because $(f \circ g)(x) = f(g(x))$. Thus G acts by left on F via $\theta\colon G \to \mathrm{Homeo}(F, F)$.

A bundle morphism

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\phi} & E(\xi') \\ \downarrow{\scriptstyle p(\xi)} & & \downarrow{\scriptstyle p(\xi')} \\ B(\xi) & \xrightarrow{\bar{\phi}} & B(\xi') \end{array}$$

is call an *isomorphism* if both ϕ and $\bar{\phi}$ are homeomorphisms. (**Note.** this means that $(\phi^{-1}, (\bar{\phi})^{-1})$ are continuous.) In this case, we write $\xi \cong \xi'$.

Theorem 4.9. *Let ξ be a fibre bundle given by $p\colon E \to B$ with fibre F and bundle group G. Let ξ^G be the principal G-bundle constructed in Proposition 4.7 according to a set of transitions functions to ξ Then $\xi^G[F] \cong \xi$.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for ξ. We write $\tilde{\phi}_\alpha$ for ϕ_α in the proof of Proposition 4.7. Consider the map θ_α given by the composite:

$$\pi^{-1}(U_\alpha) \times_G F \xleftarrow[\cong]{\tilde{\phi}_\alpha \times \mathrm{id}_F} (U_\alpha \times G \times \alpha)_G \times F = U_\alpha \times F \xrightarrow[\cong]{\phi_\alpha} p^{-1}(U_\alpha).$$

From the commutative diagram

$$\begin{array}{ccc}
((U_\alpha \cap U_\beta) \times G \times \beta) \times_G F & \xrightarrow[((b,g',\beta),y) \mapsto ((b, g_{\alpha\beta}(b)g', \alpha), y)]{(b,g',y) \mapsto (b, g', g_{\alpha\beta}(b)(y))} & ((U_\alpha \cap U_\beta) \times G \times \alpha) \times_G F \\
\Vert & & \Vert \\
(U_\alpha \cap U_\beta) \times F & \xrightarrow{(b,y) \mapsto (b, g_{\alpha\beta}(b,y))} & (U_\alpha \cap U_\beta) \times F \\
\cong \downarrow \phi_\beta & & \cong \downarrow \phi_\alpha \\
p^{-1}(U_\alpha \cap U_\beta) & = & p^{-1}(U_\alpha \cap U_\beta),
\end{array}$$

the map θ_α induces a bundle map

$$\begin{array}{ccc}
E^G \times_G F & \xrightarrow{\theta} & E(\xi) \\
\downarrow & & \downarrow \\
B(\xi) & = & B(\xi).
\end{array}$$

This is an bundle isomorphism because θ is one-to-one and onto, and θ is a local homeomorphic by restricting each chart. The assertion follows. $\square$

This theorem tells that any fibre bundle with a bundle group G is an induced fibre bundle of a principal G-bundle. Thus, for classifying fibre bundles over a fixed base space B, it suffices to classify the principal G-bundles over B. The latter is actually done by the homotopy classes from B to the classifying space BG of G. (There are few assumptions on the topology on B such as B is paracompact.) The theory for classifying fibre bundles is also called *(unstable) K-theory*, which is one of important applications of homotopy theory to geometry. Rough introduction to this theory is as follows:

There exists a *universal G-bundle* ω_G as $\pi\colon EG \to BG$. Given any principal G-bundle ξ over B, there exists a (continuous) map $f\colon B \to BG$ such that ξ, as a principal G-bundle, is isomorphic to the pull-back bundle $f^*\omega_G$ given by

$$\begin{array}{ccc} E(f^*\omega_G)=\{(x,y)\in B\times EG\mid f(x)=\pi(y)\} & \longrightarrow & EG \\ \downarrow (x,y)\mapsto x & & \downarrow \pi \\ B & \xrightarrow{\quad f\quad} & BG. \end{array}$$

Moreover, for continuous maps $f,g\colon B\to BG$, $f^*\omega_G\cong g^*\omega_G$ if and only if $f\simeq g$, that is, there is a continuous map (called *homotopy*) $F\colon B\times[0,1]\to BG$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$. In other words, the set of homotopy classes $[B,BG]$ is one-to-one correspondent to the set of isomorphic classes of principal G-bundles over G.

