Solutions to Homework 1

Question 1. The tangent space to S^1 at a point (a, b) is a one-dimensional subspace of $\mathbb{R}^2$. Explicitly calculate the subspace in terms of a and b. [The answer is obviously the space spanned by $(-b, a)$, but prove it.]

Solution: Let $f(x, y) = x^2+y^2-1$. Then, S^1 is the level surface given by $f(x, y) = 0$. At the point (a, b), the gradient of f is

$$\nabla f(a, b) = (2a, 2b).$$

By definition, the tangent space at (a, b) is the set of vectors $(u, v) \in \mathbb{R}^2$ passing through (a, b) such that $\nabla f(a, b) \cdot (u, v) = 0$, i.e., $2au + 2bv = 0$. Thus, $(u, v) = t(-b, a)$, where $t \in \mathbb{R}$. Consequently, the tangent space to the surface at the point (a, b) is the one-dimensional subspace of $\mathbb{R}^2$ spanned by the vector $(-b, a)$.

Question 2. Exhibit a basis for $T_P(S^2)$ at arbitrary point $P = (a, b, c) \in S^2$. [Consider S^2 is the surface given $x^2 + y^2 + z^2 = 1$.]

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Then, S^2 is the level surface given by $f(x, y, z) = 0$. Clearly, the gradient of f at any point $P = (a, b, c)$ is

$$\nabla f(a, b, c) = (2a, 2b, 2c).$$

The tangent space to S^2 at P is the set of vectors $(u, v, w) \in \mathbb{R}^3$ passing through P such that $\nabla f(a, b, c) \cdot (u, v, w) = 0$, which gives

$$2au + 2bv + 2cw = 0.$$

Solving, we obtain $(u, v, w) = s(-b, a, 0) + t(-c, 0, a), s, t \in \mathbb{R}$. Consequently, a basis for the tangent space is $\{(-b, a, 0), (-c, 0, a)\}$.

Question 3. What is the tangent space to the paraboloid defined by

$$x^2 + y^2 - z^2 = a$$

at $(\sqrt{a}, 0, 0)$, where $a > 0$? What does it happen when $a = 0$?

Solution: Let $f(x, y, z) = x^2 + y^2 - z^2$. Then,

$$\nabla f = (2x, 2y, -2z).$$

At the point $P = (\sqrt{a}, 0, 0)$, where $a > 0$, The gradient of f is $\nabla f(P) = (2\sqrt{a}, 00)$. The tangent space at P is the set of vectors $(u, v, w) \in \mathbb{R}^3$ through P such that $\nabla f(P) \cdot (u, v, w) = 0$, or equivalently, $\sqrt{a}u = 0$. Thus, $u = 0$ and the tangent space is the 2-dimensional subspace of $\mathbb{R}^3$ spanned by the vectors $(0, 1, 0)$ and $(0, 0, 1)$. Now, if $a = 0$, then $z^2 = x^2 + y^2$ and∇f is 0 at $(0, 0, 0)$. In fact, the point $(0, 0, 0)$ is the common vertex of the two opposite cones.

Question 4. Let M be the intersection of two level surfaces $f(x^1, \ldots, x^n) = c$ and $g(x^1, \ldots, x^n) = d$. Given a point P in M, assume that the gradients ∇f and ∇g are linearly independent at P, find the tangent space to M and P. What would happen if ∇f and ∇g are linearly dependent, but both of them are non-zero?

