Question 1:

Let $\{e_1, \ldots, e_m\}$ be the basis for V and $\{e_1^*, \ldots, e_m^*\}$ be the dual basis.
Let $\omega = f_1 \wedge \cdots \wedge f_m$, where $f_i = f_{i1}e_1^* + \cdots + f_{im}e_m^*$.
Thus, $\omega = \det(f_{ij})e_1^* \wedge \cdots \wedge e_m^*$.

Then, $\alpha^*(\omega)(u_1, \ldots, u_m) = \omega(T\alpha(u_1), \ldots, T\alpha(u_m))$ for $u_1, \ldots, u_m \in V$.
Since $T\alpha = \alpha$, we have:

$$\begin{aligned}
\alpha^*(\omega)(u_1, \ldots, u_m) &= \omega(\alpha(u_1), \ldots, \alpha(u_m)) \\
&= \det(f_{ij})(e_1^* \wedge \cdots \wedge e_m^*)(\alpha(u_1), \ldots, \alpha(u_m)) \\
&= \det(f_{ij})\det(\alpha)\det(U), \text{ where } U = (u_1 \cdots u_m) \\
\alpha^*(\omega)(u_1, \ldots, u_m) &= \det(\alpha)\omega(u_1, \ldots, u_m)
\end{aligned}$$

Question 2:

- $\phi_1 \wedge \cdots \wedge \phi_k = 0$ *iff* $\{\phi_1, \ldots, \phi_k\}$ *is linearly dependent.*

($\Rightarrow$) Suppose $\phi_1 \wedge \cdots \wedge \phi_k = 0$. Assume to the contrary $\{\phi_1, \ldots, \phi_k\}$ is linear independent. Then, the space Λ^1 spanned by $\{\phi_1, \ldots, \phi_k$ is of dimension k. By proposition (7.9.) the dimension of space Λ^k, spanned by $\{\phi_1 \wedge \cdots \wedge \phi_k\}$, is $\binom{k}{k} = 1$. A contradiction that $\phi_1 \wedge \cdots \wedge \phi_k = 0$. Therefore, $\{\phi_1, \ldots, \phi_k\}$ is linear dependent.

($\Leftarrow$) If $\{\phi_1, \ldots, \phi_k\}$ is linearly dependent then there is ϕ_i such that ϕ_i is a linear combination of $\{\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_k\}$. Therefore,
$\phi_1 \wedge \cdots \wedge \phi_k = \phi_1 \wedge \cdots \wedge (\sum_{j \neq i} a_j\phi_j) \wedge \cdots \wedge \phi_k = 0 + \cdots + 0 = 0$.

- $\phi_1 \wedge \cdots \wedge \phi_k = \psi_1 \wedge \cdots \wedge \psi_k$ *iff* $\phi_i = \sum_j a_{ij}\psi_j$ *and* $\det(a_{ij}) = 1$.

($\Rightarrow$) For each ψ_i, $\psi_i \wedge \psi_1 \wedge \cdots \psi_k = \psi_i \wedge \phi_1 \wedge \cdots \phi_k = 0$. Thus, $\{\psi_i, \phi_1, \ldots, \phi_k\}$ is linear dependent. Since $\{\phi_1, \ldots, \phi_k\}$ is linear indepedent ψ_i is a linear combination of $\{\phi_1, \ldots, \phi_k\}$. Denote

$$\psi_i = a_{i1}\phi_1 + \cdots + a_{ik}\phi_k.$$

Then,

$$\begin{aligned}
\phi_1 \wedge \cdots \wedge \phi_k &= \psi_1 \wedge \cdots \wedge \psi_k && (1)\\
&= \bigwedge_i (a_{i1}\phi_1 + \cdots + a_{ik}\phi_k) && (2)\\
&= \det(a_{ij})\ \phi_1 \wedge \cdots \wedge \phi_k && (3)
\end{aligned}$$

So, $\det(a_{ij}) = 1$.

($\Leftarrow$) As the equations 1, 2, 3 shown above.