Seminar Topic: The classification of principal G-bundles and fibre bundles. (References: for instance [8, pp.48-58] Or [9, 10].)

4.4. **Vector Bundles.** Let $\mathbb{F}$ denote $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$-the real, complex or quaternion numbers. An n-dimensional $\mathbb{F}$-*vector bundle* is a fibre bundle ξ given by $p\colon E\to B$ with fibre $\mathbb{F}^n$ and an atlas $\{(U_\alpha,\phi_\alpha)\}$ in which each fibre $p^{-1}(b)$, $b\in B$, has the structure of vector space over $\mathbb{F}$ such that each homeomorphism $\phi_\alpha\colon U_\alpha\times\mathbb{F}^n\to p^{-1}(U_\alpha)$ has the property that

$$\phi_\alpha|_{\{b\}\times\mathbb{F}^n}\colon\{b\}\times F^n\longrightarrow p^{-1}(b)$$

is a vector space isomorphism for each $b\in U_\alpha$.

Let ξ be a vector bundle. From the composite

$$(U_\alpha\cap U_\beta)\times\mathbb{F}^n\xrightarrow{\phi_\beta}p^{-1}(U_\alpha\cap U_\beta)\xrightarrow{\phi_\alpha^{-1}}(U_\alpha\cap U_\beta)\times\mathbb{F}^n,$$

the transition functions

$$g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow \mathrm{Homeo}(\mathbb{F}^n,\mathbb{F}^n)$$

have that property that, for each $x\in U_{\alpha\beta}$,

$$g_{\alpha\beta}(x)\colon\mathbb{F}^n\longrightarrow\mathbb{F}^n$$

is a **linear isomorphism**. It follows that the bundle group for a vector bundle can be chosen as the general linear group $\mathrm{GL}_n(\mathbb{F})$. By Theorem 4.9, we have the following.

Proposition 4.10. *Let ξ be an n-dimensional $\mathbb{F}$-vector space over B. Then there exists a principal $\mathrm{GL}_n(\mathbb{F})$-bundle $\xi^{\mathrm{GL}_n(\mathbb{F})}$ over B such that $\xi\cong\xi^{\mathrm{GL}_n(\mathbb{F})}[\mathbb{F}^n]$. Conversely, for any principal $\mathrm{GL}_n(\mathbb{F})$-bundle over B, $\xi^{\mathrm{GL}_n(\mathbb{F})}[\mathbb{F}^n]$ is an n-dimensional $\mathbb{F}$-vector bundle over B.* □

In other words, the total spaces of all vector bundles are just given by $E(\xi^{\mathrm{GL}_n(\mathbb{F})})\times_{\mathrm{GL}_n(\mathbb{F})}\mathbb{F}^n$.

4.5. **The Construction of Gauss Maps.** The *Grassmann manifold* $G_{n,m}(\mathbb{F})$ is the set of n-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}^m$, that is, all n-$\mathbb{F}$-planes through the origin, with the topology described as in the topic on the examples of Manifolds. (If $m=\infty$, $\mathbb{F}^\infty=\bigoplus_{j=1}^{\infty}\mathbb{F}$.) Let

$$E(\gamma_n^m)=\{(V,x)\in G_{n,m}(\mathbb{F})\times\mathbb{F}^m\mid x\in V\}.$$

Exercise 4.3. Show that

$$p\colon E(\gamma_n^m)\to G_{n,m}(\mathbb{F}) \qquad (V,x)\mapsto V$$

is an n-dimensional $\mathbb{F}$-vector bundle, denoted by γ_n^m. [Hint: By reading the topic on the examples of manifolds, check that $V_{n,m}(\mathbb{F})\to G_{n,m}(\mathbb{F})$ is a principal $O(n,\mathbb{F})$, where $O(n,\mathbb{R})=O(n)$, $O(n,\mathbb{C})=U(n)$ and $O(n,\mathbb{H})=\mathrm{Sp}(n)$. Then check that $E(\gamma_n^m)=V_{n,m}(\mathbb{F})\times_{O(n,\mathbb{F})}\mathbb{F}^n$.]

