LECTURE NOTES ON DIFFERENTIABLE MANIFOLDS

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Contents

1. Tangent Spaces, Vector Fields in $\mathbb{R}^n$ and the Inverse Mapping Theorem

1.1. **Tangent Space to a Level Surface.** Let γ be a curve in $\mathbb{R}^n$: $\gamma\colon t \mapsto (\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t))$. (A curve can be described as a vector-valued function. Converse a vector-valued function gives a curve in $\mathbb{R}^n$.) The *tangent line* at the point $\gamma(t_0)$ is given with the direction

$$\frac{d\gamma}{dt}(t_0) = \left(\frac{d\gamma^1}{dt}(t_0), \ldots, \frac{d\gamma^n}{t}(t_0)\right).$$

(Certainly we need to assume that the derivatives exist. We may talk about *smooth curves*, that is, the curves with all continuous higher derivatives.)

Consider the level surface $f(x^1, x^2, \ldots, x^n) = c$ of a differentiable function f, where x^i refers to i-th coordinate. The *gradient vector* of f at a point $P = (x^1(P), x^2(P), \ldots, x^n(P))$ is

$$\nabla f = (\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}).$$

Given a vector $\vec{u} = (u^1, \ldots, u^n)$, the *directional derivative* is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x^1} u^1 + \cdots + \frac{\partial f}{\partial x^n} u^n.$$

The *tangent space* at the point P on the level surface $f(x^1, \ldots, x^n) = c$ is the $(n-1)$-dimensional (if $\nabla f \neq 0$) space through P normal to the gradient ∇f. In other words, the tangent space is given by the equation

$$\frac{\partial f}{\partial x^1}(P)(x^1 - x^1(P)) + \cdots + \frac{\partial f}{\partial x^n}(P)(x^n - x^n(P)) = 0.$$

From the geometric views, the tangent space *should* consist of all tangents to the smooth curves **on the level surface** through the point P. Assume that γ is a curve through P (when $t = t_0$) that lies in the level surface $f(x^1, \ldots, x^n) = c$, that is

$$f(\gamma^1(t), \gamma^2(t), \ldots, \gamma^n(t)) = c.$$

By taking derivatives on both sides,

$$\frac{\partial f}{\partial x^1}(P)(\gamma^1)'(t_0) + \cdots + \frac{\partial f}{\partial x^n}(P)(\gamma^n)'(t_0) = 0$$

and so the tangent line of γ is really normal (orthogonal) to ∇f. When γ *runs over all possible curves* on the level surface through the point P, then we obtain the tangent space at the point P.

Roughly speaking, *a tangent space is a vector space attached to a point in the surface.*

How to obtain the tangent space: *take all tangent lines of smooth curve through this point on the surface.*

1.2. **Tangent Space and Vectors Fields on $\mathbb{R}^n$.** Now consider the tangent space of $\mathbb{R}^n$. According to the ideas in the previous subsection, first we assume a given point $P \in \mathbb{R}^n$. Then we consider all smooth curves passes through P and then take the tangent lines from the smooth curves. The obtained vector space at the point P is the n-dimensional space. But we can look at in a little detail.

Let γ be a smooth curve through P. We may assume that $\gamma(0) = P$. Let ω be another smooth curve with $\omega(0) = P$. γ is called to be *equivalent* to ω if the directives $\gamma'(0) = \omega'(0)$. The tangent space of $\mathbb{R}^n$ at P, denoted by $T_P(\mathbb{R}^n)$, is then the set of equivalence class of all smooth curves through P.

Let $T(\mathbb{R}^n) = \bigcup\limits_{P\in\mathbb{R}^n} T_P(\mathbb{R}^n)$, called the tangent bundle of $\mathbb{R}^n$. If S is a region of $\mathbb{R}^n$, let $T(S) = \bigcup\limits_{P\in S} T_P(S)$, called the tangent bundle of S.

Note. Each $T_P(\mathbb{R}^n)$ is an n-dimensional vector space, but $T(S)$ is *not* a vector space. In other words, $T(S)$ is obtained by attaching a vector space $T_P(\mathbb{R}^n)$ to each point P in S. Also S is assumed to be a region of $\mathbb{R}^n$, otherwise the tangent space of S (for instance S is a level surface) could be a proper subspace of $T_P(\mathbb{R}^n)$.

If γ is a smooth curve from P to $\mathbb{Q}$ in $\mathbb{R}^n$, then the tangent space $T_P(\mathbb{R}^n)$ moves along γ to $T_Q(\mathbb{R}^n)$. The direction for this moving is given $\gamma'(t)$, which introduces the following important concept.

Definition 1.1. A *vector field V* on a region S of $\mathbb{R}^n$ is a smooth map (also called C^∞-map)

$$V\colon S \to T(S) \quad P \mapsto \vec{v}(P).$$

Let $V\colon P \mapsto \vec{v}(P)$ and $W\colon P \mapsto \vec{w}(P)$ be two vector fields and let $f\colon S \to \mathbb{R}$ be a smooth function. Then $V + W\colon P \mapsto \vec{v}(P) + \vec{w}(P)$ and $fV\colon P \mapsto f(P)\vec{v}(P)$ give (pointwise) addition and scalar multiplication structure on vector fields.

1.3. **Operator Representations of Vector Fields.** Let J be an open interval containing 0 and let $\gamma\colon J \to \mathbb{R}^n$ be a smooth curve with $\gamma(0) = P$. Let $f = f(x^1, \ldots, x^n)$ be a smooth function defined on a neighborhood of P. Assume that the range of γ is contained in the domain of f. By applying the chain rule to the composite $T = f \circ \gamma\colon J \to \mathbb{R}$,

$$D_\gamma(f) := \frac{dT}{dt} = \sum_{i=1}^{n} \frac{d\gamma^i(t)}{dt} \frac{\partial f}{\partial x^i}\bigg|_{x^i=\gamma^i(t)}$$

Proposition 1.2.

$$D_\gamma(af + bg) = aD_\gamma(f) + bD_\gamma(g), \quad \textit{where } a, b \textit{ are constant.}$$

$$D_\gamma(fg) = D_\gamma(f)g + fD_\gamma(g).$$

Let $C^\infty(\mathbb{R}^n)$ denote the set of smooth functions on $\mathbb{R}^n$. An operation D on $C^\infty(\mathbb{R}^n)$ is called a *derivation* if D maps $C^\infty(\mathbb{R}^n)$ to $C^\infty(R^n)$ and satisfies the conditions

$$D(af + bg) = aD(f) + bD(g), \quad \text{where } a, b \text{ are constant.}$$

$$D(fg) = D(f)g + fD(g).$$

Example: For $1 \leq i \leq n$,

$$\partial_i \colon f \mapsto \frac{\partial f}{\partial x^i}$$

is a derivation.

Proposition 1.3. *Let D be any derivation on $C^\infty(\mathbb{R}^n)$. Given any point P in $\mathbb{R}^n$. Then there exist real numbers $a^1, a^2, \ldots, a^n \in \mathbb{R}$ such that*

$$D(f)(P) = \sum_{i=1}^{n} a^i \partial_i(f)(P)$$

for any $f \in C^\infty(\mathbb{R}^n)$, where a^i depends on D and P but is independent on f.

Proof. Write x for $(x^1, \ldots, x^n)$. Define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(t(x-P)+P)dt.$$

Then

$$f(x) - f(P) = \int_0^1 \frac{d}{dt} f(t(x-P)+P)dt$$

$$= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(t(x-P)+P) \cdot (x^i - x^i(P))dt$$

$$= \sum_{i=1}^{n}(x^i - x^i(P)) \int_0^1 \frac{\partial f}{\partial x^i}(t(x-P)+P)dt = \sum_{i=1}^{n}(x^i - x^i(P))g_i(x).$$

Since D is a derivation, $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1)$ and so $D(1) = 0$. It follows that $D(c) = 0$ for any constant c. By applying D to the above equations,

$$D(f(x)) = D(f(x) - f(P)) = \sum_{i=1}^{n} D(x^i - x^i(P))g_i(x) + (x^i - x^i(P))D(g_i(x))$$

$$= \sum_{i=1}^{n} D(x^i)g_i(x) + (x^i - x^i(P))D(g_i(x))$$

because $D(f(P)) = D(x^i(P)) = 0$. Let $a^i = D(x^i)(P)$ which only depends on D and P. By evaluating at P,

$$D(f)(P) = \sum_{i=1}^{n} D(x^i)(P)g_i(P) + 0 = \sum_{i=1}^{n} a^i g_i(P).$$

Since

$$g_i(P) = \int_0^1 \frac{\partial f}{\partial x^i}(t(P-P)+P)dt = \int_0^1 \frac{\partial f}{\partial x^i}(P)dt = \frac{\partial f}{\partial x^i}(P) = \partial_i(f)(P),$$

$$D(f)(P) = \sum_{i=1}^{n} a^i \partial_i(f)(P),$$

which is the conclusion. ∎

From this proposition, we can give a new way to looking at vector fields:

Given a vector fields $P \mapsto \vec{v}(P) = (v^1(P), v^2(P), \ldots, v^n(P))$, a derivation

$$D_{\vec{v}} = \sum_{i=1}^{n} v^i(P) \cdot \partial_i$$

on $C^\infty(\mathbb{R}^n)$ is called an *operator representation* of the vector field $P \mapsto \vec{v}(P)$.

Note. The operation $v^i(x)\partial_i$ is given as follows: for any $f \in C^\infty(\mathbb{R}^n)$,

$$D_{\vec{v}}(f)(P) = \sum_{i=1}^{n} v^i(P) \cdot \partial_i(f)(P)$$

for any P.

From this new view, the tangent spaces $T(\mathbb{R}^n)$ admits a basis $\{\partial_1, \partial_2, \ldots, \partial_n\}$.

1.4. **Integral Curves.** Let $V : \mathbf{x} \mapsto \vec{v}(\mathbf{x})$ be a (smooth) vector field on an neighborhood U of P. An *integral curve* to V is a smooth curve $\mathbf{s} : (-\delta, \epsilon) \to U$, defined for suitable $\delta, \epsilon > 0$, such that

$$\mathbf{s}'(t) = \vec{v}(\mathbf{s}(t))$$

for $-\delta < t < \epsilon$.

Theorem 1.4. *Let $V : \mathbf{x} \mapsto \vec{v}(\mathbf{x})$ be a (smooth) vector field on an neighborhood U of P. Then there exists an integral curve to V through P. Any two such curves agree on their common domain.*

Proof. The proof is given by assuming the fundamental existence and uniqueness theorem for systems of first order differential equations.

