2003/2004 Semester II MA5210 Differentiable Manifolds

Homework 6

Question 1.

Let α be a linear map of the m-dimensional vector space V into itself, and let $\omega \in \Lambda^m(V^*)$. Calculate $\alpha^*(\omega)$.

Question 2.

Show that, for 1-forms $\{\phi_1, \dots, \phi_k\}$, $\phi_1 \wedge \cdots \wedge \phi_k = 0$ if and only if $\{\phi_1, \dots, \phi_k\}$ are linearly dependent. If they are linearly independent, prove that $\phi_1 \wedge \cdots \wedge \phi_k = \psi_1 \wedge \cdots \psi_k$ if and only if $\phi_i = \sum\limits_j a_{ij}\psi_j$ with $\det(a_{ij}) = 1$.

[Hint for the second part: if $\phi_1 \wedge \cdots \wedge \phi_k = \psi_1 \wedge \cdots \psi_k \neq 0$, then, from the first part, $\psi_1, \dots, \psi_k$ are linearly independent. Moreover from $\phi_i \wedge \psi_1 \wedge \cdots \wedge \psi_k = \phi_i \wedge \phi_1 \wedge \cdots \wedge \phi_k = 0$, $\{\phi_i, \psi_1, \dots, \psi_k\}$ are linearly dependent.]

Question 3.

The Hodge star isomorphism $*$ from $\Omega^k(\mathbb{R}^m)$ to $\Omega^{m-k}(\mathbb{R}^m)$ is defined by mapping the basic k-form $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ to $\epsilon_\sigma dx^{j_1} \wedge \cdots \wedge dx^{j_{m-k}}$ where $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_{m-k}$, $(i_1, i_2, \dots, i_k, j_1, \dots, j_{m-k})$ is the permutation σ of $(1, 2, \dots, m)$ and ϵ_σ is the sign of σ. Let

$$\omega = a_{12}dx^1 \wedge dx^2 + a_{13}dx^1 \wedge dx^3 + a_{23}dx^2 \wedge dx^3 \in \Omega^2(\mathbb{R}^3).$$

Calculate $*\omega$. What is $*\omega$ if $\omega \in \Omega^2(\mathbb{R}^4)$?

[Answer: for the first part: $*\omega = a_{23}dx^1 - a_{13}dx^2 + a_{12}dx^3$, and for the second part, $*\omega = (a_{23}dx^1 - a_{13}dx^2 + a_{12}dx^3) \wedge dx^4$.

Question 4.

We may use the standard inner product on $\mathbb{R}^n$ to define an isomorphism between $\mathbb{R}^n$ and its dual and hence a 1-1 correspondence between vector fields and 1-forms, where the vector field X on $U \subseteq \mathbb{R}^m$ corresponds to the 1-form $\omega = \vartheta(X)$ defined by

$$\omega_p(Y) = \langle X(p), Y \rangle, \qquad \text{for each } Y \in T_p(\mathbb{R}^m).$$

i) Show that, if $f\colon U \to \mathbb{R}$, then the vector field $\vartheta^{-1}(df)$ is

$$\text{grad} f = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i} e_i,$$

where e_i is the standard basis of $\mathbb{R}^m$.

ii) If $X(x)$ and $Y(x)$ are vector fields $\sum\limits_{i=1}^{3} a^i(x)e_i$ and $\sum\limits_{i=1}^{3} b^i(x)e_i$ on $U \subseteq \mathbb{R}^3$, calculate $\vartheta^{-1} * d\vartheta(X)$ and $\vartheta^{-1} * (\vartheta(X) \wedge \vartheta(Y))$, where $*$ is defined in Problem 3.

Question 5.

Let

$$\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$$

be 1-form on $\mathbb{R}^3$ such that $d\omega = 0$. Show that $\omega = df$ where

$$f(x,y,z) = \int_0^1 \{xa(tx,ty,tz) + yb(tx,ty,tz) + zc(tx,ty,tz)\}dt.$$

[Hint:

$$a(x,y,z) = \int_0^1 \frac{d}{dt}\{ta(tx,ty,tz)\}dt$$

$$= \int_0^1 t\{x\partial_1 a(tx,ty,tz) + y\partial_2 a(tx,ty,tz) + z\partial_3 a(tx,ty,tz)\}dt + \int_0^1 a(tx,ty,tz)dt.$$

Figure out the partial derivatives of f and use $d\omega = 0$ to find the relations between the partial derivatives of a, b and c.]

Question 6.

Let M be a compact 3-dimensional smooth submanifold-with-boundary of $\mathbb{R}^3$, $f\colon M \to \mathbb{R}^3$ be the inclusion and

$$\omega = \frac{1}{3}\{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy\}.$$

i) Show that $d(\omega/r^3) = 0$ on $\mathbb{R}^3 \smallsetminus \{0\}$, where $r^2 = x^2 + y^2 + z^2$.

ii) Show that

$$\int_{\partial M} \iota^* f^*(\omega) = \mathrm{vol}(M),$$

and deduce that there is a 2-form η on S^2 such that $d\eta = 0$ but $\eta \neq d\phi$ for any 1-form ϕ.

[Hint for part (ii):

$$\int_{\partial M} \iota^* f^*(\omega) = \int_M df^*(\omega) = \int_M f^* d\omega$$

by Stokes' Theorem. Check that $d\omega = dx \wedge dy \wedge dz$. For the rest part, let $M = D^3$ and choose $\eta = \iota^* f^* \omega$. Check that $d\eta = 0$. Show that $\eta \neq d\phi$ for any 1-form ϕ by finding a contradiction that, if so, then $\int_{S^2} \eta = 0$ by Stokes' Theorem but $\int_{S^2} \eta = \mathrm{vol}(S^2)$.]