A *Gauss map* of an n-dimensional $\mathbb{F}$-vector bundle in $\mathbb{F}^m$ ($n\le m\le\infty$) is a (continuous) map $g\colon E(\xi)\to\mathbb{F}^m$ such that g restricted to each fibre is a linear monomorphism.
Example. The map

$$q\colon E(\gamma_n^m)\to\mathbb{F}^m \qquad (V,x)\mapsto x$$

is a Gauss map.

Proposition 4.11. *Let ξ be an n-dimensional $\mathbb{F}$-vector bundle.*

1) *If there is a vector bundle morphism*

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\ u\ } & E(\gamma_n^m) \\ \Big\downarrow{\scriptstyle p(\xi)} & & \Big\downarrow{\scriptstyle p(\gamma_n^m)} \\ B(\xi) & \xrightarrow{\ f\ } & G_{n,m}(\mathbb{F}) \end{array}$$

that is an isomorphism when restricted to any fibre of ξ, then $q\circ u\colon E(\xi)\to\mathbb{F}^m$ is a Gauss map.

2) *If there is a Gauss map $g\colon E(\xi)\to\mathbb{F}^m$, then there is a vector bundle morphism $(u,f)\colon\xi\to\gamma_n^m$ such that $qu=g$.*

Proof. (1) is obvious. (2). For each $b\in B(\xi)$, $g(p(\xi)^{-1}(b))$ is an n-dimensional $\mathbb{F}$-subspace of $\mathbb{F}^m$ and so a point in $G_{n,m}(\mathbb{F})$. Define the functions

$$f\colon B(\xi)\to G_{n,m}(\mathbb{F}) \qquad f(b)=g(p(\xi)^{-1}(b)),$$
$$u\colon E(\xi)\to E(\gamma_n^m) \qquad u(z)=(f(p(z)),g(z)).$$

The functions f and u are well-defined. For checking the continuity of f and u, one can look at a local coordinate of ξ and so we may assume that ξ is a trivial bundle, namely, $g\colon B(\xi)\times\mathbb{F}^n\to\mathbb{F}^m$ restricted to each fibre is a linear monomorphism. Let $\{e_1,\ldots,e_n\}$ be the standard $\mathbb{F}$-bases for $\mathbb{F}^n$. Then the map

$$h\colon B\longrightarrow\mathbb{F}^m\times\cdots\times\mathbb{F}^m \qquad b\mapsto(g(b,e_1),g(b,e_2),\ldots,g(b,e_n))$$

is continuous. Since g restricted to each fibre is a monomorphism, the vectors

$$\{g(b,e_1),g(b,e_2),\ldots,g(b,e_n)\}$$

are linearly independent and so

$$(g(b,e_1),g(b,e_2),\ldots,g(b,e_n))\in\tilde V_{n,m}(\mathbb{F})$$

for each b, where $V_{n,m}(\mathbb{F})$ is the open Stiefel manifold over $\mathbb{F}$. Thus

$$h\colon B\longrightarrow\tilde V_{n,m}(\mathbb{F}) \qquad b\mapsto(g(b,e_1),g(b,e_2),\ldots,g(b,e_n))$$

is continuous and so the composite

$$f\colon B \xrightarrow{h} \tilde{V}_{n,m}(\mathbb{F}) \xrightarrow{\text{quotient}} G_{n,m}(\mathbb{F})$$

is continuous. The function u is continuous because the composite

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\quad u \quad} & E(\gamma_n^m) \\ \Big\downarrow \Delta & & \Big\downarrow \\ E(\xi)\times E(\xi) & \xrightarrow{(f\circ p(\xi))\times g} & G_{n,m}(\mathbb{F})\times \mathbb{F}^m \end{array}$$

is continuous. This finishes the proof. □

Let ξ be a vector bundle and let $f\colon X \to B(\xi)$ be a (continuous) map. Then the *induced vector* $f^*\xi$ is the pull-back

$$\begin{array}{ccc} E(f^*\xi) = \{(x,y)\in X\times E(\xi)\ | f(x) = p(y)\} & \longrightarrow & E(\xi) \\ \Big\downarrow & & \Big\downarrow p \\ X & \xrightarrow{\quad f \quad} & B(\xi). \end{array}$$

Proposition 4.12. *There exists a Gauss map $g\colon E(\xi) \to \mathbb{F}^m$ $(n \le m \le \infty)$ if and only if*

$$\xi \cong f^*(\gamma_n^m)$$

over $B(\xi)$ for some map $f\colon B(\xi) \to G_{n,m}(\mathbb{F})$.