The requirement for a curve $\mathbf{s}(t) = (s^1(t), \ldots, s^n(t))$ to be an integral curve is:

$$\begin{cases} \frac{ds^1(t)}{dt} = v^1(s^1(t), s^2(t), \ldots, s^n(t)) \\ \frac{ds^2(t)}{dt} = v^2(s^1(t), s^2(t), \ldots, s^n(t)) \\ \cdots\cdots\cdots \\ \frac{ds^n(t)}{dt} = v^n(s^1(t), s^2(t), \ldots, s^n(t)) \end{cases}$$

with the initial conditions

$$\mathbf{s}(0) = P \qquad (s^1(0), s^2(0), \ldots, s^n(0)) = (x^1(P), x^2(P), \ldots, x^n(P))$$

$$\mathbf{s}'(0) = \vec{v}(P) \qquad \left(\frac{ds^1}{dt}(0), \ldots, \frac{ds^n}{dt}(0)\right) = (v^1(P), \ldots, v^n(P)).$$

Thus the statement follows from the fundamental theorem of first order ODE. □

Example 1.5. Let $n = 2$ and let $V : P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$, where $v^1(x, y) = x$ and $v^2(x, y) = y$. Given a point $P = (a^1, a^2)$, the equation for the integral curve $\mathbf{s}(t) = (x(t), y(t))$ is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = y(t) \end{cases}$$

with initial conditions $(x(0), y(0)) = (a^1, a^2)$ and $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, a^2)$. Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^t).$$

Example 1.6. Let $n = 2$ and let $V\colon P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$, where $v^1(x,y) = x$ and $v^2(x,y) = -y$. Given a point $P = (a^1, a^2)$, the equation for the integral curve $\mathbf{s}(t) = (x(t), y(t))$ is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = -y(t) \end{cases}$$

with initial conditions $(x(0), y(0)) = (a^1, a^2)$ and $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, -a^2)$. Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^{-t}).$$

1.5. **Implicit- and Inverse-Mapping Theorems.**

Theorem 1.7. *Let D be an open region in $\mathbb{R}^{n+1}$ and let F be a function well-defined on D with continuous partial derivatives. Let $(x_0^1, x_0^2, \ldots, x_0^n, z_0)$ be a point in D where*

$$F(x_0^1, x_0^2, \ldots, x_0^n, z_0) = 0 \qquad \frac{\partial F}{\partial z}(x_0^1, x_0^2, \ldots, x_0^n, z_0) \neq 0.$$

Then there is a neighborhood $N_\epsilon(z_0) \subseteq \mathbb{R}$, a neighborhood $N_\delta(x_0^1, \ldots, x_0^n) \subseteq \mathbb{R}^n$, and a **unique** *function $z = g(x^1, x^2, \ldots, x^n)$ defined for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ with values $z \in N_\epsilon(z_0)$ such that*

1) *$z_0 = g(x_0^1, x_0^2, \ldots, x_0^n)$ and*
$$F(x^1, x^2, \ldots, x^n, g(x^1, \ldots, x^n)) = 0$$
for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.
2) *g has continuous partial derivatives with*
$$\frac{\partial g}{\partial x^i}(x^1, \ldots, x^n) = -\frac{F_{x^i}(x^1, \ldots, x^n, z)}{F_z(x^1, \ldots, x^n, z)}$$
for all $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ where $z = g(x^1, \ldots, x^n)$.
3) *If F is smooth on D, then $z = g(x^1, \ldots, x^n)$ is smooth on $N_\delta(x_0^1, \ldots, x_0^n)$.*

Proof. **Step 1.** We may assume that $\frac{\partial F}{\partial z}(x_0^1, x_0^2, \ldots, x_0^n, z_0) > 0$. Since F_z is continuous, there exists a neighborhood $N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0)$ in which F_z is continuous and positive. Thus for fixed $(x^1, \ldots, x^n)$, F is strictly increasing on z in this neighborhood. It follows that there exists $c > 0$ such that

$$F(x_0^1, x_0^2, \ldots, x_0^n, z_0 - c) < 0 \qquad F(x_0^1, x_0^2, \ldots, x_0^n, z_0 + c) > 0$$

with

$$(x_0^1, x_0^2, \ldots, x_0^n, z_0 - c), (x_0^1, x_0^2, \ldots, x_0^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0).$$

Step 2. By the continuity of F, there exists a small $\delta > 0$ such that

$$F(x^1, x^2, \ldots, x^n, z_0 - c) < 0 \qquad F(x^1, x^2, \ldots, x^n, z_0 + c) > 0$$

with

$$(x^1, x^2, \ldots, x^n, z_0 - c), (x^1, x^2, \ldots, x^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \ldots, x_0^n, z_0)$$

for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$.

Step 3. Fixed $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$, F is continuous and strictly increasing on z. There is a **unique** z, $z_0 - c < z < z_0 + c$, such that

$$F(x^1, \ldots, x^n, z) = 0.$$

This defines a function $z = g(x^1, \ldots, x^n)$ for $(x^1, \ldots, x^n) \in N_\delta(x_0^1, \ldots, x_0^n)$ with values $z \in (z_0 - c, z_0 + c)$.

Step 4. Prove that $z = g(x^1, \ldots, x^n)$ is continuous. Let $(x_1^1, \ldots, x_1^n) \in N_\delta(x_0^1, \ldots, x_0^n)$. Let $(x_1^1(k), \ldots, x_1^n(k))$ be any sequence in $N_\delta(x_0^1, \ldots, x_0^n)$ converging to $(x_1^1, \ldots, x_1^n)$. Let A be any subsequential limit of $\{z_k = g(x_1^1(k), \ldots, x_1^n(k))\}$, that is $A = \lim_{s\to\infty} z_{k_s}$. Then, by the continuity of F,

$$\begin{aligned}0 &= \lim_{s\to\infty} F(x_1^1(k_s), \ldots, x_1^n(k_s), z_{k_s})\\ &= F(\lim_{s\to\infty} x_1^1(k_s), \ldots, \lim_{s\to\infty} x_1^n(k_s), \lim_{s\to\infty} z_{k_s})\\ &= F(x_1^1, \ldots, x_1^n, A).\end{aligned}$$

By the unique solution of the equation, $A = g(x_1^1, \ldots, x_1^n)$. Thus $\{z_k\}$ converges $g(x_1^1, \ldots, x_1^n)$ and so g is continuous.

Step 5. Compute the partial derivatives $\frac{\partial z}{\partial x_i}$. Let h be small enough. Let

$$z + k = g(x^1, \ldots, x^{i-1}, x^i + h, x^{i+1}, \ldots, x^n),$$

that is

$$F(x^1, \ldots, x^i + h, \ldots, x^n, z + k) = 0$$

with $z_0 - c < z + k < z_0 + c$. Then

$$\begin{aligned}0 &= F(x^1, \ldots, x^i + h, \ldots, x^n, z + k) - F(x^1, \ldots, x^n, z)\\ &= F_{x^i}(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})h + F_z(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})k\end{aligned}$$

by the mean value theorem (Consider the function

$$\phi(t) = F(x^1, \ldots, x^i + th, \ldots, x^n, z + tk)$$

for $0 \le t \le 1$. Then $\phi(1) - \phi(0) = \phi'(\xi)(1-0)$.), where $\tilde{x}^i$ is between x^i and $x^i + h$, and $\tilde{z}$ is between z and $z + k$. Now

$$\begin{aligned}\frac{\partial g}{\partial x_i} &= \lim_{h\to 0} \frac{g(x^1, \ldots, x^{i-1}, x^i + h, x^{i+1}, \ldots, x^n) - z}{h} = \lim_{h\to 0} \frac{k}{h}\\ &= -\lim_{h\to 0} \frac{F_{x^i}(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})}{F_z(x^1, \ldots, \tilde{x}^i, \ldots, x^n, \tilde{z})}. = -\frac{F_{x_i}}{F_z},\end{aligned}$$

where $\tilde{z} \to z$ as $h \to 0$ because g is continuous (and so $k \to 0$ as $h \to 0$).

Step 6. Since F_z is not zero in this small neighborhood, g_{x_i} is continuous for each i. If F is smooth, then all higher derivatives of g are continuous and so g is also smooth. □

Theorem 1.8 (Implicit Function Theorem)**.** *Let D be an open region in $\mathbb{R}^{m+n}$ and let $F_1, F_2, \ldots, F_n$ be functions well-defined on D with continuous partial derivatives. Let $(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n)$ be a point in D where*

$$\begin{cases} F_1(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n) = 0\\ F_2(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n) = 0\\ \quad\cdots\cdots\cdots\cdots\\ F_n(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n) = 0\end{cases}$$

and the Jacobian

$$J = \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u^1, u^2, \ldots, u^n)} = \det\left(\frac{\partial F_i}{\partial u^j}\right) \neq 0$$

at the point $(x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n)$*. Then there are neighborhoods* $N_\delta(x_0^1, \ldots, x_0^m)$, $N_{\epsilon_1}(u_0^1)$, $N_{\epsilon_2}(u_0^2)$, $\ldots$, $N_{\epsilon_n}(u_0^n)$*, and* **unique** *functions*

$$\begin{cases} u^1 = g_1(x^1, x^2, \ldots, x^m) \\ u^2 = g_2(x^1, x^2, \ldots, x^m) \\ \cdots\cdots\cdots\cdots \\ u^n = g_n(x^1, x^2, \ldots, x^m) \end{cases}$$

defined for $(x^1, \ldots, x^m) \in N_\delta(x_0^1, \ldots, x_0^m)$ *with values* $u^1 \in N_{\epsilon_1}(u_0^1), \ldots, u^n \in N_{\epsilon_n}(u_0^n)$ *such that*

1) $u_0^i = g_i(x_0^1, x_0^2, \ldots, x_0^m)$ *and*

$$F_i(x^1, x^2, \ldots, x^n, g_i(x^1, \ldots, x^m)) = 0$$

for all $1 \leq i \leq n$ *and all* $(x^1, \ldots, x^m) \in N_\delta(x_0^1, \ldots, x_0^m)$*.*

2) *Each* g_i *has continuous partial derivatives with*

$$\frac{\partial g_i}{\partial x^j}(x^1, \ldots, x^m) = -\frac{1}{J} \cdot \frac{\partial(F_1, \ldots, F_n)}{\partial(u^1, u^2, \ldots, u^{j-1}, x^j, u^{j+1}, \ldots, u^n)}$$

for all $(x^1, \ldots, x^m) \in N_\delta(x_0^1, \ldots, x_0^m)$ *where* $u^i = g_i(x^1, \ldots, x^m)$*.*

3) *If each* F_i *is smooth on* D*, then each* $u^i = g_i(x^1, \ldots, x^m)$ *is smooth on* $N_\delta(x_0^1, \ldots, x_0^m)$*.*

Sketch of Proof. The proof is given by induction on n. Assume that the statement holds for $n-1$ with $n > 1$. (We already prove that the statement holds for $n = 1$.) Since the matrix

$$\left(\frac{\partial F_i}{\partial u^j}\right)$$

is invertible at the point $P = (x_0^1, x_0^2, \ldots, x_0^m, u_0^1, u_0^2, \ldots, u_0^n)$ (because the determinant is not zero), we may assume that

$$\frac{\partial F_n}{\partial u^n}(P) \neq 0.$$

(The entries in the last column can not be all 0 and so, if $\frac{\partial F_i}{\partial u^n}(P) \neq 0$, we can interchange F_i and F_n.)

From the previous theorem, there is a solution

$$u^n = g_n(x^1, \ldots, x^m, u^1, \ldots, u^{n-1})$$

to the last equation. Consider

$$\begin{cases} G_1 = F_1(x^1, \ldots, x^m, u^1, \ldots, u^{n-1}, g_n) \\ G_2 = F_2(x^1, \ldots, x^m, u^1, \ldots, u^{n-1}, g_n) \\ \cdots\cdots\cdots\cdots \\ G_{n-1} = F_{n-1}(x^1, \ldots, x^m, u^1, \ldots, u^{n-1}, g_n). \end{cases}$$

Then

$$\frac{\partial G_i}{\partial u^j} = \frac{\partial F_i}{\partial u^j} + \frac{\partial F_i}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j}$$

for $1 \leq i, j \leq n-1$, where

$$\frac{\partial F_n}{\partial u^j} + \frac{\partial F_n}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j} = 0.$$

Let

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial g_n}{\partial u^1} & \frac{\partial g_n}{\partial u^2} & \frac{\partial g_n}{\partial u^3} & \cdots & \frac{\partial g_n}{\partial u^{n-1}} & 1 \end{pmatrix}$$

Then

$$\left(\frac{\partial F_i}{\partial u^j}\right) \cdot B = \begin{pmatrix} \left(\frac{\partial G_i}{\partial u^j}\right)_{n-1,n-1} & * \\ 0 & \frac{\partial F_n}{\partial u^n} \end{pmatrix}.$$

By taking the determinant,

$$J = \frac{\partial(F_1, \ldots, F_n)}{\partial(u^1, \ldots, u^n)} = \frac{\partial F_n}{\partial u^n} \cdot \frac{\partial(G_1, \ldots, G_{n-1})}{\partial(u^1, \ldots, u^{n-1})}.$$

Thus $\frac{\partial(G_1,\ldots,G_{n-1})}{\partial(u^1,\ldots,u^{n-1})} \neq 0$ at P and, by induction, there are solutions

$$u^i = g_i(x^1, \ldots, x^m)$$

for $1 \leq i \leq n-1$. □

Theorem 1.9 (Inverse Mapping Theorem)**.** *Let D be an open region in $\mathbb{R}^n$. Let*

$$\begin{cases} x^1 = f_1(u^1, \ldots, u^n) \\ x^2 = f_2(u^1, \ldots, u^n) \\ \qquad \cdots\cdots \\ x^n = f_n(u^1, \ldots, u^n) \end{cases}$$

be functions defined on D with continuous partial derivatives. Let $(u_0^1, \ldots, u_0^n) \in D$ satisfy $x_0^i = f_i(u_0^1, \ldots, u_0^n)$ and the Jacobian

$$\frac{\partial(x^1, \ldots, x^n)}{\partial(u^1, \ldots, u^n)} \neq 0 \qquad \text{at} \quad (u_0^1, \ldots, u_0^n).$$

Then there are neighborhood $N_\delta(x_0^1, \ldots, x_0^n)$ and $N_\epsilon(u_0^1, \ldots, u_0^n)$ such that

$$\begin{cases} u^1 = f_1^{-1}(x^1, \ldots, x^n) \\ u^2 = f_2^{-1}(x^1, \ldots, x^n) \\ \qquad \cdots\cdots \\ u^n = f_n^{-1}(x^1, \ldots, x^n) \end{cases}$$

is well-defined and has continuous partial derivatives on $N_\delta(x_0^1, \ldots, x_0^n)$ with values in $N_\epsilon(u_0^1, \ldots, u_0^n)$. Moreover if each f_i is smooth, then each f_i^{-1} is smooth.

