0.1 The answers for Homework 4

Q1 proof: To prove $T(M \times N)$ is diffeomorphic to $T(M) \times T(N)$ we should try to find one smooth map

$$f : T(M) \times T(N) \to T(M \times N)$$

s.t. f is one to one and onto, and f^{-1} should be smooth. We will give the map f locally and prove it is well defined on whole $T(M) \times T(N)$.
Suppose (U_α, ϕ_α) and $(V_\alpha, \varphi_\alpha)$ are charts of M and N Then $(T_{\phi_\alpha}, T(U_\alpha))$ is one chart of $T(M)$ and $(T_{\varphi_\alpha}, T(V_\alpha))$ is one chart of $T(N)$. so we can get the map

$$f((U_\alpha, \phi_\alpha, v_M) \times (V_\alpha, \varphi_\alpha, v_N)) = (U_\alpha \times V_\alpha, \phi_\alpha \times \varphi_\alpha, v_M \times v_N)$$

Now suppose $(\theta_\alpha, W_\alpha)$ is one chart of $M \times N$, then there is $U_\beta \subset M$ and $V_\beta \subset N$ s.t. $U_\beta \times V_\beta \subset M \times N$ and $(\phi_\beta(P), \varphi_\beta(Q)) = \theta(P, Q)$. Define $v_M^{'} = p_m(w)$ and $v_N^{'} = p_{m+n,n}(w)$(here $p_m(x_1, ..., x_{m+n}) = (x_1, ..., x_m)$ and $p_{m+n,n}(x_1, ..., x_{m+n}) = (x_{m+1}, ..., x_{m+n})$
If $(W_\alpha, \theta_\alpha, w) \in T(M \times N)$, define the map

$$g(W_\alpha, \theta_\alpha, w) = (U_\beta \times V_\beta, \phi_\beta \times \varphi_\beta, v_M^{'} \times v_N^{'})$$

After changing the chart of β to α as follows, we can get g is the inverse of f.
Next we will prove that f and g are well defined globally. It means that f and g are independent of the choice of the chart. Suppose there are charts (U_β, ϕ_β) and (V_β, φ_β) of M and N. If $(U_\alpha, \phi_\alpha, v_M) \simeq (U_\beta, \phi_\beta, v_M^{'})$ and $(V_\alpha, \varphi_\alpha, v_N) \simeq (V_\beta, \varphi_\beta, v_N^{'})$ it is easy to see that

$$((U_\alpha \cap U_\beta) \times (V_\alpha \cap \varphi_\beta), \phi_\alpha \times \varphi_\alpha, v_M \times v_N) \simeq ((U_\alpha \cap U_\beta) \times (V_\alpha \cap \varphi_\beta), \phi_\beta \times \varphi_\beta, v_M^{'} \times v_N^{'})$$

so f is well defined. It is same to g. □

Q2 proof: Suppose $P(t) = e^{it}$, let $v_0(t) = P^{'}(t) = ie^{it}$ then we can define the map $f : T(S^1) \to S^1 \times R^1$ as

$$f(P(t), v_P) = (P(t), < v_P, v_0(t) >)$$

$<>$ is the inter product of two vectors.
f is differential, one to one and onto. The inverse of f is g as

$$g(P(t), \lambda) = (P(t), \lambda v_0(t))$$

g is also smooth. $\square$

Q3 proof: We know

$$T(U_\alpha) \cong T(\phi_\alpha(U)) \cong \phi_\alpha(U) \times R^m$$

So the map $\pi : T(M) \to M$ is smooth(here we should check that π is well defined on $T(M)$). Use the T to act on π

$$T\pi : T(T(M)) \to T(M)$$

By **Q1**, there is

$$T(T(U_a)) \cong T(T(\phi_\alpha(U))) \cong T(\phi_\alpha(U) \times R^m) \cong T(U_\alpha) \times R^m$$

$$T\pi((P, U_\alpha, v), y) = (P, U_\alpha, v)$$

On U_α, $T\pi$ is surjective and it is well defined globally, so π is submersion. $\square$

Q4 proof: Suppose (U, ϕ) is one local chart of S^2, then we have

$$T(U) \cong T(\phi(U)) \cong \phi(U) \times R^2$$

is the local chart of $T(S^2)$, then we can get the local chart for $\tau(S^2)$: $\phi(U) \times S^1$. Now we should prove that for each point $P \in \tau(S^2)$, there exists a chart (V, φ) about P in $T(S^2)$ s.t.