Proof. $\Longleftarrow$ is obvious.

$\Longrightarrow$ Assume that ξ has a Gauss map g. From Part (2) of Proposition 4.11, there is a commutative diagram

$$\begin{array}{ccc} E(\xi) & \xrightarrow{u} & E(\gamma_n^m) \\ \Big\downarrow p(\xi) & & \Big\downarrow p(\gamma_n^m) \\ B(\xi) & \xrightarrow{f} & G_{n,m}(\mathbb{F}). \end{array}$$

Since $E(f^*\gamma_n^m)$ is defined to be the pull-back, there is commutative diagram

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\tilde{u}} & E(f^*\gamma_n^m) \\ \Big\downarrow p(\xi) & & \Big\downarrow p \\ B(\xi) & = & B(\xi), \end{array}$$

where $\tilde{u}$ restricted to each fibre is a linear isomorphism because both vector-bundle has the same dimension and the Gauss map g restricted to each fibre is a monomorphism. It follows that

$\tilde{u}: E(\xi) \to E(f^*\gamma_n^m)$ is one-to-one and onto. Moreover $\tilde{u}$ is a homeomorphism by considering a local coordinate. □

We are going to construct a Gauss map for each vector bundle over a paracompact space. First, we need some preliminary results for bundles over paracompact spaces. (For further information on paracompact spaces, one can see [3, 162-169].

A family of $\mathcal{C} = \{C_\alpha \mid J\}$ of subsets of a space X is called *locally finite* if each $x \in X$ admits a neighborhood W_x such that $W_x \cap C_\alpha \neq \emptyset$ for only finitely many indices $\alpha \in J$. Let $\mathcal{U} = \{U_\alpha\}$ and $\mathcal{V} = \{V_\beta\}$ be two open covers of X. $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ if for each β, $V_\beta \subseteq U_\alpha$ for some α.

A Hausdorff space X is called *paracompact* if it is regular and if every open cover of X admits a locally finite refinement.

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ be an open cover of a space X. A *partition of unity*, subordinate to $\mathcal{U}$, is a collection $\{\lambda_\alpha \mid \alpha \in J\}$ of continuous functions $\lambda_\alpha : X \to [0,1]$ such that

1) The support
$$\mathrm{supp}(\lambda_\alpha) \subseteq U_\alpha$$
for each α, where the support
$$\mathrm{supp}(\lambda_\alpha) = \overline{\{x \in X \mid \lambda_\alpha(x) \neq 0\}}$$
is the closure of the subset of X on which $\lambda_\alpha \neq 0$;
2) for each $x \in X$, there is a neighborhood W_x of x such that $\lambda_{\lambda_\alpha}|_{W_x} \not\equiv 0$ for only finitely many indices $\alpha \in J$. (In other words, the supports of λ_α's are locally finite.)
3) The equation
$$\sum_{\alpha \in J} \lambda_\alpha(x) = 1$$
for all $x \in X$, where the summation is well-defined for each given x because there are only finitely many non-zeros.

We give the following well-known theorem without proof. One may read a proof in [2, pp.17-20].

Theorem 4.13. *If X is a paracompact space and $\mathcal{U} = \{U_\alpha\}$ is an open cover of X, then there exists a partition of unity subordinate to $\mathcal{U}$.* □

Lemma 4.14. *Let ξ be a fibre bundle over a paracompact space B. Then ξ admits an atlas with countable charts.*

Proof. Let $\{(U_\alpha, \phi_\alpha \mid \alpha \in J\}$ be an atlas for ξ. We are going to find another atlas with countable charts.

