Homework 2

Question 1. Let M and N be smooth manifolds.

1) Show that $T_{(x,y)}(M \times N) = T_x(M) \times T_y(N)$.
2) Let $p\colon M \times N \to M$ be the projection map $(x, y) \mapsto x$. Prove that
$$Tp\colon T_x(M) \times T_y(N) \to T_x(M)$$
is the analogous projection $(\vec{v}, \vec{w}) \mapsto \vec{v}$.
3) Fixing any $y \in N$, let $j\colon M \to M \times N$ be the inclusion $x \mapsto (x, y)$. Show that $Tj(\vec{v}) = (\vec{v}, 0)$.
4) Let $f\colon M \to M'$ and $g\colon N \to N'$ be smooth maps. Prove that $T(f \times g)_{(x,y)} = Tf_x \times Tg_y$.

Question 2. Let M be a smooth manifold.

1) Let $\Delta\colon M \to M \times M$ be the diagonal map $x \mapsto (x, x)$. Prove that $T\Delta_x(\vec{v}) = (\vec{v}, \vec{v})$.
2) Let $\Delta(M) = \{(x, x) \mid x \in M\} \subseteq M \times M$ be the diagonal. Show that the tangent space $T_{(x,x)}(\Delta(M))$ is the diagonal of $T_x(M) \times T_x(M)$.

Question 3. Prove the following statements:

1) If f and g are immersions, then so is $f \times g$.
2) If f and g are immersions, then so is $g \circ f$.
3) If f an immersion, then so is f restricted to any submanifold of its domain.
4) If $\dim M = \dim N$, then immersions $f\colon M \to N$ are the same as local diffeomorphisms.

Question 4. Check the map
$$\mathbb{R}^1 \to \mathbb{R}^2, \qquad t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right)$$
is an embedding. Prove that its image is one nappe of the hyperbola $x^2 - y^2 = 1$.

Question 5. The smooth links can be regarded as 1-dimensional submanifolds of $\mathbb{R}^3$. The links can be also regarded as embeddings of disjoint union of finite copies of S^1 into $\mathbb{R}^3$. Draw a nontrivial links consisting of 3 components with the property that it becomes a trivial link after removing any one of the links. [This kind of links is called *Brunnian links*.]

Let $f\colon M \to N$ be a smooth map. A point $y \in N$ is called a *critical value* if $Tf\colon T_x(M) \to T_y(N)$ is not surjective for some $x \in f^{-1}(y)$. (Namely, if y is not regular.)

Question 6. Check that 0 is the only critical value of the map $f\colon \mathbb{R}^3 \to \mathbb{R}^1$ defined by
$$f(x, y, z) = x^2 + y^2 - z^2.$$

Prove that if a and b are either both positive or both negative, then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic. [Hint: Consider scalar multiplication by $\sqrt{b/a}$ on $\mathbb{R}^3$.] Pictorially examine the catastrophic change in the topology of $f^{-1}(c)$ as c passes through the critical value.

Problem 7. Let M and Z be transversal submanifolds of N. Prove that if $y \in M \cap Z$, then

$$T_y(M \cap Z) = T_y(M) \cap T_y(Z).$$

Problem 8. For which values of a does the hyperboloid defined by $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversally? What does the intersection look like for different values of a?