MA 5210

Suggested Solutions to Homework 5

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Q1. Let $p : E \to B$ be the k-dimensional vector bundle ξ in question. Suppose $E = B \times \mathbb{F}^k$, then we may define $s_i : B \to E$ such that for each b, $s_i(b) = (b, e_i)$, where e_i is the vector in $\mathbb{F}^k$ with all coordinates 0 except 1 at the i-th position.

Conversely given k cross-sections $s_1, ..., s_k$ such that $s_1(b), ..., s_k(b)$ are linearly independent for each $b \in B$, we define a map $u : E \to B \times \mathbb{F}^k$ as follows: For each $z \in E$, we set $b = p(z)$. Then $z \in p^{-1}(b) = \{b\} \times \mathbb{F}^k = \mathbb{F}^k$ and so we may write $z = z_1 s_1(b) + ... + z_k s_k(b)$. for some $z_i \in \mathbb{F}$. We set $u(z) = (b, (z_1, ..., z_n))$. Clearly such a map is a homeomorphism fitting into the following commutative diagram, since it is a continuous isomorphism at each fibre.

$$\begin{array}{ccc} E & \xrightarrow{u} & B \times \mathbb{F}^k \\ \downarrow p & & \downarrow \pi \\ B & = & B \end{array}$$

Q2. So we have a morphism $u : \xi \to \eta$ of vector bundles with $u|_{F_b(\xi)} = f(b)$. The kernel is given $\bigcup_{b \in B} \ker(f(b) : F_b(\xi) \to F_b(\eta))$ and cokernel by $\bigcup_{b \in B} \mathrm{coker}(f(b) : F_b(\xi) \to F_b(\eta))$.

For each $b \in B$, let U be a neighborhood of b such that rank of $f(a)$ is constant for $a \in U$, say k. Let n and m be the dimension of ξ and η as vector bundles respectively. So we have $f(a) : F_a(\xi) = \mathbb{F}^n = V_1 \oplus V_2 \to F_a(\eta) = \mathbb{F}^m = W_1 \oplus W_2$, where $V_1 = \ker f(a)$ and $W_1 = \mathrm{im}\ f(a)$. Since rank of $f(a)$ is constant in U, we may set $\mathbb{F}^{n-k} = V_1$ for all $a \in U$. Now define a map $\varphi : U \times \mathbb{F}^{n-k} \to E(\ker u|U)$ by $\varphi(a, x) = f(a)(x)$. Clearly φ is a continuous map fitting into the following commutative diagram.

$$\begin{array}{ccc} U \times \mathbb{F}^{n-k} & \xrightarrow{\varphi} & E(\ker u|U) \\ \downarrow \pi & & \downarrow p \\ U & = & U \end{array}$$

Now for each $a \in U$, we define

$$V = \mathbb{F}^n \oplus W_2 = V_1 \oplus V_2 \oplus W_2 \xrightarrow{w_a} W_1 \oplus W_2 \oplus V_1 = \mathbb{F}^m \oplus V_1 = W$$

where $w_a|_{V_1} = (f(a)|_{V_1}) \oplus 1_{V_1}$, $w_a|_{V_2} = (f(a))|_{V_2}) \oplus 0$ and $w_a|_{W_2} = 0 \oplus 1_{W_2} \oplus 0$. It is straightforward to verify that w_a is a linear isomorphism. Note that $a \mapsto w_a$ is a continuous on U. Since the set of isomorphisms between V and W is an open subset of $\mathrm{Hom}_{\mathbb{F}}(V, W)$, there exists a neighborhood U_a of a contained in U such that w_c is a linear isomorphism for each $c \in U_a$. Now set $v_c : W \to V$ to be the inverse of w_c for each $c \in U_a$. Then $c \mapsto v_c$ is continuous on U_a.

Now for each $x \in E(\ker u|U)$, $x \in F_a(\xi)$ for some a. So we may assign $x \mapsto (a, v_a(x))$. This assignment gives a continuous inverse to φ.

For coker f, we define the map $\psi : U \times W_2 \to E(\mathrm{coker}\ u|U)$ as follows: For $a \in U$, since $\mathrm{im}(f(a)) \cap W_2 = 0$, we may assign $\psi(a, y) = y(\mathrm{mod}(\mathrm{im}\ f(a)))$. Its inverse is constructed from the projection $U \times (W_1 \oplus W_2) \to U \times W_2$ and factoring through *coker* u.

Remarks. Note that the kernel and cokernel need not be fibre bundles since we are only able to prove that they are locally trivial with respect to some fibre, since the rank is locally constant.

Endow $\mathbb{N}$ with the discrete topology, we see that the map from B to $\mathbb{N}$ given by $b \mapsto \mathrm{rank}\ f(b)$ is continuous since f is locally trivial. From here, we can deduce that in fact this map is constant at each connected component. So in fact, it follows that if B is connected, then we can indeed show that the kernel and cokernel are fibre bundle (actually vector bundles since their fibre are vector spaces!).