Proof. Let $F_i = f_i(u^1, \ldots, u^n) - x_i$. The assertion follows from the Implicit Function Theorem. □

2. Topological and Differentiable Manifolds, Diffeomorphisms, Immersions, Submersions and Submanifolds

2.1. **Topological Spaces.**

Definition 2.1. Let X be a set. A *topology* $\mathcal{U}$ for X is a collection of subsets of X satisfying

i) $\emptyset$ and X are in $\mathcal{U}$;
ii) the intersection of two members of $\mathcal{U}$ is in $\mathcal{U}$;
iii) the union of any number of members of $\mathcal{U}$ is in $\mathcal{U}$.

The set X with $\mathcal{U}$ is called a *topological space*. The members $U \in \mathcal{U}$ are called the *open sets*.

Let X be a topological space. A subset $N \subseteq X$ with $x \in N$ is called a *neighborhood* of x if there is an open set U with $x \in U \subseteq N$. For example, if X is a metric space, then the closed ball $D_\epsilon(x)$ and the open ball $B_\epsilon(x)$ are neighborhoods of x. A subset C is said to *closed* if $X \setminus C$ is open.

Definition 2.2. A function $f\colon X \to Y$ between two topological spaces is said to be *continuous* if for every open set U of Y the pre-image $f^{-1}(U)$ is open in X.

A continuous function from a topological space to a topological space is often simply called a *map*. A *space* means a *Hausdorff space*, that is, a topological spaces where any two points has disjoint neighborhoods.

Definition 2.3. Let X and Y be topological spaces. We say that X and Y are *homeomorphic* if there exist continuous functions $f\colon X \to Y, g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We write $X \cong Y$ and say that f and g are *homeomorphisms* between X and Y.

By the definition, a function $f\colon X \to Y$ is a homeomorphism if and only if

i) f is a bijective;
ii) f is continuous and
iii) f^{-1} is also continuous.

Equivalently f is a homeomorphism if and only if 1) f is a bijective, 2) f is continuous and 3) f is an open map, that is f sends open sets to open sets. Thus a homeomorphism between X and Y is a bijective between the points and the open sets of X and Y.

A very general question in topology is how to classify topological spaces under homeomorphisms. For example, we know (from complex analysis and others) that any simple closed loop is homeomorphic to the unit circle S^1. Roughly speaking topological classification of curves is known. The topological classification of (two-dimensional) surfaces is known as well. However the topological classification of 3-dimensional manifolds (we will learn manifolds later.) is quite open.

The famous Poicaré conjecture is related to this problem, which states that any simply connected 3-dimensional (topological) manifold is homeomorphic to the 3-sphere S^3. A space X is called *simply connected* if (1) X is path-connected (that is, given any two points, there is a continuous path joining them) and (2) the fundamental group $\pi_1(X)$ is trivial (roughly speaking, any loop can be deformed to be the constant loop in X). The *manifolds* are the objects that we are going to discuss in this course.

2.2. **Topological Manifolds.**

A Hausdorff space M is called a *(topological) n-manifold* if each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. Roughly speaking, an n-manifold is *locally* $\mathbb{R}^n$. Sometimes M is denoted as M^n for mentioning the dimension of M.

(**Note.** If you are not familiar with topological spaces, you just think that M is a subspace of $\mathbb{R}^N$ for a large N.)

For example, $\mathbb{R}^n$ and the n-sphere S^n is an n-manifold. A 2-dimensional manifold is called a *surface*. The objects traditionally called 'surfaces in 3-space' can be made into manifolds in a standard way. The compact surfaces have been classified as spheres or projective planes with various numbers of handles attached.

By the definition of manifold, the closed n-disk D^n is not an n-manifold because it has the 'boundary' S^{n-1}. D^n is an example of 'manifolds with boundary'. We give the definition of manifold with boundary as follows.

A Hausdorff space M is called an *n-manifold with boundary* ($n \geq 1$) if each point in M has a neighborhood homeomorphic to an open set in the half space

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

Manifold is one of models that we can do calculus 'locally'. By means of calculus, we need local coordinate systems. Let $x \in M$. By the definition, there is a an open neighborhood $U(x)$ of x and a homeomorphism ϕ_x from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The collection $\{(U(x), \phi_x) | x \in M\}$ has the property that 1) $\{U(x) | x \in M\}$ is an open cover and 2) ϕ_x is a homeomorphism from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The subspace $\phi_x(U_x)$ in $\mathbb{R}^n_+$ plays a role as a local coordinate system. The collection $\{(U(x), \phi_x) | x \in M\}$ is somewhat too large and we may like less local coordinate systems. This can be done as follows.

Let M be a space. A *chart* of M is a pair (U, ϕ) such that 1) U is an open set in M and 2) ϕ is a homeomorphism from U onto an open set in $\mathbb{R}^n_+$. The map

$$\phi \colon U \to \mathbb{R}^n_+$$

can be given by n coordinate functions $\phi_1, \ldots, \phi_n$. If P denotes a point of U, these functions are often written as

$$x^1(P), x^2(P), \ldots, x^n(P)$$

or simply $x^1, x^2, \ldots, x^n$. They are called *local coordinates* on the manifold.

An *atlas* for M means a collection of charts $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ such that $\{U_\alpha | \alpha \in J\}$ is an open cover of M.

Proposition 2.4. *A Hausdorff space M is a manifold (with boundary) if and only if M has an atlas.*

Proof. Suppose that M is a manifold. Then the collection $\{(U(x), \phi_x) | x \in M\}$ is an atlas. Conversely suppose that M has an atlas. For any $x \in M$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x that is homeomorphic to an open set in $\mathbb{R}^n_+$. Thus M is a manifold. □

We define a subset ∂M as follows: $x \in \partial M$ if there is a chart (U_α, ϕ_α) such that $x \in U_\alpha$ and $\phi_\alpha(x) \in \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 0\}$. ∂M is called the boundary of M. For example the boundary of D^n is S^{n-1}.

Proposition 2.5. *Let M be a n-manifold with boundary. Then ∂M is an $(n-1)$-manifold without boundary.*

Proof. Let $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ be an atlas for M. Let $J' \subseteq J$ be the set of indices such that $U_\alpha \cap \partial M \neq \emptyset$ if $\alpha \in J'$. Then Clearly

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M} | \alpha \in J'\}$$

can be made into an atlas for ∂M. $\square$

Note. The key point here is that if U is open in $\mathbb{R}^n_+$, then $U \cap \mathbb{R}^{n-1}$ is also open because: Since U is open in $\mathbb{R}^n_+$, there is an open subset V of $\mathbb{R}^n$ such that $U = V \cap \mathbb{R}^n_+$. Now if $x \in U \cap \mathbb{R}^{n-1}$, there is an open disk $E_\epsilon(x) \subseteq V$ and so

$$E_\epsilon(x) \cap \mathbb{R}^{n-1} \subseteq V \cap \mathbb{R}^{n-1} = U \cap \mathbb{R}^{n-1}$$

is an open $(n-1)$-dimensional ϵ-disk in $\mathbb{R}^{n-1}$ centered at x.

2.3. **Differentiable Manifolds.**

Definition 2.6. A Hausdorff space M is called a *differential manifold of class C^k (with boundary)* if there is an atlas of M

$$\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$$

such that

For any $\alpha, \beta \in J$, the composites

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$

is differentiable of class C^k.

The atlas $\{(U_\alpha, \phi_\alpha | \alpha \in J\}$ is called a *differential atlas of class C^k* on M.

(**Note.** Assume that M is a subspace of $\mathbb{R}^N$ with $N >> 0$. If M has an atlas $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ such that each $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n_+$ is differentiable of class C^k, then M is a differentiable manifold of class C^k. This is the definition of differentiable (smooth) manifolds in [6] as in the beginning they already assume that M is a subspace of $\mathbb{R}^N$ with N large. In our definition (the usual definition of differentiable manifolds using charts), we only assume that M is a (Hausdorff) topological space and so ϕ_α is *only an identification* of an abstract U_α with an open subset of $\mathbb{R}^n_+$. In this case we can not talk differentiability of ϕ_α unless U_α is regarded as a subspace of a (large dimensional) Euclidian space.)

Two differential atlases of class C^k $\{(U_\alpha, \phi_\alpha) | \alpha \in I\}$ and $\{(V_\beta, \psi_\beta) | \beta \in J\}$ are called *equivalent* if

$$\{(U_\alpha, \phi_\alpha) | \alpha \in I\} \cup \{(V_\beta, \psi_\beta) | \beta \in J\}$$

is again a differential atlas of class C^k (this is an equivalence relation). A *differential structure of class C^k* on M is an equivalence class of differential atlases of class C^k on M. Thus a differential manifold of class C^k means a manifold with a differential structure of class C^k. A *smooth* manifold means a differential manifold of class C^∞.

Note: A general manifold is also called *topological manifold.* Kervaire and Milnor [4] have shown that the topological sphere S^7 has 28 distinct oriented smooth structures.

Definition 2.7. let M and N be smooth manifolds (with boundary) of dimensions m and n respectively. A map $f \colon M \to N$ is called *smooth* if for some smooth atlases $\{(U_\alpha, \phi_\alpha | \alpha \in I\}$ for M and $\{(V_\beta, \psi_\beta)\beta \in J\}$ for N the functions

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}|_{\phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha)} \colon \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n_+$$

are of class C^∞.

Proposition 2.8. *If $f\colon M \to N$ is smooth with respect to atlases*

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\}, \quad \{(V_\beta, \phi_\beta | \beta \in J\}$$

for M, N then it is smooth with respect to equivalent atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \quad \{(V'_\gamma, \eta_\gamma | \beta \in J'\}$$

Proof. Since f is smooth with respect with the atlases

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\}, \ \{(V_\beta, \phi_\beta | \beta \in J\},$$

f is smooth with respect to the smooth atlases

$$\{(U_\alpha, \phi_\alpha | \alpha \in I\} \cup \{(U'_\delta, \theta_\delta | \alpha \in I'\}, \quad \{(V_\beta, \phi_\beta | \beta \in J\} \cup \{(V'_\gamma, \eta_\gamma | \beta \in J'\}$$

by look at the local coordinate systems. Thus f is smooth with respect to the atlases

$$\{(U'_\delta, \theta_\delta | \alpha \in I'\}, \ \{(V'_\gamma, \eta_\gamma | \beta \in J'\}.$$

□

Thus the definition of smooth maps between two smooth manifolds is independent of choice of atlas.

Definition 2.9. A smooth map $f\colon M \to N$ is called a *diffeomorphism* if f is one-to-one and onto, and if the inverse $f^{-1}\colon N \to M$ is also smooth.

Definition 2.10. Let M be a smooth n-manifold, possibly with boundary. A subset X is called a *properly embedded submanifold* of dimension $k \leq n$ if X is a closed in M and, for each $P \in X$, there exists a chart (U, ϕ) about P in M such that

$$\phi(U \cap X) = \phi(U) \cap \mathbb{R}^k_+,$$

where $\mathbb{R}^k_+ \subseteq \mathbb{R}^n_+$ is the standard inclusion.

Note. In the above definition, the collection $\{(U \cap X, \phi|_{U\cap X})\}$ is an atlas for making X to a smooth k-manifold with boundary $\partial X = X \cap \partial M$.

If $\partial M = \emptyset$, by dropping the requirement that X is a closed subset but keeping the requirement on local charts, X is called simply a *submanifold* of M.