$$\varphi(V \cap \tau(S^2)) = \varphi(V) \cap R^3_+$$

If select $V = \phi(U) \times R^2$, then we need $\varphi(\phi(U) \times S^1) = \varphi(\phi(U) \times R^2) \cap R^3_+$ suppose φ is diffeomorphism from $\phi(U) \times R^2$ to $\phi(U) \times R^2$, then we have $\varphi(\phi(U) \times S^1) = \phi(U) \times R$, now our task becomes find the diffeomorphism from R^2 to R^2 s.t. S^1 mapping to $R \times 0$. Construct the smooth map as the following step

1) maps R^2 to S^2
2) $f : S^2 \to S^2$ as $f(P) = f(x_P, y_P, z_P) = (z_P, y_P, -x_P)$
3) maps S^2 to R^2

It is is easy to check it is a smooth map from R^2 to R^2 s.t. S^1 maps to

$R \times 0$ □

Q5 proof: Suppose $(P, v_P) \in T(S^{n+q})|S^n$,then $P = (x_1, ..., x_n, 0, ..., 0)$, so $v_P = (y_1, ..., y_n, z_1, ..., z_q)$ fits for $(x_1, ..., x_n) \perp (y_1, ..., y_n)$ and $(z_1, ..., z_q)$ is any vector in R^q. If P and v_P are represented as above, we construct the bundle morphism $(f, u) : T(S^{n+q}) \to T(S^n) \oplus \theta^q$ as

$$(f, u)(v_P, P) = ((y_1, ..., y_n) \oplus (z_1, ..., z_q), (x_1, ..., x_n))$$

Easy to prove it is bundle isomorphism. □

Q6 proof: Consider the case of S^3. Suppose $h = a + bi + cj + dk \in H^1$, we construct the following maps:

$$f_1(h) = hi$$

$$f_2(h) = hj$$

$$f_3(h) = hk$$

maps f_1, f_2, f_3 back to S^3, we can get the 3 unit orthogonal vector fields. Then use the **proposition 5.9**, we can get the the 3 unit orthogonal vector fields in S^{4n-1} □

Q7 proof: The main idea is in the exercise 5.1. We need to check that Xf is well defined globally. Suppose it is well defined locally on chart (U, ϕ), then if (V, φ) is another chart, we should prove that $Xf(P)$ is agreed under the two charts.

$$(Xf)_U(P) = \Sigma_{i=1}^m \xi_U^i(P) \partial(f \circ \phi^{-1})/\partial x^i(0)$$

$$(Xf)_V(P) = \Sigma_{i=1}^m \xi_V^i(P) \partial(f \circ \varphi^{-1})/\partial y^i(0)$$

We know that $\phi \circ \varphi^{-1}$ is the map from R^m to R^m.

$$D(\varphi \circ \phi^{-1})(\partial(f \circ \varphi^{-1})/\partial y^i(0)) = \partial(f \circ \varphi^{-1}(\varphi \circ \phi^{-1})/\partial x^i(0)) = \partial(f \circ \phi^{-1})/\partial x^i(0)$$

□

Q8 proof: $(1),(2),(3)$ are just from definition.

(4) :

$$\begin{aligned}[fX,gY](h) &= fX(gY(h)) - gY(fX(h))\\ &= fgX(Y(h)) + fX(g)Y(h) - gfY(X(h)) - gY(f)X(h)\\ &= fg[X,Y](h) + f(Xg)Y(h) - gY(f)X(h)\end{aligned}$$

(5):
$[X,[Y,Z]](h) = X(Y(Z(h))) - X(Z(Y(h))) - Y(Z(X(h))) + Z(Y(X(h)))$
$[Y,[Z,X]](h) = Y(Z(X(h))) - Y(X(Z(h))) - Z(X(Y(h))) + X(Z(Y(h)))$
$[Z,[X,Y]](h) = Z(X(Y(h))) - Z(Y(X(h))) - X(Y(Z(h))) + Y(X(Z(h)))$

Plus together and use the property of (3) we get

$$[X,[Y,Z]](h) + [Y,[Z,X]](h) + [Z,[X,Y]](h) = 0$$

□