Q3. (a) Let s be a cross-section of $\xi \oplus \eta$. Then $s : B \to E(\xi \oplus \eta) \subseteq E(\xi) \times E(\eta)$. Define s_1 to be the composition $B \xrightarrow{s} E(\xi) \times E(\eta) \to E(\xi)$. This can be seen easily to be a cross-section of ξ. Similarly we have s_2, the composite $B \xrightarrow{s} E(\xi) \times E(\eta) \to E(\eta)$, to be a cross-section of η. It follows that the assignment $s \mapsto (s_1, s_2)$ gives an injective $C^0(B)$-map from $\Gamma(\xi \oplus \eta)$ to $\Gamma(\xi) \oplus \Gamma(\eta)$. To see that this map is surjective, let s_1 and s_2 be cross-sections of ξ and η respectively. We define $s : B \to E(\xi) \times E(\eta)$ by $s(b) = (s_1(b), s_2(b))$. Since $p(\xi)s_1(b) = b = p(\eta)s_2(b)$, we have $s(B) \subseteq E(\xi \oplus \eta)$ and s is a cross-section of $\xi \oplus \eta$ with the assignment (s_1, s_2).

(b) Suppose $u : \xi \cong \eta$, then define $\phi : \Gamma(\xi) \to \Gamma(\eta)$ by $\phi(s) = u \circ s$. This is a $C^0(B)$-map with inverse given by $s' \mapsto u^{-1} \circ s'$ and so $\Gamma(\xi) \cong \Gamma(\eta)$.

Conversely suppose $\varphi : \Gamma(\xi) \cong \Gamma(\eta)$ is an isomorphism of $C^0(B)$-modules. Let $x \in E(\xi)$ with $p(\xi)(x) = b$. We can always define a local cross-section s' on an open set U containing b such that $s'(b) = x$. Since B is compact Hausdorff, we have neighborhoods of x, V and W such that $\bar{V} \subseteq U$ and $\bar{W} \subseteq V$. Let $f \in C^0(B)$ with $f|_{\bar{W}} = 1$ and $f|_{B-V} = 0$. Define $s(a) = f(a)s'(a)$ if $a \in U$ and $s(a) = 0$ if $a \notin U$. Plainly s is a cross-section of ξ with $s(b) = x$. We define the map $u : E(\xi) \to E(\eta)$ by $u(x) = \varphi(s)(b)$. We now want to show that this map is well-defined and continuous. Note that by the definition of vector bundles, we can always find local cross-sections $s_1, ..., s_n$ such that $s_1(b), ..., s_n(b)$ are linearly independent. By similar argument as above, we may extend each of this local cross-sections to cross-sections, which we still denote by $s_1, ..., s_n$ such that $s_1(b), ..., s_n(b)$ are linearly independent.

Now we shall show that u is well-defined. Suppose t is another cross-section such that $t(b) = x$. Set $r = s - t$ and clearly r is a cross-section with $r(b) = 0$. We may write $r(a) = \sum g_i(b)s_b$ for a near b, $g_i(b) \in \mathbb{R}$. Let $f_i \in C^0(B)$ such that $f_i = g_i$ in a neighborhood of b. (The construction of such f_i is similar to that of the extension of the s_i's) Therefore, $r' = r - \sum f_i s_i$ vanishes in a neighborhood U of b. Let U_0 be a neighborhood of b such that $\bar{U_0} \subseteq U$. Now let $f \in C^0(B)$ such that $f(b) = 0$ and 1 on $B - U_0$. Then $r = fr' + \sum f_i s_i = \sum f_i s_i$ on B. Since $r(b) = 0 = f(b)$, we have $\sum f_i(b)s_i(b) =$ implying $f_i(b) = 0$ by the independence of $s_i(b)$. Now $\varphi(r) = f\varphi(r') + \sum f_i\varphi(s_i)$ since φ is $C^0(B)$-linear. Hence $\varphi(r)(b) = 0$. i.e $\varphi(s)(b) = \varphi(t)(b)$.

To establish continuity, let $y \in E(\xi)$ be such that $p(\xi)(y) = a$ is in some neighborhood of b, then we have $y = \sum h_i(a)s_i(a)$ where $h_i \in C^0(B)$. So $u(y) = \sum h_i(a)\varphi(s_i)(a)$. Since $\varphi(s_i)$ is a cross-section of η and all the terms in the sum are continuous in y, u is continuous.

Finally it is clear that the following diagram commutes, thus yielding a vector

$$\begin{array}{ccc} E(\xi) & \xrightarrow{u} & E(\eta) \\ \Big\downarrow{\scriptstyle p(\xi)} & & \Big\downarrow{\scriptstyle p(\eta)} \\ B & = & B \end{array}$$

bundle morphism.

Repeat the above construction for $\varphi^{-1} : \Gamma(\eta) \cong \Gamma(\xi)$ and we will obtain a vector bundle morphism which is the inverse of the above, thus proving what we want.