2.4. **Tangent Space.** Let S be an open region of $\mathbb{R}^n$. Recall that, for $P \in S$, the tangent space $T_P(S)$ is just the n-dimensional vector space by putting the origin at P. Let T be an open region of $\mathbb{R}^m$ and let $f = (f_1, \ldots, f_m)\colon S \to T$ be a smooth map. Then f induces a *linear transformation*

$$Tf\colon T_P(S) \to T_{f(P)}(T)$$

given by

$$Tf(\vec{v}) = \left(\frac{\partial f_i}{\partial x^j}\right)_{m\times n} \cdot \begin{pmatrix} v^1 \\ v^2 \\ \cdots \\ v^n \end{pmatrix}_{n\times 1} = \begin{pmatrix} v^1\partial_1(f_1) + v^2\partial_2(f_1) + \cdots + v^n\partial_n(f_1) \\ v^1\partial_1(f_2) + v^2\partial_2(f_2) + \cdots + v^n\partial_n(f_2) \\ \cdots\cdots\cdots\cdots \\ v^1\partial_1(f_m) + v^2\partial_2(f_m) + \cdots + v^n\partial_n(f_m) \end{pmatrix},$$

namely Tf is obtained by taking directional derivatives of $(f_1, \ldots, f_m)$ along vector $\vec{v}$ for any $\vec{v} \in T_P(S)$.

Now we are going to define the *tangent space* to a (differentiable) manifold M at a point P as follows:

First we consider the set

$$\mathcal{T}_P = \{(U, \phi, \vec{v}) \mid P \in U, \quad (U, \phi) \text{ is a chart } \ \vec{v} \in T(\phi(P))(\phi(U))\}.$$

The point is that there are possibly many charts around P. Each chart creates an n-dimension vector space. So we need to define an *equivalence relation* in $\mathcal{T}_P$ such that, $\mathcal{T}_P$ modulo these relations is only one copy of n-dimensional vector space which is also independent on the choice of charts.

Let $(U, \phi, \vec{v})$ and $(V, \psi, \vec{w})$ be two elements in $\mathcal{T}_P$. That is (U, ϕ) and (V, ψ) are two charts with $P \in U$ and $P \in V$. By the definition,

$$\psi \circ \phi^{-1} \colon \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

is diffeomorphism and so it induces an isomorphism of vector spaces

$$T(\psi \circ \phi^{-1}) \colon T_{\phi(P)}(\phi(U \cap V)) \longrightarrow T_{\psi(P)}(\psi(U \cap V)).$$

Now $(U, \phi, \vec{v})$ is called equivalent to $(V, \psi, \vec{w})$, denoted by $(U, \phi, \vec{v}) \sim (V, \psi, \vec{w})$, if

$$T(\psi \circ \phi^{-1})(\vec{v}) = \vec{w}.$$

Define $T_P(M)$ to be the quotient

$$T_P(M) = \mathcal{T}_P / \sim .$$

Exercise 2.1. Let M be a differentiable n-manifold and let P be any point in M. Prove that $T_P(M)$ is an n-dimensional vector space. [Hint: Fixed a chart (U, ϕ) and defined

$$a(U, \phi, \vec{v}) + b(U, \phi, \vec{w}) := (U, \phi, a\vec{v} + b\vec{w}).$$

Now given any $(V, \psi, \vec{x}), (\tilde{V}, \tilde{\psi}, \vec{y}) \in \mathcal{T}_P$, consider the map

$$\phi \circ \psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V) \qquad \phi \circ \tilde{\psi}^{-1} \colon \tilde{\psi}(U \cap \tilde{V}) \to \phi(U \cap \tilde{V})$$

and define

$$a(V, \psi, \vec{x}) + b(\tilde{V}, \tilde{\psi}, \vec{y}) = (U, \phi, aT(\phi \circ \psi^{-1})(\vec{x}) + bT(\phi \circ \tilde{\psi}^{-1})(\vec{y})).$$

Then prove that this operation gives a well-defined vector space structure on T_P, that is, independent on the equivalence relation.]

The tangent space $T_P(M)$, as a vector space, can be described as follows: given any chart (U, ϕ) with $P \in U$, there is a unique isomorphism

$$T_\phi \colon T_P(M) \to T_{\phi(P)}(\phi(U)).$$

by choosing $(U, \phi, \vec{v})$ as representatives for its equivalence class. If (V, ψ) is another chart with $P \in V$, then there is a commutative diagram

$$\begin{array}{ccc} T_P(M) & \xrightarrow[\cong]{T_\phi} & T_{\phi(P)}(\phi(U \cap V)) \\ \Vert & & \downarrow T(\psi \circ \phi^{-1}) \\ T_P(M) & \xrightarrow[\cong]{T_\psi} & T_{\psi(P)}(\psi(U \cap V)), \end{array} \tag{1}$$

where $T(\psi \circ \phi^{-1})$ is the linear isomorphism induced by the Jacobian matrix of the differentiable map $\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$.

Exercise 2.2. Let $f\colon M \to N$ be a smooth map, where M and N need not to have the same dimension. Prove that there is a unique linear transformation

$$Tf\colon T_P(M) \longrightarrow T_{f(P)}(N)$$

such that the diagram

$$\begin{array}{ccc}
T_P(M) & \xrightarrow[\cong]{T_\phi} & T_{\phi(P)}(\phi(U)) \\
\Big\downarrow{\scriptstyle Tf} & & \Big\downarrow{\scriptstyle T(\psi\circ f\circ\phi^{-1})} \\
T_{f(P)}(N) & \xrightarrow[\cong]{T_\psi} & T_{\psi(f(P))}(\psi(V))
\end{array}$$

commutes for any chart (U,ϕ) with $P \in U$ and any chart (V,ψ) with $f(P) \in V$. [First fix a choice of (U,ϕ) with $P \in U$ and (V,ψ) with $f(P) \in V$, the linear transformation Tf is uniquely defined by the above diagram. Then use Diagram (1) to check that Tf is independent on choices of charts.

2.5. **Immersions.** A smooth map $f\colon M \to N$ is called *immersion* at P if the linear transformation

$$Tf\colon T_P(M) \to T_{f(P)}(M)$$

is injective.

Theorem 2.11 (Local Immersion Theorem)**.** *Suppose that $f\colon M^m \to N^n$ is immersion at P. Then there exist charts (U,ϕ) about P and (V,ψ) about $f(P)$ such that the diagram*

$$\begin{array}{ccc}
U & \xrightarrow{\qquad f|_U \qquad} & V \\
{\scriptstyle \phi(P)=0}\Big\downarrow{\scriptstyle\phi} & & {\scriptstyle \psi(f(P))=0}\Big\downarrow{\scriptstyle\psi} \\
\mathbb{R}^m & \xhookrightarrow{\text{canonical coordinate inclusion}} & \mathbb{R}^n
\end{array}$$

commutes.

Proof. We may assume that $\phi(P) = 0$ and $\psi(f(P)) = 0$. (Otherwise replacing ϕ and ψ by $\phi - \phi(P)$ and $\psi - \psi(f(P))$, respectively.)

Consider the commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{f|_U} & V \\
\Big\downarrow{\scriptstyle\phi} & & \Big\downarrow{\scriptstyle\psi} \\
\phi(U) & \xrightarrow{g=\psi\circ f\circ\phi^{-1}} & \psi(V) \\
\Big\downarrow & & \Big\downarrow \\
\mathbb{R}^m & & \mathbb{R}^n.
\end{array}$$

By the assumption,

$$Tg\colon T_0(\phi(U)) \longrightarrow T_0(\psi(V))$$

is an injective linear transformation and so

$$\operatorname{rank}(Tg) = m$$

at the origin. The matric for Tg is

$$\begin{pmatrix} \frac{\partial g^1}{\partial x^1} & \frac{\partial g^1}{\partial x^2} & \cdots & \frac{\partial g^1}{\partial x^m} \\ \frac{\partial g^2}{\partial x^1} & \frac{\partial g^2}{\partial x^2} & \cdots & \frac{\partial g^2}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g^m}{\partial x^1} & \frac{\partial g^m}{\partial x^2} & \cdots & \frac{\partial g^m}{\partial x^m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g^n}{\partial x^1} & \frac{\partial g^n}{\partial x^2} & \cdots & \frac{\partial g^n}{\partial x^m} \end{pmatrix}. \tag{2}$$

By changing basis of $\mathbb{R}^n$ (corresponding to change the rows), we may assume that the first m rows forms an invertible matrix $A_{m\times m}$ at the origin.

Define a function

$$h = (h^1, h^2, \ldots, h^n)\colon \phi(U) \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$$

by setting

$$h^i(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = g^i(x^1, \ldots, x^m)$$

for $1 \leq i \leq m$ and

$$h^i(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = g^i(x^1, \ldots, x^m) + x^i$$

for $m+1 \leq i \leq n$. Then Jacobian matrix of h is

$$\begin{pmatrix} A_{m\times m} & 0_{m\times(n-m)} \\ B_{(n-m)\times m} & I_{n-m} \end{pmatrix},$$

where B is taken from $(m+1)$-st row to n-th row in the matrix (2). Thus the Jacobian of h is not zero at the origin. By the Inverse Mapping Theorem, h is an diffeomorphism in a small neighborhood of the origin. It follows that there exist open neighborhoods $\tilde{U} \subseteq U$ of P and $\tilde{V} \subseteq V$ of $f(P)$ such that the following diagram commutes

$$
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\quad f|_{\tilde{U}} \quad} & \tilde{V} \\
\downarrow \phi|_{\tilde{U}} & & \downarrow \psi|_{\tilde{V}} \\
\phi(\tilde{U}) & \xrightarrow{\quad g=\psi\circ f\circ\phi^{-1} \quad} & \psi(\tilde{V}) \\
\| & & \cong \downarrow h^{-1} \\
\phi(\tilde{U})\times 0 & \hookrightarrow & \phi(\tilde{U})\times U_2 \\
\downarrow & & \downarrow \\
\mathbb{R}^m=\mathbb{R}^m\times 0 & \hookrightarrow & \mathbb{R}^n,
\end{array}
$$

where U_2 is a small neighborhood of the origin in $\mathbb{R}^{n-m}$. □

Theorem 2.12. *Let $f\colon M\to N$ be a smooth map. Suppose that*

1) *f is immersion at every point $P\in M$,*
2) *f is one-to-one and*
3) *$f\colon M\to f(M)$ is a homeomorphism.*

Then $f(M)$ is a smooth submanifold of M and $f\colon M\to f(M)$ is a diffeomorphism.

Note. In Condition 3, we need that if U is an open subset of M, then there is an open subset V of N such that $V\cap f(M)=f(U)$.

Proof. For any point P in M, we can choose the charts as in Theorem 2.11. By Condition 3, $f(U)$ is an open subset of $f(M)$. The charts $\{(f(U),\psi|_{f(U)})\}$ gives an atlas for $f(M)$ such that $f(M)$ is a submanifold of M. Now $f\colon M\to f(M)$ is a diffeomorphism because it is locally diffeomorphism and the inverse exists. □

Condition 3 is important in this theorem, namely an injective immersion need not give a diffeomorphism with its image. (Construct an example for this.) An injective immersion satisfying condition 3 is called an *embedding.*

2.6. **Submersions.** A smooth map $f\colon M\to N$ is called *submersion* at P if the linear transformation

$$Tf\colon T_P(M)\to T_{f(P)}(M)$$

is surjective.

Theorem 2.13 (Local Submersion Theorem)**.** *Suppose that $f\colon M^m \to N^n$ is submersion at P. Then there exist charts (U,ϕ) about P and (V,ψ) about $f(P)$ such that the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\quad f|_U \quad} & V \\ \phi(P)=0 \Big\downarrow \phi & & \psi(f(P))=0 \Big\downarrow \psi \\ \mathbb{R}^m & \xhookrightarrow{\text{canonical coordinate proj.}} & \mathbb{R}^n \end{array}$$

commutes.

For a smooth map of manifolds $f\colon M \to N$, a point $Q \in N$ is called *regular* if $Tf\colon T_P(M) \to T_Q(N)$ is surjective for every $P \in f^{-1}(Q)$, the pre-image of Q.

Theorem 2.14 (Pre-image Theorem)**.** *Let $f\colon M \to N$ be a smooth map and let $Q \in N$ such that $f^{-1}(Q)$ is not empty. Suppose that Q is regular. Then $f^{-1}(Q)$ is a submanifold of M with* $\dim f^{-1}(Q) = \dim M - \dim N$.

