Homework 1

Question 1. The tangent space to S^1 at a point (a, b) is a one-dimensional subspace of $\mathbb{R}^2$. Explicitly calculate the subspace in terms of a and b. [The answer is obviously the space spanned by $(-b, a)$, but prove it.]

Question 2. Exhibit a basis for $T_P(S^2)$ at arbitrary point $P = (a, b, c) \in S^2$. [Consider S^2 is the surface given $x^2 + y^2 + z^2 = 1$.]

Question 3. What is the tangent space to the paraboloid defined by

$$x^2 + y^2 - z^2 = a$$

at $(\sqrt{a}, 0, 0)$, where $a > 0$? What does it happen when $a = 0$?

Question 4. Let M be the intersection of two level surfaces $f(x^1, \ldots, x^n) = c$ and $g(x^1, \ldots, x^n) = d$. Given a point P in M, assume that the gradients ∇f and ∇g are linearly independent at P, find the tangent space to M and P. What would happen if ∇f and ∇g are linearly dependent, but both of them are non-zero?

Question 5. Let $V\colon P \mapsto (P, \vec{v}(P))$, where $\vec{v}(x, y) = (-y, x)$, be a vector field in $\mathbb{R}^2$. Find the integral curve of V through the point (a, b) at $t = 0$.

Question 6. Compute the Jacobian of each of the following transformation. Determine where local inverses exist.

(a) $x = e^u \cos v,\ y = e^u \sin v$;
(b) $x = u^2 - v^2,\ y = 2uv$;
(c) $x = u^2 - uv,\ y = v - u$;
(d) $x = \sin(u + v),\ y = \cos(u + v)$.