(c) Let ζ^n denoted the n-dimensional trivial vector bundle. It can be checked that $\zeta^n = \oplus_{i=1}^n \zeta^1$ by induction and $\Gamma(\zeta^1) \cong C^0(B)$. So if ξ is trivial, then by (a) and (b), $\Gamma(\xi) \cong \Gamma(\zeta^1)^n \cong C^0(B)^n$ and so is free. Conversely if $\Gamma(\xi)$ is free, then $\Gamma(\xi) \cong C^0(B)^n \cong \Gamma(\zeta^n)$. By part (b), this implies $\xi \cong \zeta^n$ and so ξ is trivial.

Q4. We still denote the matrix representation of ϕ with respect to a basis of V by ϕ. Then for every $x, y \in W$, we have $< x, \phi y >=< \phi^*(x), y >= (\phi^* x)^t y = x^t(\phi^*)^t y =< x, (\phi^*)^t(y) >$. Since inner product is nondegenerate, $\phi = (\phi^*)^t$ or $\phi^t = \phi^*$. So $\mathrm{id} = \mathrm{id}^* = (\phi\phi^{-1})^* = (\phi^{-1})^*\phi^*$. i.e $(\phi^{-1})^* = (\phi^*)^{-1} = (\phi^t)^{-1} = (\phi^{-1})^t$.

Q5. Let x_1 and x_2 be an orthonormal basis for $\mathbb{R}^2$ (eg. (0,1) and (1,0)). Denote the corresponding dual basis of $(\mathbb{R}^2)^*$ by dx_1 and dx_2. By definition of Riemann metric, we may write $g(x) = \sum_{1\le i,j\le 2} g_{ij}(x)dx_i \otimes dx_j$ for $x \in \mathbb{R}^2$, since each $g(x)$ is bilinear. Also by the symmetric property, we have $g_{12}(x) = g(x)(dx_1 \otimes dx_2) = g(x)(dx_2 \otimes dx_1) = g_{21}(x)$. Also $g_{11}(x) = g(x)(dx_1 \otimes dx_1) \ge 0$ by the positive definite property. Similarly we have $g_{22}(x) \ge 0$.

Remarks. If we replace $\mathbb{R}^2$ by $\mathbb{R}^n$, then by a similar argument as above choosing an orthonormal basis for $\mathbb{R}^n$, the Riemann metric is given by $g(x) = \sum_{i,j} g_{ij}(x)dx_i \otimes dx_j$ with $g_{ij}(x) = g_{ji}(x)$ and $g_{ii}(x) \ge 0$.

Q6. Let $i : \mathbb{Q}\times\mathbb{Q} \to \mathbb{Q}$ be given by $(a, b) \mapsto ab$. Clearly this is middle linear. Now let A be an abelian group and $f : \mathbb{Q} \times \mathbb{Q} \to A$ a middle linear map. Define $\tilde{f} : \mathbb{Q} \to A$ by $\tilde{f}(a) = f(a, 1)$. Clearly $\tilde{f}$ is an abelian group homomorphism (or $\mathbb{Z}$-module homomorphism). Now suppose g is another $\mathbb{Z}$-map from $\mathbb{Q}$ to A such that $g \circ i = f$, then for each $a \in \mathbb{Q}$, $g(a) = g \circ i(a, 1) = f(a, 1) = \tilde{f}(a)$. Hence $\tilde{f}$ is the unique map such the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Q}\times\mathbb{Q} & \xrightarrow{\ i\ } & \mathbb{Q} \\ \Big\| & & \Big\downarrow \exists!\tilde{f} \\ \mathbb{Q}\times\mathbb{Q} & \xrightarrow{\ f\ } & A \end{array}$$

So $\mathbb{Q}$ satisfies the universal property for tensor products and thus $\mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $i : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/(n, m)\mathbb{Z}$ be given by $(a, b) \mapsto ab$. Clearly this is middle linear. Now let A be an abelian group and $f : \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to A$ a middle linear map. Define $\tilde{f} : \mathbb{Z}/(n, m)\mathbb{Z} \to A$ by $\tilde{f}(a) = f(a, 1)$. Clearly $\tilde{f}$ is an abelian group homomorphism (or $\mathbb{Z}$-module homomorphism). Now suppose g is another $\mathbb{Z}$-map from $\mathbb{Z}/(n, m)\mathbb{Z}$ to A such that $g \circ i = f$, then for each $a \in \mathbb{Z}/(n, m)\mathbb{Z}$, $g(a) = g \circ i(a, 1) = f(a, 1) = \tilde{f}(a)$. Hence $\tilde{f}$ is the unique map such the following diagram commutes.

$$\begin{array}{ccc} \mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z} & \xrightarrow{\ i\ } & \mathbb{Z}/(n,m)\mathbb{Z} \\ \Big\| & & \Big\downarrow \exists!\tilde{f} \\ \mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z} & \xrightarrow{\ f\ } & A \end{array}$$

So $\mathbb{Z}/(n, m)\mathbb{Z}$ satisfies the universal property for tensor products and thus $\mathbb{Z}/(n, m)\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. In particular, if $(n, m) = 1$, $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.