Proof. From the above theorem, for any $P \in f^{-1}(Q)$,

$$\phi|_{f^{-1}(Q)}\colon f^{-1}(Q) \cap U \longrightarrow \mathbb{R}^{m-n}$$

gives a chart about P. □

Let Z be a submanifold of N. A smooth map $f\colon M \to N$ is said to be *transversal* to Z if

$$\mathrm{Im}(Tf\colon T_P(M) \to T_{f(P)}(N)) + T_{f(P)}(Z) = T_{f(P)}(N)$$

for every $x \in f^{-1}(Z)$.

Theorem 2.15. *If a smooth map $f\colon M \to N$ is transversal to a submanifold $Z \subseteq N$, then $f^{-1}(Z)$ is a submanifold of M. Moreover the codimension of $f^{-1}(Z)$ in M equals to the codimension of Z in N.*

Proof. Given $P \in f^{-1}(Z)$, since Z is a submanifold, there is a chart (V,ψ) of N about $f(P)$ such that $V = V_1 \times V_2$ with $V_1 = V \cap Z$ and $(V_1, \psi|_{V_1})$ is a chart of Z about $f(P)$. By the assumption, the composite

$$f^{-1}(V) \xrightarrow{f|_{f^{-1}(V)}} V \xrightarrow{\text{proj.}} V_2$$

is submersion. By the Pre-image Theorem, $f^{-1}(V) \cap f^{-1}(Z)$ is a submanifold of the open subset $f^{-1}(V)$ of M and so there is a chart about P such that Z is a submanifold of M.

With respect to the assertion about the codimensions,

$$\mathrm{codim}(f^{-1}(Z)) = \dim V_2 = \mathrm{codim}(Z).$$

□

Consider the special case that both M and Z are submanifolds of N. Then the transversal condition is

$$T_P(M) + T_P(Z) = T_P(N)$$

for any $P \in M \cap Z$.

Corollary 2.16. *The intersection of two transversal submanifolds of N is again a submanifold. Moreover*

$$\operatorname{codim}(M \cap Z) = \operatorname{codim}(M) + \operatorname{codim}(Z)$$

in N.

3. Examples of Manifolds

3.1. **Open Stiefel Manifolds and Grassmann Manifolds.** The *open Stiefel manifold* is the space of k-tuples of linearly independent vectors in $\mathbb{R}^n$:

$$\tilde{V}_{k,n} = \{(\vec{v}_1, \ldots, \vec{v}_k)^T \mid \vec{v}_i \in \mathbb{R}^n, \ \{\vec{v}_1, \ldots, \vec{v}_k\} \text{ linearly independent}\},$$

where $\tilde{V}_{k,n}$ is considered as the subspace of $k \times n$ matrixes $M(k,n) \cong \mathbb{R}^{kn}$. Since $\tilde{V}_{k,n}$ is an open subset of $M(k,n) = \mathbb{R}^{kn}$, $\tilde{V}_{k,n}$ is an open submanifold of $\mathbb{R}^{kn}$.

The *Grassmann manifold* $G_{k,n}$ is the set of k-dimensional subspaces of $\mathbb{R}^n$, that is, all k-planes through the origin. Let

$$\pi \colon \tilde{V}_{k,n} \to G_{k,n}$$

be the quotient by sending k-tuples of linearly independent vectors to the k-planes spanned by k vectors. The topology in $G_{k,n}$ is given by quotient topology of π, namely, U is an open set of $G_{k,n}$ if and only if $\pi^{-1}(U)$ is open in $\tilde{V}_{k,n}$.

For $(\vec{v}_1, \ldots, \vec{v}_k)^T \in \tilde{V}_{k,n}$, write $\langle \vec{v}_1, \ldots, \vec{v}_k \rangle$ for the k-plane spanned by $\vec{v}_1, \ldots, \vec{v}_k$. Observe that two k-tuples $(\vec{v}_1, \ldots, \vec{v}_k)^T$ and $(\vec{w}_1, \ldots, \vec{w}_k)^T$ spanned the same k-plane if and only if each of them is basis for the common plane, if and only if there is nonsingular $k \times k$ matrix P such that

$$P(\vec{v}_1, \ldots, \vec{v}_k)^T = (\vec{w}_1, \ldots, \vec{w}_k)^T.$$

This gives the identification rule for the Grassmann manifold $G_{k,n}$. Let $\mathrm{GL}_k(\mathbb{R})$ be the space of general linear groups on $\mathbb{R}^k$, that is, $\mathrm{GL}_k(\mathbb{R})$ consists of $k \times k$ nonsingular matrixes, which is an open subset of $M(k,k) = \mathbb{R}^{k^2}$. Then $G_{k,n}$ is the quotient of $\tilde{V}_{k,n}$ by the action of $\mathrm{GL}_k(\mathbb{R})$.

First we prove that $G_{k,n}$ is Hausdorff. If $k = n$, then $G_{n,n}$ is only one point. So we assume that $k < n$. Given an k-plane X and $\vec{w} \in \mathbb{R}^n$, let $\rho_{\vec{w}}$ be the square of the Euclidian distance from $\vec{w}$ to X. Let $\{e_1, \ldots, e_k\}$ be the orthogonal basis for X, then

$$\rho_{\vec{w}}(X) = \vec{w} \cdot \vec{w} - \sum_{j=1}^{k} (\vec{w} \cdot e_j)^2.$$

Fixing any $\vec{w} \in \mathbb{R}^n$, we obtain the continuous map

$$\rho_{\vec{w}} \colon G_{k,n} \longrightarrow \mathbb{R}$$

because $\rho_{\vec{w}} \circ \pi \colon \tilde{V}_{k,n} \to \mathbb{R}$ is continuous and $G_{k,n}$ given by the quotient topology. (Here we use the property of quotient topology that any function f from the quotient space $G_{k,n}$ to any space is continuous if and only if $f \circ \pi$ from $\tilde{V}_{k,n}$ to that space is continuous.) Given any two distinct points X and Y in $G_{k,n}$, we can choose a $\vec{w}$ such that $\rho_{\vec{w}}(X) \neq \rho_{\vec{w}}(Y)$. Let V_1 and V_2 be disjoint open subsets of $\mathbb{R}$ such that $\rho_{\vec{w}}(X) \in V_1$ and $\rho_{\vec{w}}(Y) \in V_2$. Then $\rho_{\vec{w}}^{-1}(V_1)$ and $\rho_{\vec{w}}^{-1}(V_2)$ are two open subset of $G_{k,n}$ that separate X and Y, and so $G_{k,n}$ is Hausdorff.

Now we check that $G_{k,n}$ is manifold of dimension $k(n-k)$ by showing that, for any X in $G_{k,n}$, there is an open neighborhood U_X of α such that $U_X \cong \mathbb{R}^{k(n-k)}$.

Let $X \in G_{k,n}$ be spanned by $(\vec{v}_1, \ldots, \vec{v}_k)^T$. There exists a nonsingular $n \times n$ matrix Q such that

$$(\vec{v}_1, \ldots, \vec{v}_k)^T = (I_k, 0)Q,$$

where I_k is the unit $k \times k$-matrix. Fixing Q, define

$$X_\alpha = \{(P_k, B_{k,n-k})Q \mid \det(P_k) \neq 0, B_{k,n-k} \in M(k, n-k)\} \subseteq \tilde{V}_{k,n}.$$

Then E_X is an open subset of $\tilde{V}_{k,n}$. Let $U_X = \pi(E_X) \subseteq G_{k,n}$. Since

$$\pi^{-1}(U_X) = E_X$$

is open in $\tilde{V}_{k,n}$, U_X is open in $G_{k,n}$ with $X \in U_X$. From the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}_k(\mathbb{R}) \times M(k, n-k) & \xrightarrow[\cong]{(P,A) \mapsto (P, PA)Q} & E_X \\ \Big\downarrow \text{proj.} & & \Big\downarrow \pi \\ M(k, n-k) & \xrightarrow[\phi_X^{-1}]{A \mapsto \langle (I_k, A)Q \rangle} & U_X, \end{array}$$

U_X is homeomorphic to $M(k, n-k) = \mathbb{R}^{k(n-k)}$ and so $G_{k,n}$ is a (topological) manifold.

For checking that $G_{k,n}$ is a smooth manifold, let X and $Y \in G_{k,n}$ be spanned by $(\vec{v}_1, \ldots, \vec{v}_k)^T$ and $(\vec{w}_1, \ldots, \vec{w}_k)^T$, respectively. There exists nonsingular $n \times n$ matrixes Q and $\tilde{Q}$ such that

$$(\vec{v}_1, \ldots, \vec{v}_k)^T = (I_k, 0)Q, \quad (\vec{w}_1, \ldots, \vec{w}_k)^T = (I_k, 0)\tilde{Q}.$$

Consider the maps:

$$M(k, n-k) \xrightarrow[\phi_X^{-1}]{} U_X \qquad A \mapsto \langle (I_k, A)Q \rangle$$

$$M(k, n-k) \xrightarrow[\phi_Y^{-1}]{} U_Y \qquad A \mapsto \langle (I_k, A)\tilde{Q} \rangle.$$

If $Z \in U_X \cap U_Y$, then

$$Z = \langle (I_k, A_Z)Q \rangle = \langle (I_k, B_Z)\tilde{Q} \rangle$$

for unique $A, B \in M(k, n-k)$. It follows that there is a nonsingular $k \times k$ matrix P such that

$$(I_k, B_Z)\tilde{Q} = P(I_k, A_Z)Q \Leftrightarrow (I_k, B_Z) = P(I_k, A_Z)Q\tilde{Q}^{-1}.$$

Let

$$T = Q\tilde{Q}^{-1} = \begin{pmatrix} T_{11} & T_{12} \\ & \\ T_{21} & T_{22} \end{pmatrix}.$$

Then

$$(I_k, B_Z) = (P, PA_Z)T = (PT_{11} + PA_ZT_{21}, PT_{12} + PA_ZT_{22})$$

$$\begin{cases} I_k = P(T_{11} + A_ZT_{21}) \\ \\ B_Z = P(T_{12} + A_ZT_{22}). \end{cases}$$

It follows that

$$Z \in U_X \cap U_Y \text{ if and only if } \det(T_{11} + A_ZT_{21}) \neq 0 \text{ (that is, } T_{11} + A_ZT_{21} \text{ is invertible).}$$

From the above, the composite

$$\phi_X(U_X\cap U_Y)\xrightarrow{\phi_X^{-1}} U_X\cap U_Y \xrightarrow{\phi_Y} M(k,n)$$

is given by

$$A\mapsto (T_{11}+AT_{21})^{-1}(T_{12}+AT_{22}),$$

which is smooth. Thus $G_{k,n}$ is a smooth manifold.

As a special case, $G_{1,n}$ is the space of lines (through the origin) of $\mathbb{R}^n$, which is also called *projective space* denoted by $\mathbb{RP}^{n-1}$. From the above, $\mathbb{RP}^{n-1}$ is a manifold of dimension $n-1$.

3.2. **Stiefel Manifold.** The *Stiefel manifold*, denoted by $V_{k,n}$, is defined to be the set of k orthogonal unit vectors in $\mathbb{R}^n$ with topology given as a subspace of $\tilde{V}_{k,n}\subseteq M(k,n)$. Thus

$$V_{k,n}=\{A\in M(k,n)\mid A\cdot A^T=I_k\}.$$

We prove that $V_{k,n}$ is a smooth submanifold of $M(k,n)$ by using Pre-image Theorem.

Let $S(k)$ be the space of symmetric matrixes. Then $S(k)\cong\mathbb{R}^{\frac{(k+1)k}{2}}$ is a smooth manifold of dimension. Consider the map

$$f\colon M(k,n)\to S(k)\qquad A\mapsto AA^T.$$

For any $A\in M(k,n)$, $Tf_A\colon T_A(M(k,n))\to T_{f(A)}(S(k))$ is given by setting $Tf_A(B)$ is the directional derivative along B for any $B\in T_A(M(k,n))$, that is,

$$\begin{aligned}Tf_A(B)&=\lim_{s\to0}\frac{f(A+sB)-f(A)}{s}\\&=\lim_{s\to0}\frac{(A+sB)(A+sB)^T-AA^T}{s}\\&=\lim_{s\to0}\frac{AA^T+sAB^T+sBA^T+s^2BB^T-AA^T}{s}=AB^T+BA^T.\end{aligned}$$

We check that $Tf_A\colon T_A(M(k,n))\to T_{f(A)}(S(k))$ is surjective for any $A\in f^{-1}(I_k)$.