Question 3:

1. Sign of permutation from (1 2 3) to (1 2 3) is positive.
2. Sign of permutation from (1 2 3) to (1 3 2) is negative.
3. Sign of permutation from (1 2 3) to (2 3 1) is positive.

So the Hodge star mapping in $\mathbb{R}^3$ is the following:

1. $dx^1 \wedge dx^2$ to dx^3.
2. $dx^1 \wedge dx^3$ to $-dx^2$.
3. $dx^2 \wedge dx^3$ to dx^1.

So $*(a_{12}dx^1 \wedge dx^2 + a_{13}dx^1 \wedge dx^3 + a_{23}dx^2 \wedge dx^3) = a_{12}dx^3 - a_{13}dx^2 + a_{23}dx^1$.

For premutation of four elements:

1. Sign of permutation from (1 2 3 4) to (1 2 3 4) is positive.
2. Sign of permutation from (1 2 3 4) to (1 3 2 4) is negative.
3. Sign of permutation from (1 2 3 4) to (2 3 1 4) is positive.

So the Hodge star mapping in $\mathbb{R}^4$ is the following:

1. $dx^1 \wedge dx^2$ to $dx^3 \wedge dx^4$.
2. $dx^1 \wedge dx^3$ to $-dx^2 \wedge dx^4$.
3. $dx^2 \wedge dx^3$ to $dx^1 \wedge dx^4$.

So $*(a_{12}dx^1 \wedge dx^2 + a_{13}dx^1 \wedge dx^3 + a_{23}dx^2 \wedge dx^3) = a_{12}dx^3 \wedge dx^4 - a_{13}dx^2 \wedge dx^4 + a_{23}dx^1 \wedge dx^4$.

Question 4:

Let $\{e_1, \ldots, e_m\}$ be the standard basis for $\mathbb{R}^m$ and $\{e_1^*, \ldots, e_m^*\}$ be the dual basis.

- *If* $f : U \mapsto \mathbb{R}$ *then* $\vartheta^{-1}(df) = \text{grad } f = \sum_{i=1}^m \frac{\partial f}{\partial x_i} e_i$.

Let $X(p) = \sum_{i=1}^{m} X_i(p)e_i$ be vector field such that $\vartheta(X) = df$ and $\vec{v} = \sum_{i=1}^{m} v_i e_i \in \mathbb{R}^m$ be vector variable.

$$\begin{aligned}
\langle X(p), \vec{v}\rangle &= df_p \cdot \vec{v} = \begin{pmatrix} \left.\frac{\partial f}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial f}{\partial x_m}\right|_p \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \\
\sum_{i=1}^{m} X_i(p)v_i &= \sum_{i=1}^{m} \left.\frac{\partial f}{\partial x_i}\right|_p v_i \\
X_i(p) &= \frac{\partial f}{\partial x_i}, \text{ for each } i = 1, \ldots, m.
\end{aligned}$$

So, $X = \text{grad } f$.

• *$X(x) = \sum_{i=1}^{3} a^i(x)e_i$ and $Y(x) = \sum_{i=1}^{3} b^i(x)e_i$ be vector fields on $U \subseteq \mathbb{R}^3$. Calculate $\vartheta^{-1} * d\vartheta(X)$ and $\vartheta^{-1} * (\vartheta(X) \wedge \vartheta(Y))$.*

The following is the step-by-step computation. The answer is on number (iv) and (vii)

(i) $\vartheta(X)(x) = \sum_{i=1}^{3} a_i(x)e_i^*$ and $\vartheta(Y)(x) = \sum_{i=1}^{3} b_i(x)e_i^*$.

(ii) $d\vartheta(X) = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial a_i}{\partial x_j}(e_j^* \wedge e_i^*) =$
$(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2})(e_1^* \wedge e_2^*) + (\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3})(e_1^* \wedge e_3^*) + (\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3})(e_2^* \wedge e_3^*)$.

(iii) $*d\vartheta(X) = (\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2})(e_3^*) - (\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3})(e_2^*) + (\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3})(e_1^*)$.

(iv) $\vartheta^{-1} * d\vartheta(X) = (\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3})e_1 + (\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1})e_2 + (\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2})e_3$.