By Theorem 4.13, there is a partition of unity $\{\lambda_\alpha \mid \alpha \in J\}$ subordinate to $\{U_\alpha \mid \alpha \in J\}$. Let
$$V_\alpha = \lambda_\alpha^{-1}(0,1] = \{b \in B \mid \lambda_\alpha(b) > 0\}.$$
Then, by the definition of partition of unity, $V_\alpha \subseteq \bar{V}_\alpha \subseteq U_\alpha$. For each $b \in B$, let
$$S(b) = \{\alpha \in J \mid \lambda_\alpha(b) > 0\}.$$
Then, by the definition of partition of unity, $S(b)$ is a finite subset of J.

Now for each finite subset S of J define
$$W(S) = \{b \in B \mid \lambda_\alpha(b) > \lambda_\beta(b) \text{ for each } \alpha \in S \text{ and } \beta \notin S\}$$

$$= \bigcap_{\substack{\alpha \in S \\ \beta \notin S}} (\lambda_\alpha - \lambda_\beta)^{-1}(0, 1].$$

Then $W(S)$ is open because for each $b \in W(S)$, by definition of partition of unity, there exists a neighborhood W_b of b such that there are finitely many supports intersect with W_b; and so the above (possibly infinite) intersection of open sets restricted to W_b is only a finite intersection of open sets.

Let S and S' be two subsets of J such that $S \neq S'$ and $|S| = |S'| = m > 0$, where $|S|$ is the number of elements in S. Then there exist $\alpha \in S \smallsetminus S'$ and $\beta \in S' \smallsetminus S$ because $S \neq S'$ but S and S' has the same number elements. We claim that

$$W(S) \cap W(S') = \emptyset.$$

Otherwise there exists $b \in W(S) \cap W(S')$. By definition $W(S)$, $\lambda_\alpha(b) > \lambda_\beta(b)$ because $\alpha \in S$ and $\beta \notin S$. On the other hand, $\lambda_\beta(b) > \lambda_\alpha(b)$ because $b \in W(S')$, $\beta \in S'$ and $\alpha \notin S'$.

Now define

$$W_m = \bigcup_{\substack{b \in B \\ |S| = m}} W(S(b))$$

for each $m \geq 1$. We prove that (1) $\{W_m \mid m = 1, \ldots\}$ is an open cover of B; and (2) ξ restricted to W_m is a trivial bundle for each m. (Then $\{W_m\}$ induces an atlas for ξ.)

To check $\{W_m\}$ is an open cover, note that each W_m is open. For each $b \in B$, $S(b)$ is a finite set and $b \in W(S(b))$ because $\lambda_\beta(b) = 0$ for $\beta \notin S(b)$ and $\lambda_\alpha(b) > 0$ for $\alpha \in S(b)$. Let $m = |S(b)|$, then $b \in W_m$ and so $\{W_m\}$ is an open cover of B.

Now check that ξ restricted to W_m is trivial. From the above, W_m is a disjoint union of $W(S(b))$. It suffices to check that ξ restricted to each $W(S(b))$ is trivial. Fixing $\alpha \in S(b)$, for any $x \in W(S(b))$, then

$$\lambda_\alpha(x) > \lambda_\beta(x)$$

for any $\beta \notin S$. In particular, $\lambda_\alpha(x) > 0$ for any $x \in W(S(b))$. It follows that $W(S(b)) \subseteq V_\alpha \subseteq U_\alpha$. Since ξ restricted to U_α is trivial, ξ restricted to $W(S(b))$ is trivial. This finishes the proof. □

Note. From the proof, if for each $b \in B$ there are at most k sets U_α with $b \in U_\alpha$, then B admits an atlas of finite (at most k) charts. [In this case, check that $W_j = \emptyset$ for $j > k$.]

Theorem 4.15. *Any n-dimensional $\mathbb{F}$-vector bundle ξ over a paracompact space B has a Gauss map $g \colon E(\xi) \to \mathbb{F}^\infty$. Moreover, if ξ has an atlas of k charts, then ξ has a Gauss map $g \colon E(\xi) \to \mathbb{F}^{kn}$.*