By the identification of $M(k,n)$ and $S(k)$ with Euclidian spaces, $T_A(M(k,n))=M(k,n)$ and $T_{f(A)}(S(k))=S(k))$. Let $A\in f^{-1}(I_k)$ and let $C\in T_{f(A)}(S(k))$. Define

$$B=\frac{1}{2}CA\in T_A(M(k,n)).$$

Then

$$Tf_A(B)=AB^T+BA^T=\frac12AA^TC^T+\frac12CAA^T\overset{AA^T=I_k}{=\!=\!=}\frac12C^T+\frac12C\overset{C=C^T}{=\!=\!=}C.$$

Thus $Tf\colon T_A(M(k,n))\to T_{f(A)}(S(k))$ is onto and so I_k is a regular value of f. Thus, by Pre-image Theorem, $V_{k,n}=f^{-1}(I_k)$ is a smooth submanifold of $M(k,n)$ of dimension

$$kn-\frac{(k+1)k}{2}=\frac{k(2n-k-1)}{2}.$$

Special Cases: When $k=n$, then $V_{n,n}=O(n)$ the orthogonal group. From the above, $O(n)$ is a (smooth) manifold of dimension $\frac{n(n-1)}{2}$. (**Note.** $O(n)$ is a Lie group, namely, a smooth manifold plus a topological group such that the multiplication and inverse are smooth.)

When $k=1$, then $V_{1,n}=S^{n-1}$ which is manifold of dimension $n-1$.

When $k=n-1$, then $V_{n-1,n}$ is a manifold of dimension $\frac{(n-1)n}{2}$. One can check that

$$V_{n-1,n}\cong SO(n)$$

the subgroup of $O(n)$ with determinant 1. In general case, $V_{k,n} = O(n)/O(n-k)$.

As a space, $V_{k,n}$ is compact. This follows from that $V_{k,n}$ is a closed subspace of the k-fold Cartesian product of S^{n-1} because $V_{k,n}$ is given by k unit vectors $(\vec{v}_1, \ldots, \vec{v}_k)^T$ in $\mathbb{R}^n$ that are solutions to $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$, and the fact that any closed subspace of compact Hausdorff space is compact. The composite

$$V_{k,n} \hookrightarrow \tilde{V}_{k,n} \xrightarrow{\pi} G_{k,n}$$

is onto and so the Grassmann manifold $G_{k,n}$ is also compact. Moreover the above composite is a smooth map because π is smooth and $V_{k,n}$ is a submanifold. This gives the diagram

$$\begin{array}{ccccc} V_{k,n} & \xrightarrow[\hookrightarrow]{\text{submanifold}} & M(k,n) & \xrightarrow[A \mapsto AA^T]{\text{submersion at } I_k} & S(k) \\ \Big\downarrow{\scriptstyle \text{smooth}} & & & & \\ G_{k,n} & & & & \end{array}$$

Note. By the construction, $G_{k,n}$ is the quotient of $V_{k,n}$ by the action of $O(k)$. This gives identifications:

$$G_{k,n} = V_{k,n}/O(k) = O(n)/(O(k) \times O(n-k)).$$

4. Fibre Bundles and Vector Bundles

4.1. **Fibre Bundles.** A *bundle* means a triple (E, p, B), where $p\colon E \to B$ is a (continuous) map. The space B is called the *base space*, the space E is called the *total space*, and the map p is called the *projection* of the bundle. For each $b \in B$, $p^{-1}(b)$ is called the *fibre* of the bundle over $b \in B$.

Intuitively, a bundle can be thought as a union of fibres $f^{-1}(b)$ for $b \in B$ parameterized by B and *glued together* by the topology of the space E. Usually a Greek letter ($\xi, \eta, \zeta, \lambda$, etc) is used to denote a bundle; then $E(\xi)$ denotes the total space of ξ, and $B(\xi)$ denotes the base space of ξ.

A *morphism* of bundles $(\phi, \bar{\phi})\colon \xi \to \xi'$ is a pair of (continuous) maps $\phi\colon E(\xi) \to E(\xi')$ and $\bar{\phi}\colon B(\xi) \to B(\xi')$ such that the diagram

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\phi} & E(\xi') \\ \Big\downarrow{\scriptstyle p(\xi)} & & \Big\downarrow{\scriptstyle p(\xi')} \\ B(\xi) & \xrightarrow{\bar{\phi}} & B(\xi') \end{array}$$

commutes.

The trivial bundle is the projection of the Cartesian product:

$$p\colon B \times F \to B, \qquad (x, y) \mapsto x.$$

Roughly speaking, a *fibre bundle* $p\colon E \to B$ is a "locally trivial" bundle with a "fixed fibre" F. More precisely, for any $x \in B$, there exists an open neighborhood U of x such that $p^{-1}(U)$ is a trivial bundle, in other words, there is a homeomorphism $\phi_U \colon p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi_x} & p^{-1}(U) \\ \Big\downarrow \pi_1 & & \Big\downarrow p \\ U & =\!=\!=\!= & U \end{array}$$

commutes, that is, $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$.

Similar to manifolds, we can use "chart" to describe fibre bundles. A *chart* (U, ϕ) for a bundle $p \colon E \to B$ is (1) an open set U of B and (2) a homeomorphism $\phi \colon U \times F \to p^{-1}(U)$ such that $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$. An *atlas* is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ is an open covering of B.

Proposition 4.1. *A bundle $p \colon E \to B$ is a fibre bundle with fibre F if and only if it has an atlas.*

Proof. Suppose that $p \colon E \to B$ is a fibre bundle. Then the collection $\{(U(x), \phi_x) | x \in B\}$ is an atlas.

Conversely suppose that $p \colon E \to B$ has an atlas. For any $x \in B$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x with the property that $p|_{p^{-1}(U)} \colon p^{-1}(U_\alpha) \to U_\alpha$ is a trivial bundle. Thus $p \colon E \to B$ is a fibre bundle. □

Let ξ be a fibre bundle with fibre F and an atlas $\{(U_\alpha, \phi_\alpha)\}$. The composite

$$\phi_\alpha^{-1} \circ \phi_\beta \colon (U_\alpha \cap U_\beta) \times F \xrightarrow{\phi_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} (U_\alpha \cap U_\beta) \times F$$

has the property that

$$\phi_\alpha^{-1} \circ \phi_\beta(x, y) = (x, g_{\alpha\beta}(x, y))$$

for any $x \in U_\alpha \cap U_\beta$ and $y \in F$. Consider the continuous map $g_{\alpha\beta} \colon U_{\alpha\beta} \times F \to F$. Fixing any x, $g_{\alpha\beta}(x, -) \colon F \to F$, $y \mapsto g_{\alpha\beta}(x, y)$ is a homeomorphism with inverse given by $g_{\beta\alpha}(x, -)$. This gives a *transition function*

$$g_{\alpha\beta} \colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F, F),$$

where $\mathrm{Homeo}(F, F)$ is the group of all homeomorphisms from F to F.

Exercise 4.1. *Prove that the transition functions $\{g_{\alpha\beta}\}$ satisfy the following equation*

$$g_{\alpha\beta}(x) \circ g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{3}$$

By choosing $\alpha = \beta = \gamma$, $g_{\alpha\alpha}(x) \circ g_{\alpha\alpha}(x) = g_{\alpha\alpha}(x)$ and so

$$g_{\alpha\alpha}(x) = x \qquad x \in U_\alpha \tag{4}$$

.

By choosing $\alpha = \gamma$, $g_{\alpha\beta}(x) \circ g_{\beta\alpha}(x) = g_{\alpha\alpha}(x) = x$ and so

$$g_{\beta\alpha}(x) = g_{\alpha\beta}(x)^{-1} \qquad x \in U_\alpha \cap U_\beta. \tag{5}$$

We need to introduce a *topology* on $\mathrm{Homeo}(F, F)$ such that the transition functions $g_{\alpha\beta}$ are continuous. The topology on $\mathrm{Homeo}(F, F)$ is given by *compact-open topology* briefly reviewed as follows:

Let X and Y be topological spaces. Let $\mathrm{Map}(X, Y)$ denote the set of all continuous maps from X to Y. Given any compact set K of X and any open set U of Y, let

$$W_{K,U} = \{f \in \mathrm{Map}(X, Y) \mid f(K) \subseteq U\}.$$

Then the *compact-open topology* is generated by $W_{K,U}$, that is, an open set in $\mathrm{Map}(X,Y)$ is an arbitrary union of a finite intersection of subsets with the form $W_{K,U}$.

$\mathrm{Map}(F,F)$ be the set of all continuous maps from F to F with compact open topology. Then $\mathrm{Homeo}(F,F)$ is a subset of $\mathrm{Map}(F,F)$ with subspace topology.

Proposition 4.2. *If* $\mathrm{Homeo}(F,F)$ *has the compact-open topology, then the transition functions* $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\mathrm{Homeo}(F,F)$ *are continuous.*

Proof. Given $W_{K,U}$, we show that $g^{-1}_{\alpha\beta}(W_{K,U})$ is open in $U_\alpha\cap U_\beta$. Let $x_0\in U_\alpha\cap U_\beta$ such that $g_{\alpha\beta}(x_0)\in W_{(K,U)}$. We need to show that there is a neighborhood V is x_0 such that $g_{\alpha\beta}(V)\subseteq W_{K,U}$, or $g_{\alpha\beta}(V\times K)\subseteq U$. Since U is open and $g_{\alpha\beta}\colon (U_\alpha\cap U_\beta)\times F\to F$ is continuous, $g^{-1}(U)$ is an open set of $(U_\alpha\cap U_\beta)\times F$ with $x_0\times K\subseteq g^{-1}_{\alpha\beta}(U)$. For each $y\in K$, there exist open neighborhoods $V(y)$ of x and $N(y)$ of y such that $V(x)\times N(y)\subseteq g^{-1}_{\alpha\beta}(U)$. Since $\{N(y)\mid y\in K\}$ is an open cover of the compact set K, there is a finite cover $\{N(y_1),\ldots,N(y_n)\}$ of K. Let $V=\bigcap\limits_{j=1}^{n}V(y_j)$. Then $V\times K\subseteq g^{-1}_{\alpha\beta}(U)$ and so $g_{\alpha\beta}(V)\subseteq W_{K,U}$. □

Proposition 4.3. *If F regular and locally compact, then the composition and evaluation maps*

$$\mathrm{Homeo}(F,F)\times\mathrm{Homeo}(F,F)\longrightarrow\mathrm{Homeo}(F,F)\qquad (g,f)\mapsto f\circ g$$
$$\mathrm{Homeo}(F,F)\times F\longrightarrow F\qquad (f,y)\mapsto f(y)$$

are continuous.

Proof. Suppose that $f\circ g\in W_{K,U}$. Then $f(g(K))\subseteq U$, or $g(K)\subseteq f^{-1}(U)$, and the latter is open. Since F is regular and locally compact, there is an open set V such that

$$g(K)\subseteq V\subseteq\bar V\subseteq f^{-1}(U)$$

and the closure $\bar V$ is compact. If $g'\in W_{K,V}$ and $f'\in W_{\bar V,U}$, then $f'\circ g'\in W_{K,U}$. Thus $W_{K,V}$ and $W_{\bar V,U}$ are neighborhoods of g and f whose composition product lies in $W_{K,U}$. This implies that $\mathrm{Homeo}(F,F)\times\mathrm{Homeo}(F,F)\to\mathrm{Homeo}(F,F)$ is continuous.

Let U be an open set of F and let $f_0(y_0)\in U$ or $y_0\in f_0^{-1}(U)$. Since F is regular an locally compact, there is a neighborhood V of y_0 such that $\bar V$ is compact and $y_0\in V\subseteq\bar V\subseteq g_0^{-1}(U)$. If $g\in W_{\bar V,U}$ and $y\in V$, then $g(y)\in U$ and so the evaluation map $\mathrm{Homeo}(F,F)\times F\to F$ is continuous. □

Proposition 4.4. *If F is compact Hausdorff, then the inverse map*

$$\mathrm{Homeo}(F,F)\longrightarrow\mathrm{Homeo}(F,F)\qquad f\mapsto f^{-1}$$

is continuous.

