2003/2004 Semester II MA5210 Differentiable Manifolds

Homework 4

Question 1. Prove that a k-dimensional vector bundle ξ is trivial if and only if it has k cross-sections $s_1, \ldots, s_k$ such that each $s_1(b), \ldots, s_k(b)$ are linearly independent for each $b \in B$.

Question 2.
Let ξ and η be vector bundles over B and let f be a cross-section of the bundle $\mathrm{Hom}(\xi, \eta)$. If the rank of the linear transformation

$$f(b)\colon F_b(\xi) \longrightarrow F_b(\eta)$$

is locally constant as a function of b, define the kernel $\mathrm{Ker}_f \subseteq \xi$ and the cokernel Coker_f, and prove that they are locally trivial.

Question 3.
Let B be a compact Hausdorff space and let $C^0(B)$ be the ring of continuous real valued functions on B. For any vector bundle ξ over B let $\Gamma(\xi)$ denote the $C^0(B)$-module consisting of all cross-sections of ξ.

a) Show that $\Gamma(\xi \oplus \eta) \cong \Gamma(\xi) \oplus \Gamma(\eta)$.
b) Show that $\xi \cong \eta$ if and only if $\Gamma(\xi) \cong \Gamma(\eta)$ as $C^0(B)$-modules.
c) Show that ξ is trivial if and only if $\Gamma(\xi)$ is a free $C^0(B)$-bundle.

[Hint: These are some statements for special cases from the paper: R. Swan, *vector bundles and projective modules*, Trans. Ameri. Math. Soc., 105 (1962), 264-277.]

Question 4.
Let $\phi\colon V \to W$ be linear isomorphism of finite dimensional vector spaces. Show that the matrix of $(\phi^{-1})*\colon W^* \to V^*$ is the transpose of the inverse of the matrix of ϕ. [**Note.** You may use the formula $\langle \phi^* x, y \rangle = \langle x, \phi y \rangle$.]

Recall that a Riemann metric is a tensor field $g \in T_2^0(M)$ such that for each m, g_m is an inner product, that is, positive definite symmetric and bilinear.

Question 5.
Determine Riemann metrics $g \in T_2^0(\mathbb{R}^2) = (\mathbb{R}^2)^* \otimes (\mathbb{R}^2)^*$.

Question 6. Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \qquad \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(n,m)\mathbb{Z},$$

where (n, m) is the greatest common factor of m and n.