2003/2004 Semester II MA5210 Differentiable Manifolds

Homework 3

Let ξ be a fibre bundle and let $f\colon X \to B$ be a map. Recall that the induced bundle $f^*\xi$ over X is obtained from the pull-back

$$\begin{array}{ccc} E(f^*(\xi)) & \longrightarrow & E(\xi) \\ \Big\downarrow{\scriptstyle p(f^*(\xi))} & & \Big\downarrow{\scriptstyle p(\xi)} \\ X & \xrightarrow{\ f\ } & B(\xi). \end{array}$$

Question 1. Let ξ be a fibre bundle given by $p\colon E \to B$ and let $f\colon X \to B$ be any (continuous) map. Suppose that $\{U_\alpha\}$ is an open cover of B such that ξ restricted to each U_α is trivial. Show that $f * \xi$ restricted to each $f^{-1}(U_\alpha)$ is trivial. From this, show that if ξ has an atlas of countable (finite) charts, then so does any induced fibre bundle of ξ.

Question 2. Let X be a G-space. Suppose that $\pi\colon X \to X/G$ is a principal G-bundle with a (continuous) cross-section $s\colon X/G \to X$ (that is $\pi \circ s = \mathrm{id}_{X/G}$). Show that $\pi\colon X \to X/G$ is a trivial bundle.

Question 3. Let G and H be two closed subgroups of a topological group Γ. Suppose that $\Gamma/(H \cap G) \to \Gamma/G$ has local cross-sections. Show that $\Gamma/(H \cap G) \to \Gamma/G$ is a fibre bundle with fibre $G/(H \cap G)$.

Question 4.

1) Show that $V_{n,m}(\mathbb{F}) = O(m,\mathbb{F})/O(m-n,\mathbb{F}) \to G_{n,m}(\mathbb{F}) = O(m,\mathbb{F})/(O(n,\mathbb{F}) \times O(m-n,\mathbb{F}))$ is a principal $O(n,\mathbb{F})$-bundle. [Hint: Read the topic on examples of manifolds for constructing local cross-sections.]
2) Show that there is a principal S^1-bundle $\eta\colon S^3 \longrightarrow S^2$. This fibre bundle is called Hopf fibration. [Hint: From part (1), there is a principal $O(1,\mathbb{C}) = U(1) = S^1$ bundle $V_{1,m}(\mathbb{C}) = S^{2m-1} \to G_{1,m}(\mathbb{C}) = \mathbb{C}\mathrm{P}^{m-1}$. Then show that $\mathbb{C}\mathrm{P}^1 \cong S^2$.]
3) Let
$$E(\gamma_n^m) = \{(V,x) \in G_{n,m}(\mathbb{F}) \times \mathbb{F}^m \mid x \in V\}.$$
Show that
$$p\colon E(\gamma_n^m) \to G_{n,m}(\mathbb{F}) \qquad (V,x) \mapsto V$$
is an n-dimensional $\mathbb{F}$-vector bundle, denoted by γ_n^m. [Hint: Check that $E(\gamma_n^m) = V_{n,m}(\mathbb{F}) \times_{O(n,\mathbb{F})} \mathbb{F}^n$.]

Note to Question 4. Similar to part (2), there is another Hopf fibration $S^7 \to S^4$ which is a principal S^3-bundle by taking $\mathbb{F} = \mathbb{H}$, $n = 1$ and $m = 2$ from part (1). By taking $n = 1$ from part (1), one obtains other fibre bundles (principal G-bundles):

$$S^{m-1} \longrightarrow \mathbb{R}\mathrm{P}^{m-1} \quad G = O(1) = \mathbb{Z}/2;$$
$$S^{2m-1} \longrightarrow \mathbb{C}\mathrm{P}^{m-1} \quad G = U(1) = S^1;$$

$$S^{4m-1} \longrightarrow \mathbb{H}\mathrm{P}^{m-1} \quad G = \mathrm{Sp}(1) = S^3.$$

Question 5.

1) Show that ξ given by $S^1 \to S^1, \quad z \mapsto z^2$ is a principal $\mathbb{Z}/2$-bundle.
2) Let ξ be given in part (1) and let $\mathbb{Z}/2 = \{\tau, 1\}$ act on $I = [-1, 1]$ by $\tau(t) = -t$. Show that the total space of the induced fibre bundle $E(\xi[I])$ is the M obius band.
3) Let ξ be given in part (1) and let $\mathbb{Z}/2 = \{\tau, 1\}$ act on S^1 by $\tau(z) = z^{-1}$. Show that the total space of the induced fibre bundle $E(\xi[S^1])$ is the Klein bottle.

Question 6.

1) Let ξ be a fibre bundle over $B \times [0, 1]$ with fibre F. Suppose that there exists t such that $\xi|_{B\times[0,t]}$ and $\xi|_{B\times[t,1]}$ are trivial. Show that ξ is a trivial bundle. [Hint: Let $\phi_1 \colon E(\xi|_{B\times[0,t]}) \to B\times[0, t]\times F$ and $\phi_2 \colon E(\xi|_{B\times[t,1]}) \to B\times[t, 1]\times F$ be trivializations. Then
$$\phi_2 \circ \phi_1^{-1} \colon B \times \{t\} \times F \longrightarrow B \times \{t\} \times F \qquad (b, t, y) \mapsto (b, t, \theta_b(y)$$
is a homeomorphism, where $\theta_b(y) \colon F \to F$ is a homeomorphism depending continuously on b. Define a function
$$\chi \colon B \times [t, 1] \times F \longrightarrow B \times [t, 1] \times F \qquad (b, s, y) \mapsto (b, s, \theta_b^{-1}(y)).$$
Check that χ is a homeomorphism. Prove that the trivializations
$$\chi \circ \phi_2 \colon E(\xi|_{B\times[t,1]}) \longrightarrow B \times [t, 1] \times F \qquad \phi_1 \colon E(\xi|_{B\times[0,t]}) \longrightarrow B \times [0, t] \times F$$
induce a trivialization $E(\xi) \to B\times[0, 1]\times F$ by checking that $\chi\circ\phi_2(z) = \phi_1(z)$ for $z \in E(\xi|_{B\times\{t\}})$.]
2) Let ξ be a fibre bundle over $B \times [0, 1]$ with fibre F. Suppose that there is finite cover $\{V_i\}$ of $[0, 1]$ such that $\xi|_{B\times V_i}$ is trivial for each i. Show that ξ is a trivial bundle. [Hint: Prove it by induction.]
3) Let ξ be a fibre bundle over $B \times [0, 1]$ with fibre F. Show that there is an open cover $\{U_\alpha\}$ of B such that $\xi|_{U_\alpha\times[0,1]}$ is trivial. In particular, any fibre bundle over $[0, 1]$ is trivial.