Proof. Suppose that $g_0^{-1}\in W_{K,U}$. Then $g_0^{-1}(K)\subseteq U$ or $K\subseteq g_0(U)$. It follows that

$$F\smallsetminus K\supseteq F\smallsetminus g_0(U)=g_0(F\smallsetminus U)$$

because g_0 is a homeomorphism. Note that $F\smallsetminus U$ is compact, $F\smallsetminus K$ is open and $g_0\in W_{F\smallsetminus U,F\smallsetminus K}$. If $g\in W_{F\smallsetminus U,F\smallsetminus K}$, then, from the above arguments, $g^{-1}\in W_{K,U}$ and hence the result. □

Note. If F is regular and locally compact, then $\mathrm{Homeo}(F,F)$ is a topological monoid, namely compact-open topology only fails in the continuity of g^{-1}. A modification on compact-open topology eliminates this defect [1].

4.2. **G-Spaces and Principal G-Bundles.** Let G be a topological group and let X be a space. A right G-action on X means a(continuous) map $\mu\colon X\times G\to X, (x,g)\mapsto x\cdot g$ such that $x\cdot 1=x$ and $(x\cdot g)\cdot h=x\cdot(gh)$. In this case, we call X a (right) G-space. Let X and Y be (right) G-spaces. A continuous map $f\colon X\to Y$ is called a G-map if $f(x\cdot g)=f(x)\cdot g$ for any $x\in X$ and $g\in G$. Let X/G be the set of G-orbits xG, $x\in X$, with quotient topology.

Proposition 4.5. *Let X be a G-space.*

1) *For fixing any $g\in G$, the map $x\mapsto x\cdot g$ is a homeomorphism.*
2) *The projection $\pi\colon X\to X/G$ is an open map.*

Proof. (1). The inverse is given by $x\mapsto x\cdot g^{-1}$.

(2) If U is an open set of X,

$$\pi^{-1}(\pi(U))=\bigcup_{g\in G}U\cdot g$$

is open because it is a union of open sets, and so $\pi(U)$ is open by quotient topology. Thus π is an open map. □

We are going to find some conditions such that $\pi\colon X\to X/G$ has canonical fibre bundle structure with fibre G. Given any point $\bar{x}\in X/G$, choose $x\in X$ such that $\pi(x)=\bar{x}$. Then

$$\pi^{-1}(\bar{x})=\{x\cdot g\mid g\in G\}=G/H_x,$$

where $H_x=\{g\in G\mid x\cdot g=x\}$.

For having constant fibre G, we need to assume that the G-action on X is free, namely

$$x\cdot g=x\quad\Longrightarrow\quad g=1$$

for any $x\in X$. This is equivalent to the property that

$$x\cdot g=x\cdot h\quad\Longrightarrow\quad g=h$$

for any $x\in X$. In this case we call X a *free G-space.*

Since a fibre bundle is locally trivial (locally Cartesian product), there is always a local cross-section from the base space to the total space. Our second condition is that the projection $\pi\colon X\to X/G$ has local cross-sections. More precisely, for any $\bar{x}\in X/G$, there is an open neighborhood $U(\bar{x})$ with a continuous map $s_{\bar{x}}\colon U(\bar{x})\to X$ such that $\pi\circ s_{\bar{x}}=\mathrm{id}_{U(\bar{x})}$.

(**Note.** For every point $\bar{x}$, we can always choose a pre-image of π, the local cross-section means the pre-images can be chosen "continuously" in a neighborhood. This property depends on the topology structure of X and X/G.)

Assume that X is a (right) free G-space with local cross-sections to $\pi\colon X\to X/G$. Let $\bar{x}$ be any point in X/G. Let $U(\bar{x})$ be a neighborhood of $\bar{x}$ with a (continuous) crosse-section $s_{\bar{x}}\colon U(\bar{x})\to X$. Define

$$\phi_{\bar{x}}\colon U(\bar{x})\times G\longrightarrow\pi^{-1}(U(\bar{x}))\qquad(\bar{y},g)\longrightarrow s_{\bar{x}}(\bar{y})\cdot g$$

for any $y\in U(\bar{x})$.

Exercise 4.2. Let X be a (right) free G-space with local cross-sections to $\pi\colon X\to X/G$. Then the continuous map $\phi_{\bar{x}}\colon U(\bar{x})\times G\to\pi^{-1}(U(\bar{x}))$ is one-to-one and onto. □

We need to find the third condition such that $\phi_{\bar{x}}$ is a homeomorphism. Let

$$X^*=\{(x,x\cdot g)\mid x\in X,g\in G\}\subseteq X\times X.$$

A function

$$\tau\colon X^* \longrightarrow G$$

such that

$$x \cdot \tau(x, x') = x' \qquad \text{for all } (x, x') \in X$$

is called a *translation function.* (**Note.** If X is a free G-space, then translation function is unique because, for any $(x, x') \in X^*$, there is a unique $g \in G$ such that $x' = x \cdot g$, and so, by definition, $\tau(x, x') = g$.)

Proposition 4.6. *Let X be a (right) free G-space with local cross-sections to $\pi\colon X \to X/G$. Then the following statements are equivalent each other:*

1) *The translation function $\tau\colon X^* \to G$ is continuous.*
2) *For any $\bar{x} \in X/G$, the map $\phi_{\bar{x}}\colon U(\bar{x}) \times G \to \pi^{-1}(U(\bar{x}))$ is a homeomorphism.*
3) *There is an atlas $\{(U_\alpha, \phi_\alpha\}$ of X/G such that the homeomorphisms*

$$\phi_\alpha\colon U_\alpha \times G \longrightarrow \pi^{-1}(U_\alpha)$$

satisfy the condition $\phi_\alpha(\bar{y}, gh) = \phi_\alpha(\bar{y}, g) \cdot h$, that is ϕ_α is a homeomorphism of G-spaces.

Proof. (1) $\Longrightarrow$ (2). Consider the (continuous) map

$$\theta\colon \pi^{-1}(U(\bar{x})) \longrightarrow U(\bar{x}) \times G \qquad z \mapsto (\pi(z), \tau(s_{\bar{x}}(\pi(z)), z)).$$

Then

$$\theta \circ \phi_{\bar{x}}(\bar{y}, g) = \theta(s_{\bar{x}}(\bar{y}) \cdot g) = (\bar{y}, \tau(s_{\bar{x}}(\bar{y}), s_{\bar{x}}(\bar{y}) \cdot g)) = (\bar{y}, g),$$

$$\phi_{\bar{x}} \circ \theta(z) = \phi_{\bar{x}}(\pi(z), \tau(s_{\bar{x}}(\pi(z)), z)) = s_{\bar{x}}(\pi(z)) \cdot \tau(s_{\bar{x}}(\pi(z)), z) = z.$$

Thus $\phi_{\bar{x}}$ is a homeomorphism.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Note that the translation function is unique for free G-spaces. It suffices to show that the restriction

$$\tau(X)\colon X^* \cap (\pi^{-1}(U_\alpha) \times \pi^{-1}(U_\alpha)) = \left(\pi^{-1}(U_\alpha)\right)^* \longrightarrow G$$

is continuous. Consider the commutative diagram

$$\begin{array}{ccc}
(U_\alpha \times G)^* & \xrightarrow[\cong]{\phi_\alpha^*} & \left(\pi^{-1}(U_\alpha)\right)^* \\
\Big\downarrow{\scriptstyle \tau(U_\alpha \times G)} & & \Big\downarrow{\scriptstyle \tau(X)} \\
G & = & G.
\end{array}$$

Since

$$\tau(U_\alpha \times G)((\bar{y}, g), (\bar{y}, h)) = g^{-1}h$$

is continuous, the translation function restricted to $\left(\pi^{-1}(U_\alpha)\right)^*$

$$\tau(X) = \tau(U_\alpha \times G) \circ ((\phi_\alpha)^*)^{-1}$$

is continuous for each α and so $\tau(X)$ is continuous. □

Now we give the definition. A *principal G-bundle* is a free G-space X such that

$$\pi\colon X \to X/G$$

has local cross-sections and one of the (equivalent) conditions in Proposition 4.6 holds.

Example. Let Γ be a topological group and let G be a closed subgroup. Then the action of G on Γ given by $(a, g) \mapsto ag$ for $a \in \Gamma$ and $g \in G$ is free. Then translation function is given by $\tau(a, b) = a^{-1}b$, which is continuous. Thus $\Gamma \to \Gamma/G$ is principal G-bundle if and only if it has local cross-sections.

4.3. **The Associated Principal G-Bundles of Fibre Bundles.** We come back to look at fibre bundles ξ given by $p\colon E \to B$ with fibre F. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas and let

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F, F)$$

be the transition functions. A topological group G is called a *group of the bundle* ξ if

1) There is a group homomorphism
$$\theta\colon G \longrightarrow \mathrm{Homeo}(F, F).$$
2) There exists an atlas of ξ such that the transition functions $g_{\alpha\beta}$ lift to G via θ, that is, there is commutative diagram
$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & \mathrm{Homeo}(F, F) \\ \Vert & & \big\uparrow \theta \\ U_\alpha \cap U_\beta & \xrightarrow{\quad g_{\alpha\beta} \quad} & G, \end{array}$$
(where we use the same notation $g_{\alpha\beta}$.)
3) The transition functions
$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$$
are continuous.
4) The G-action on F via θ is continuous, that is, the composite
$$G \times F \xrightarrow{\theta \times \mathrm{id}_F} \mathrm{Homeo}(F, F) \times F \xrightarrow{\text{evaluation}} F$$
is continuous.

We write $\bar{\xi} = \{(U_\alpha, g_{\alpha\beta})\}$ for the set of transition functions to the atlas $\{(U_\alpha, \phi_\alpha)\}$.

Note. In Steenrod's definition [13, p.7], θ is assume to be a monomorphism (equivalently, the G-action on F is effective, that is, if $y \cdot g = y$ for all $y \in F$, then $g = 1$.).

We are going to construct a principal G-bundle $\pi\colon E^G \to B$. Then prove that the total space $E = F \times_G E^G$ and $p\colon E \to B$ can be obtained canonically from $\pi\colon E^G \to B$. In other words, all fibre bundles can obtained through principal G-bundles through this way. Also the topological group G plays an important role for fibre bundles. Namely, by choosing different topological groups G, we may get different properties for the fibre bundle ξ. For instance, if we can choose G to be trivial (that is, $g_{\alpha\beta}$ lifts to the trivial group), then fibre bundle is trivial. We will see that the bundle group G for n-dimensional vector bundles can be chosen as the general linear group $\mathrm{GL}_n(\mathbb{R})$. The vector bundle is *orientable* if and only if the transition functions can left to the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ matrices whose determinant is positive. If $n = 2m$, then $\mathrm{GL}_m(\mathbb{C}) \subseteq \mathrm{GL}_{2m}(\mathbb{R})$. The vector bundle admits (almost) complex structure if and only if the transition functions can left to $\mathrm{GL}_m(\mathbb{C})$. (For manifolds, one can consider the structure on the tangent bundles. For instance, an oriented manifold means its tangent bundle is oriented.)

Proposition 4.7. *If $\bar{\xi}$ is the set of transition functions for the space B and topological group G, then there is a principal G-bundle ξ^G given by*

$$\pi\colon E^G \longrightarrow B$$

and an atlas $\{(U_\alpha, \phi_\alpha)\}$ such that $\bar{\xi}$ is the set of transition functions to this atlas.