Solution: Let $P \in M$. The tangent space to the surface $f(x^1, \dots, x^n) = c$ at P is the $n-1$-dimensional subspace $T_1 = \{(u^1, \dots, u^n) \in \mathbb{R}^n \mid \frac{\partial f}{\partial x^1}(P)u^1 + \cdots + \frac{\partial f}{\partial x^n}(P)u^n = 0\}$. Similarly, the tangent space to the level surface $g(x^1, \dots, x^n) = d$ at P is the $n-1$-dimensional subspace $T_2 = \{(v^1, \dots, v^n) \in \mathbb{R}^n \mid \frac{\partial g}{\partial x^1}(P)v^1 + \cdots + \frac{\partial g}{\partial x^n}(P)v^n = 0\}$. If $\nabla f(P)$ and $\nabla g(P)$ are linearly independent, $T_1 \neq T_2$ and the tangent space to M at P is the $n-2$-dimensional subspace formed by the intersection of the two tangent spaces. On the other hand, if $\nabla f(P)$ and $\nabla g(P)$ are linearly dependent, $T_1 = T_2$ and the tangent space to M at P is this common tangent space.

Question 5. Let $V\colon P \mapsto (P, \vec{v}(P))$, where $\vec{v}(x, y) = (-y, x)$, be a vector field in $\mathbb{R}^2$. Find the integral curve of V through the point (a, b) at $t = 0$.

Solution: Let $s(t) = (x(t), y(t))$ be the integral curve to V at $t = 0$. Then,

$$s'(t) = (x'(t), y'(t)) = \vec{v}(s(t)) = (-y(t), x(t)).$$

By considering the x and y components separately, we have $x'(t) = -y(t)$ and $y'(t) = x(t)$. Solving the differential equations with initial conditions $s(0) = (a, b)$ yields

$$x(t) = a - \int_0^t y(t)dt$$

and

$$y(t) = b + \int_0^t x(t)dt.$$

Question 6. Compute the Jacobian of each of the following transformation. Determine where local inverses exist.

(a) $x = e^u \cos v,\ y = e^u \sin v$;
(b) $x = u^2 - v^2,\ y = 2uv$;
(c) $x = u^2 - uv,\ y = v - u$;
(d) $x = \sin(u + v),\ y = \cos(u + v)$.

Solution:

(a) $x = e^u \cos v,\ y = e^u \sin v$. The Jacobian is given by

$$\begin{aligned} J &= \frac{\partial(x, y)}{\partial(u, v)} \\ &= \det \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \\ &= e^{2u} \cdot \det \begin{matrix} \cos v & -\sin v \\ \sin v & \cos v \end{matrix} \\ &= \sin^2 v + \cos^2 v \\ &= 1. \end{aligned}$$

Since $J \neq 0$ for all $(u, v) \in \mathbb{R}^2$, it follows from the Inverse Mapping Theorem that the local inverse exists everywhere in $\mathbb{R}^2$.

(b) $x = u^2 - v^2,\ y = 2uv.$

$$\begin{aligned} J &= \frac{\partial(x,y)}{\partial(u,v)} \\ &= \det \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \\ &= \det \begin{matrix} 2u & -2v \\ 2v & 2u \end{matrix} \\ &= 4u^2 + 4v^2. \end{aligned}$$

For $(u,v) \neq (0,0), J \neq 0$ and so, we conclude from the Inverse Mapping Theorem that the local inverses exist.

(c) $x = u^2 - uv,\ y = v - u;$

$$\begin{aligned} J &= \frac{\partial(x,y)}{\partial(u,v)} \\ &= \det \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \\ &= \det \begin{matrix} 2u - v, -u & \\ -1 & 1 \end{matrix} \\ &= u - v. \end{aligned}$$

Clearly, $J = 0$ if and only if (u,v) lies on the line $v = u$. Hence, for points in $\mathbb{R}^2$ not on this line, we conclude from the Inverse Mapping Theorem that the local inverses exist.

(d) $x = \sin(u+v),\ y = \cos(u+v).$

$$\begin{aligned} J &= \frac{\partial(x,y)}{\partial(u,v)} \\ &= \det \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \\ &= \det \begin{matrix} \cos(u+v) & \cos(u+v) \\ -\sin(u+v) & -\sin(u+v) \end{matrix} \\ &= 0. \end{aligned}$$

Consequently, the local inverses do not exist for all points in $\mathbb{R}^2$.