Homework 4

**Question 1.** Let $M$ and $N$ be differentiable manifolds. Show that $T(M \times N)$ is diffeomorphic to $T(M) \times T(N)$.

**Question 2.** Show that the tangent bundle to $S^1$ is diffeomorphic to the cylinder $S^1 \times \mathbb{R}^1$.

**Question 3.** Prove that the projection map $\pi\colon T(M) \to M,\ \vec{v}_P \mapsto P,$ is a submersion.

**Question 4.** Let $\tau(S^2) = \{(P, \vec{v}) \in T(S^2) \mid |\vec{v}| = 1\}$ be the circle bundle of $S^2$. (**Note.** $\tau(S^2) = V_{3,2}$ is the Stiefel manifold.) Prove that $\tau(S^2)$ is a submanifold of $T(S^2)$ of dimension 3.

Let $\xi$ be a vector bundle given by $\pi\colon E \to B$ and let $B_0$ be a subspace of $B$. Then $\pi\colon \pi^{-1}(B_0) \to B_0$ is a vector bundle over $B_0$, called the *restriction* of $\xi$ to $B_0$, denoted by $\xi|B_0$.

**Question 5.** Prove that $T(S^{n+q})|S^n$ is isomorphic to $T(S^n) \oplus \theta^q$ where $\theta^q$ is the trivial bundle over $S^n$ with fibre $\mathbb{R}^q$ and $S^n \subseteq S^{n+q}$ is the standard inclusion.

**Question 6.** Using the fact that $S^{4n-1}$ is the set of unit vectors in $\mathbb{H}^n$, prove that $S^{4n-1}$ has three unit vector fields on it which are orthogonal at each point. [Hint: Do the case $S^3$ first.]

**Question 7.** Let $X$ be a vector field on $M$ and let $f\colon M \to \mathbb{R}$ be a smooth function. Prove that $Xf$ is well-defined smooth function on $M$. (Exercise 5.1 in the lecture notes.)

**Question 8.** Prove the following identities for the bracket of vector fields:

(1). $[X + Y, Z] = [X, Z] + [Y, Z]$;
(2). $[X, Y + Z] = [X, Y] + [X, Z]$;
(3). $[X, Y] = -[Y, X]$;
(4). $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$, for $f, g \in C^\infty(M)$;
(5). $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

(Exercise 5.2 in the lecture notes.)