(v) $\vartheta(X) \wedge \vartheta(Y) =$
$(a_1b_2 - a_2b_1)(e_1^* \wedge e_2^*) + (a_1b_3 - a_3b_1)(e_1^* \wedge e_3^*) + (a_2b_3 - a_3b_2)(e_2^* \wedge e_3^*)$.

(vi) $*(\vartheta(X)\wedge\vartheta(Y)) = (a_1b_2 - a_2b_1)(e_3^*) - (a_1b_3 - a_3b_1)(e_2^*) + (a_2b_3 - a_3b_2)(e_1^*)$.

(vii) $\vartheta^{-1}*(\vartheta(X)\wedge\vartheta(Y)) = (a_2b_3 - a_3b_2)e_1 - (a_1b_3 - a_3b_1)e_2 + (a_1b_2 - a_2b_1)e_3$.

Question 5:

$\omega = a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz$ with $d\omega = 0$. Show that $\omega = df$ where $f = \int_0^1 \{xa(tx,ty,tz) + yb(tx,ty,tz) + zc(tx,ty,tz)\}dt$.

• We compute $d\omega$.

$$d\omega = (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})(dx \wedge dy) + (\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})(dy \wedge dz) + (\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z})(dx \wedge dz).$$

Since $d\omega = 0$ then

$$\frac{\partial b}{\partial x} = \frac{\partial a}{\partial y} \quad ; \quad \frac{\partial c}{\partial y} = \frac{\partial b}{\partial z} \quad ; \quad \frac{\partial c}{\partial x} = \frac{\partial a}{\partial z}$$

- **Lemma** : $\frac{\partial}{\partial x}\int_0^1 a(tx,ty,tz)dt = \int_0^1 t\frac{\partial a(tx,ty,tz)}{\partial x}dt.$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial x}\int_0^1 a(tx,ty,tz)dt &= \lim_{h\to 0}\frac{\int_0^1 a(t(x+h),ty,tz)dt - \int_0^1 a(tx,ty,tz)dt}{h} \\
&= \lim_{h\to 0}\int_0^1 \frac{a(t(x+h),ty,tz) - a(tx,ty,tz)}{h}dt \\
&= \int_0^1 \lim_{h\to 0} t\frac{a(tx+th,ty,tz) - a(tx,ty,tz)}{th}dt \\
&= \int_0^1 t\frac{\partial a(tx,ty,yz)}{\partial x}dt
\end{aligned}$$

□

Simmilarly, for $b(tx,ty,tz)$ and $c(tx,ty,tz)$.

- We compute $\partial_x f = \frac{\partial f}{\partial x}$.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\partial_x\int_0^1 x\cdot a(tx,ty,tz) + y\cdot b(tx,ty,tz) + z\cdot c(tx,ty,tz)dt \\
&= \frac{\partial}{\partial x}\Big(\int_0^1 x\cdot a(tx,ty,tz)dt\Big) + \frac{\partial}{\partial x}\Big(\int_0^1 y\cdot b(tx,ty,tz)dt\Big) + \\
&\quad \frac{\partial}{\partial x}\Big(\int_0^1 z\cdot c(tx,ty,tz)dt\Big) \\
&= \int_0^1 a(tx,ty,tz)dt + x\cdot\frac{\partial}{\partial x}\Big(\int_0^1 a(tx,ty,tz)dt\Big) + \\
&\quad y\cdot\frac{\partial}{\partial x}\Big(\int_0^1 b(tx,ty,tz)dt\Big) + z\cdot\frac{\partial}{\partial x}\Big(\int_0^1 c(tx,ty,tz)dt\Big)
\end{aligned}$$

Applying the lemma above,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \int_0^1 a(tx,ty,tz)dt + x\int_0^1 t\cdot\frac{\partial a(tx,ty,tz)}{\partial x}dt + \\
&\quad y\int_0^1 t\cdot\frac{\partial b(tx,ty,tz)}{\partial x}dt + z\int_0^1 t\cdot\frac{\partial c(tx,ty,tz)}{\partial x}dt
\end{aligned}$$