Proof. Let $\{(U_i, \phi_i)\}_{1 \leq i \leq k}$ be an atlas of ξ with countable or finite charts, where k is finite or infinite. Let $\{\lambda_i\}$ be the partition of unity subordinate to $\{U_i\}$. For each i, define the map $g_i \colon E(\xi) \to \mathbb{F}^n$ as follows: g_i restricted to $p(\xi)^{-1}(U_i)$ is given by

$$g_i(z) = \lambda_i(z)(p_2 \circ \phi_i^{-1}(z)),$$

where $p_2 \circ \phi_i^{-1}$ is the composite

$$p(\xi)^{-1}(U_i) \xrightarrow{\phi_i^{-1}} U_i \times \mathbb{F}^n \xrightarrow[p_2]{\text{projection}} \mathbb{F}^n;$$

and g_i restricted to the outside of $p(\xi)^{-1}(U_i)$ is 0. Since the closure of $\lambda_i^{-1}(0,1]$ is contained in U_i, g_i is a well-defined (continuous) map. Now define

$$g\colon E(\xi) \to \bigoplus_{i=1}^{k} \mathbb{F}^n = \mathbb{F}^{kn} \qquad g(z) = \sum_{i=1}^{k} g_i(z).$$

This a well-defined (continuous) map because for each z, there is a neighborhood of z such that there are only finitely many g_i are not identically zero on it.

Since each $g_i\colon E(\xi) \to \mathbb{F}^n$ is a monomorphism (actually isomorphism) on the fibres of $E(\xi)$ over b with $\lambda_i(b) > 0$, and since the images of g_i are in complementary subspaces of $\mathbb{F}^{kn}$, the map g is a Gauss map. □

This gives the following classification theorem:

Corollary 4.16. *Every vector bundle over a paracompact space B is isomorphic to an induced vector bundle $f^*(\gamma_n^\infty)$ for some map $f\colon B \to G_{n,\infty}(\mathbb{F})$. Moreover every vector bundle over a paracompact space B with an atlas of finite charts is isomorphic to an induced vector bundle $f^*(\gamma_n^m)$ for some m and some map $f\colon B \to G_{n,m}(\mathbb{F})$.* □

Remarks: It can be proved that $f^*\gamma_n^m \cong g^*\gamma_n^m$ if and only if $f \simeq g\colon B \to G_{n,m}(\mathbb{F})$. From this, one get that the set of isomorphism classes of n-dimensional $\mathbb{F}$-vector bundles over a paracompact space B is isomorphic to the set of homotopy classes $[B, G_{n,\infty}(\mathbb{F})]$.

For instance, if $n = 1$ and $\mathbb{F} = \mathbb{R}$, $G_{1,\infty}(\mathbb{R}) \simeq BO(1) \simeq B\mathbb{Z}/2 \simeq \mathbb{R}\mathrm{P}^\infty$, where BG is so-called the classifying space of the (topological) group G, and $[B, \mathbb{R}\mathrm{P}^\infty] = H^1(B, \mathbb{Z}/2)$, which states that all line bundles are by the first cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. In particular, any real line bundles over a simply connected space is always trivial.

If $n = 1$ and $\mathbb{F} = \mathbb{C}$, then $G_{1,\infty}(\mathbb{C}) \simeq BU(1) \simeq BS^1 \simeq \mathbb{C}\mathrm{P}^\infty$, and $[B, \mathbb{C}\mathrm{P}^\infty] = H^2(B, \mathbb{Z})$, which states that all complex line bundles are by the second integral cohomology.

If $n = 1$ and $\mathbb{F} = \mathbb{H}$, then $G_{1,\infty}(\mathbb{H}) \simeq B\mathrm{Sp}(1) \simeq BS^3 \simeq \mathbb{H}\mathrm{P}^\infty$, and so $[B, G_{1,\infty}(\mathbb{H})] = [B, \mathbb{H}\mathrm{P}^\infty]$. However, the determination of $[B, \mathbb{H}\mathrm{P}^\infty]$ is very hard problem even when B are spheres. If $B = S^n$, then $[B, \mathbb{H}\mathrm{P}^\infty] = \pi_{n-1}(S^3)$ that is only known for n less than 66 or so, by a lot of computations through many papers. Some people even believe that it is impossible to compute the general homotopy groups $\pi_n(S^3)$.

Seminar Topic: Gauss Maps and the Classification of Vector Bundles. (References: for instance [8, pp.26-29,31-33].)