Applying the relations between partial derivatives of a, b and c above.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \int_0^1 a(tx,ty,tz)dt + x\int_0^1 t\cdot\frac{\partial a(tx,ty,tz)}{\partial x}dt + \\
&\quad y\int_0^1 t\cdot\frac{\partial a(tx,ty,tz)}{\partial y}dt + z\int_0^1 t\cdot\frac{\partial a(tx,ty,tz)}{\partial z}dt \\
&= \int_0^1 a(tx,ty,tz)dt + \\
&\quad \int_0^1 xt\cdot\frac{\partial a(tx,ty,tz)}{\partial x} + yt\cdot\frac{\partial a(tx,ty,tz)}{\partial y} + zt\cdot\frac{\partial a(tx,ty,tz)}{\partial z}dt \\
\frac{\partial f}{\partial x} &= a(x,y,z)
\end{aligned}$$

Similarly, $\frac{\partial f}{\partial y} = b(x,y,z)$ and $\frac{\partial f}{\partial z} = c(x,y,z)$.
So $df = a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz = \omega$.

Question 6:
M is a compact 3-dim smooth submanifold-with-boundary of $\mathbb{R}^3$ and $f : M \mapsto \mathbb{R}^3$ is the inclusion. Let $d\omega = \frac{1}{3}(xdy\wedge dz + ydz\wedge dx + zdx\wedge dy)$.

- *Show that* $d(\omega/r^3) = 0$ *on* $\mathbb{R}^3$*, where* $r^2 = x^2 + y^2 + z^2$.

$$\begin{aligned}
d(\omega/r^3) &= \frac{\partial(x/3r^3)}{\partial x}dx\wedge dy\wedge dz + \frac{\partial(y/3r^3)}{\partial y}dy\wedge dz\wedge dx + \\
&\quad \frac{\partial(z/3r^3)}{\partial z}dz\wedge dx\wedge dy \\
&= \Big(\frac{\partial(x/3r^3)}{\partial x} + \frac{\partial(y/3r^3)}{\partial y} + \frac{\partial(z/3r^3)}{\partial z}\Big)dx\wedge dy\wedge dz \\
&= \Big(\frac{1}{3}r^{-3} - x^2r^{-5} + \frac{1}{3}r^{-3} - y^2r^{-5} + \frac{1}{3}r^{-3} - z^2r^{-5}\Big)dx\wedge dy\wedge dz \\
&= (r^{-3} - r^2r^{-5})dx\wedge dy\wedge dz \\
d(\omega/r^3) &= 0
\end{aligned}$$

- *Show that* $\int_{\partial M}\iota^*f^*(\omega) = \mathrm{vol}\ (M)$ *and there is a 2-form* η *on* S^2 *such that* $d\eta = 0$ *but* $\eta \neq d\phi$ *for any 1-form* ϕ.

By Stoke's theorem(9.6), $\int_{\partial M}\iota^*f^*(\omega) = \int_M d(f^*w) = \int_M f^*d\omega$. Since $d\omega = dx\wedge dy\wedge dz$, $\int_{\partial M}\iota^*f^*(\omega) = \int_M f^*(dx\wedge dy\wedge dz)$.
By definition of f, $f^*d\omega = \omega_{f(p)}(Tf(u_1), Tf(u_2), Tf(u_3)) = \omega_p(u_1,u_2,u_3)$.
So, $\int_M f^*dx\wedge dy\wedge dz = \int_M dxdydz = \mathrm{vol}\ (M)$.

Let $M = D^3$ and $\eta = \iota^*f^*\omega$. Because η is a 2-form then $d\eta$ is 3-form. That is, $d\eta = \iota^*f^*d\omega$. Since $d\eta$ is a 3-form on S^2 and S^2 is of dimension 2, so $d\eta = 0$. Suppose $\eta = d\phi$ where ϕ is a 1-form on S^2. Then by applying corollary 9.7., $\int_{S^2} d\phi = 0$, therefore contradicts $\int_{S^2} \eta = \text{vol } (D^3)$.