A vector bundle is called *of finite type* if it has an atlas with finite charts. Given two vector bundles ξ and η over B, the *Whitney sum* $\xi \oplus \eta$ is defined to be the pull-back:

$$\begin{array}{ccc}
E(\xi\oplus\eta) & \longrightarrow & E(\xi)\times E(\eta) \\
\downarrow & & \downarrow p(\xi)\times p(\eta) \\
B & \xrightarrow[\text{diagonal}]{\Delta} & B\times B.
\end{array}$$

Intuitively, $\xi \oplus \eta$ is just the fibrewise direct sum.

Proposition 4.17. *For a vector bundle ξ over a paracompact space B, the following statement are equivalent:*

1) *The bundle ξ is of finite type.*
2) *There exists a map $f\colon B \to G_{n,m}(\mathbb{F})$ such that ξ is isomorphic to $f^*\gamma_n^m$.*
3) *There exists a vector bundle η such that the Whitney sum $\xi \oplus \eta$ is trivial.*

Proof. (1) $\Longrightarrow$ (2) follows from Corollary 4.16. (2) $\Longrightarrow$ (1) It is an exercise to check that γ_n^m is of finite type by using the property that the Grassmann manifold $G_{n,m}(\mathbb{F}) = O(m,\mathbb{F})/O(n,\mathbb{F}) \times O(m-n,\mathbb{F})$ is compact, where $O(n,\mathbb{R}) = O(n)$, $O(n,\mathbb{C}) = U(n)$ and $O(n,\mathbb{H}) = \mathrm{Sp}(n)$. It follows that $\xi \cong f^*\gamma_n^m$ is of finite type.

(2) $\Longrightarrow$ (3). Let $(\gamma_n^m)^*$ be the vector bundle given by

$$E((\gamma_n^m)^*) = \{(V,\vec{v}) \in G_{n,m}(\mathbb{F}) \times \mathbb{F}^m \mid \vec{v} \perp V\}$$

with canonical projection $E((\gamma_n^m)^*) \to G_{n,m}(\mathbb{F})$. Then $\gamma_n^m \oplus (\gamma_n^m)^*$ is an m-dimensional trivial $\mathbb{F}$-vector bundle. It follows that

$$f^*(\gamma_n^m \oplus (\gamma_n^m)^*) = f^*(\gamma_n^m) \oplus f^*((\gamma_n^m)^*)$$

is trivial. Let $\eta = f^*((\gamma_n^m)^*)$. Then $\xi \oplus \eta$ is trivial.

(3) $\Longrightarrow$ (2). The composite

$$E(\xi) \hookrightarrow E(\xi \oplus \eta) = B \times F^m \longrightarrow \mathbb{F}^m$$

is a Gauss map into finite dimensional vector space, where $m = \dim(\xi \oplus \eta)$. By Proposition 4.12, there is a map $f\colon B \to G_{n,m}$ such that $\xi \cong f^*\gamma_n^m$. □

Corollary 4.18. *Let ξ be a $\mathbb{F}$-vector bundle over a compact (Hausdorff) space B. Then there is a $\mathbb{F}$-vector bundle η such that $\xi \oplus \eta$ is trivial.* □

In the view of (stable) K-theory, the Whitney sum is an operation on vector bundles over a (fixed) base-space, where the trivial bundles (of different dimensions) are all regarded as 0. In this sense, the Whitney sum plays as an addition (that is associative and commutative with 0). The bundle ξ with property that $\xi \oplus \eta$ is trivial for some η means that ξ is invertible. Those who are interested in algebra can push notions in algebra to vector bundles by doing constructions fibrewisely. More general situation possibly is the *sheaf theory* (by removing the locally trivial condition) that is pretty useful in algebraic geometry. In algebraic topology, people also study the category whose objects are just continuous maps $f\colon E \to B$ with fixed space B, or even more general category whose objects are *diagrams* over spaces. In the terminology of fibre bundles, a map $f\colon E \to B$ is called a *bundle* (without assuming locally trivial).

5. Tangent Bundles and Vector Fields

6. Cotangent Bundles and Tensor Fields

7. Orientation of Manifolds

8. Tensor Algebras and Exterior Algebras

9. DeRham Cohomology

10. Integration on Manifolds

11. Stokes' Theorem

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