Proof. The proof is given by construction. Let

$$\bar{E} = \bigcup_\alpha U_\alpha \times G \times \alpha,$$

that is $\bar{E}$ is the disjoint union of $U_\alpha \times G$. Now define a relation on $\bar{E}$ by

$$(b, g, \alpha) \sim (b', g', \beta) \Longleftrightarrow b = b', g = g_{\alpha\beta}(b)g'.$$

This is an equivalence relation by Equations (3)-(5). Let $E^G = \bar{E}/\sim$ with quotient topology and let $\{b, g, \alpha\}$ for the class of (b, g, α) in E^G. Define $\pi\colon E^G \to B$ by

$$\pi\{b, g, \alpha\} = b,$$

then π is clearly well-defined (and so continuous). The right G-action on E^G is defined by

$$\{b, g, \alpha\} \cdot h = \{b, gh, \alpha\}.$$

This is well-defined (and so continuous) because if $(b', g', \beta) \sim (b, g, \alpha)$, then

$$(b', g'h, \beta) = (b, (g_{\alpha\beta}(b)g)h, \beta) = (b, g_{\alpha\beta}(b)(gh), \beta) \sim (b, gh, \alpha).$$

Define $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$ by setting

$$\phi_\alpha(b, g) = \{b, g, \alpha\},$$

then ϕ_α is continuous and satisfies $\pi \circ \phi_\alpha(b, g) = b$ and

$$\phi_\alpha(b, g) = \{b, 1 \cdot g, \alpha\} = \{b, 1, \alpha\} \cdot g$$

for $b \in U_\alpha$ and $g \in G$. The map ϕ_α is a homeomorphism because, for fixing α, the map

$$\coprod_\beta (U_\alpha \cap U_\beta) \times G \times \beta \longrightarrow U_\alpha \times G \qquad (b, g', \beta) \mapsto (b, g_{\alpha\beta}(b)g')$$

induces a map $\pi^{-1}(U_\alpha) \to U_\alpha \times G$ which the inverse of ϕ_α. Moreover,

$$\phi_\alpha(b, g_{\alpha\beta}(b)g) = \{b, g_{\alpha\beta}(b)g, \alpha\} = \{b, g, \beta\} = \theta_\beta(b, g)$$

for $b \in U_\alpha \cap U_\beta$ and $g \in G$. Thus the $\{(U_\alpha, g_{\alpha\beta})\}$ is the set of transition function to the atlas $\{(U_\alpha, \phi_\alpha)\}$. ∎

Let X be a right G-space and let Y be a left G-space. The product over G is defined by

$$X \times_G Y = X \times Y/(xg, y) \sim (x, gy)$$

with quotient topology. Note that the composite

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\pi_X} & X & \xrightarrow{\pi} & X/G \\ (x,y) & \mapsto & x & \mapsto & \bar{x} \end{array}$$

factors through $X \times_G Y$. Let $p\colon X \times_G Y \to X/G$ be the resulting map. For any $\bar{x} \in X/G$, choose $x \in \pi^{-1}(\bar{x}) \subseteq X$, then

$$p^{-1}(\bar{x}) = \pi^{-1}(\bar{x}) \times_G Y = x \times Y/H_x,$$

where $H_x = \{g \in G \mid xg = x\}$. Thus if X is a free right G-space, then the projection $p\colon X \times_G Y \to X/G$ has the constant fibre Y.

Proposition 4.8. *Let $\pi\colon X \to X/G$ be a (right) principal G-bundle and let Y be any left G-space. Then*

$$p\colon X \times_G Y \longrightarrow X/G$$

is a fibre bundle with fibre Y.

Proof. Consider a chart (U_α, ϕ_α) for $\pi\colon X \to X/G$. Since the homeomorphism $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$ is a G-map, there is a commutative diagram

$$\begin{array}{ccccccc} U_\alpha \times Y & = & (U_\alpha \times G) \times_G Y & \xrightarrow[\cong]{\phi_\alpha} & \pi^{-1}(U_\alpha) \times_G Y & = & p^{-1}(U_\alpha) \\ \downarrow{\scriptstyle \pi_{U_\alpha}} & & \downarrow{\scriptstyle \pi_{U_\alpha}} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ U_\alpha & = & U_\alpha & = & U_\alpha & = & U_\alpha \end{array}$$

and hence the result. □

Let ξ be a (right) principal G-bundle given by $\pi\colon X \to X/G$. Let Y be any left G-space. Then fibre bundle

$$p\colon X \times_G Y \longrightarrow X/G$$

is called *induced fibre bundle* of ξ, denoted by $\xi[Y]$.

Now let $p\colon E \to B$ is a fibre bundle with fibre F and bundle group G. Observe that the action of $\mathrm{Homeo}(F, F)$ on F is a left action because $(f \circ g)(x) = f(g(x))$. Thus G acts by left on F via $\theta\colon G \to \mathrm{Homeo}(F, F)$.

A bundle morphism

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\phi} & E(\xi') \\ \downarrow{\scriptstyle p(\xi)} & & \downarrow{\scriptstyle p(\xi')} \\ B(\xi) & \xrightarrow{\bar{\phi}} & B(\xi') \end{array}$$

is call an *isomorphism* if both ϕ and $\bar{\phi}$ are homeomorphisms. (**Note.** this means that $(\phi^{-1}, (\bar{\phi})^{-1})$ are continuous.) In this case, we write $\xi \cong \xi'$.

Theorem 4.9. *Let ξ be a fibre bundle given by $p\colon E \to B$ with fibre F and bundle group G. Let ξ^G be the principal G-bundle constructed in Proposition 4.7 according to a set of transitions functions to ξ Then $\xi^G[F] \cong \xi$.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for ξ. We write $\tilde{\phi}_\alpha$ for ϕ_α in the proof of Proposition 4.7. Consider the map θ_α given by the composite:

$$\pi^{-1}(U_\alpha) \times_G F \xleftarrow[\cong]{\tilde{\phi}_\alpha \times \mathrm{id}_F} (U_\alpha \times G \times \alpha)_G \times F = U_\alpha \times F \xrightarrow[\cong]{\phi_\alpha} p^{-1}(U_\alpha).$$

From the commutative diagram

$$\begin{array}{ccc}
((U_\alpha \cap U_\beta) \times G \times \beta) \times_G F & \xrightarrow[((b,g',\beta),y) \mapsto ((b, g_{\alpha\beta}(b)g', \alpha), y)]{(b,g',y) \mapsto (b,g',g_{\alpha\beta}(b)(y))} & ((U_\alpha \cap U_\beta) \times G \times \alpha) \times_G F \\
\| & & \| \\
(U_\alpha \cap U_\beta) \times F & \xrightarrow{(b,y) \mapsto (b, g_{\alpha\beta}(b,y))} & (U_\alpha \cap U_\beta) \times F \\
\cong \downarrow \phi_\beta & & \cong \downarrow \phi_\alpha \\
p^{-1}(U_\alpha \cap U_\beta) & = & p^{-1}(U_\alpha \cap U_\beta),
\end{array}$$

the map θ_α induces a bundle map

$$\begin{array}{ccc}
E^G \times_G F & \xrightarrow{\theta} & E(\xi) \\
\downarrow & & \downarrow \\
B(\xi) & = & B(\xi).
\end{array}$$

This is an bundle isomorphism because θ is one-to-one and onto, and θ is a local homeomorphic by restricting each chart. The assertion follows. ☐

This theorem tells that any fibre bundle with a bundle group G is an induced fibre bundle of a principal G-bundle. Thus, for classifying fibre bundles over a fixed base space B, it suffices to classify the principal G-bundles over B. The latter is actually done by the homotopy classes from B to the classifying space BG of G. (There are few assumptions on the topology on B such as B is paracompact.) The theory for classifying fibre bundles is also called *(unstable) K-theory*, which is one of important applications of homotopy theory to geometry. Rough introduction to this theory is as follows:

There exists a *universal G-bundle ω_G* as $\pi\colon EG \to BG$. Given any principal G-bundle ξ over B, there exists a (continuous) map $f\colon B \to BG$ such that ξ, as a principal G-bundle, is isomorphic to the pull-back bundle $f^*\omega_G$ given by

$$\begin{array}{ccc} E(f^*\omega_G)=\{(x,y)\in B\times EG \mid f(x)=\pi(y)\} & \longrightarrow & EG \\ \Big\downarrow (x,y)\mapsto x & & \Big\downarrow \pi \\ B & \xrightarrow{\quad f \quad} & BG. \end{array}$$

Moreover, for continuous maps $f,g\colon B\to BG$, $f^*\omega_G\cong g^*\omega_G$ if and only if $f\simeq g$, that is, there is a continuous map (called *homotopy*) $F\colon B\times[0,1]\to BG$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$. In other words, the set of homotopy classes $[B,BG]$ is one-to-one correspondent to the set of isomorphic classes of principal G-bundles over G.

Seminar Topic: The classification of principal G-bundles and fibre bundles. (References: for instance [8, pp.48-58] Or [10, 11].)

4.4. **Vector Bundles.** Let $\mathbb{F}$ denote $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$-the real, complex or quaternion numbers. An n-dimensional $\mathbb{F}$-*vector bundle* is a fibre bundle ξ given by $p\colon E\to B$ with fibre $\mathbb{F}^n$ and an atlas $\{(U_\alpha,\phi_\alpha)\}$ in which each fibre $p^{-1}(b)$, $b\in B$, has the structure of vector space over $\mathbb{F}$ such that each homeomorphism $\phi_\alpha\colon U_\alpha\times\mathbb{F}^n\to p^{-1}(U_\alpha)$ has the property that

$$\phi_\alpha|_{\{b\}\times\mathbb{F}^n}\colon \{b\}\times F^n \longrightarrow p^{-1}(b)$$

is a vector space isomorphism for each $b\in U_\alpha$.

Let ξ be a vector bundle. From the composite

$$(U_\alpha\cap U_\beta)\times\mathbb{F}^n \xrightarrow{\phi_\beta} p^{-1}(U_\alpha\cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} (U_\alpha\cap U_\beta)\times\mathbb{F}^n,$$

the transition functions

$$g_{\alpha\beta}\colon U_\alpha\cap U_\beta \longrightarrow \mathrm{Homeo}(\mathbb{F}^n,\mathbb{F}^n)$$

have that property that, for each $x\in U_{\alpha\beta}$,

$$g_{\alpha\beta}(x)\colon \mathbb{F}^n \longrightarrow \mathbb{F}^n$$

is a **linear isomorphism**. It follows that the bundle group for a vector bundle can be chosen as the general linear group $\mathrm{GL}_n(\mathbb{F})$. By Theorem 4.9, we have the following.

Proposition 4.10. *Let ξ be an n-dimensional $\mathbb{F}$-vector space over B. Then there exists a principal $\mathrm{GL}_n(\mathbb{F})$-bundle $\xi^{\mathrm{GL}_n(\mathbb{F})}$ over B such that $\xi\cong\xi^{\mathrm{GL}_n(\mathbb{F})}[\mathbb{F}^n]$. Conversely, for any principal $\mathrm{GL}_n(\mathbb{F})$-bundle over B, $\xi^{\mathrm{GL}_n(\mathbb{F})}[\mathbb{F}^n]$ is an n-dimensional $\mathbb{F}$-vector bundle over B.* □

In other words, the total spaces of all vector bundles are just given by $E(\xi^{\mathrm{GL}_n(\mathbb{F})})\times_{\mathrm{GL}_n(\mathbb{F})}\mathbb{F}^n$.

4.5. **The Construction of Gauss Maps.** The *Grassmann manifold* $G_{n,m}(\mathbb{F})$ is the set of n-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}^m$, that is, all n-$\mathbb{F}$-planes through the origin, with the topology described as in the topic on the examples of Manifolds. (If $m=\infty$, $\mathbb{F}^\infty=\bigoplus_{j=1}^{\infty}\mathbb{F}$.) Let

$$E(\gamma_n^m)=\{(V,x)\in G_{n,m}(\mathbb{F})\times\mathbb{F}^m \mid x\in V\}.$$

Exercise 4.3. Show that

$$p\colon E(\gamma_n^m) \to G_{n,m}(\mathbb{F}) \qquad (V,x) \mapsto V$$

is an n-dimensional $\mathbb{F}$-vector bundle, denoted by γ_n^m. [Hint: By reading the topic on the examples of manifolds, check that $V_{n,m}(\mathbb{F}) \to G_{n,m}(\mathbb{F})$ is a principal $O(n,\mathbb{F})$, where $O(n,\mathbb{R}) = O(n)$, $O(n,\mathbb{C}) = U(n)$ and $O(n,\mathbb{H}) = \mathrm{Sp}(n)$. Then check that $E(\gamma_n^m) = V_{n,m}(\mathbb{F}) \times_{O(n,\mathbb{F})} \mathbb{F}^n$.]

A *Gauss map* of an n-dimensional $\mathbb{F}$-vector bundle in $\mathbb{F}^m$ ($n \leq m \leq \infty$) is a (continuous) map $g\colon E(\xi) \to \mathbb{F}^m$ such that g restricted to each fibre is a linear monomorphism.
Example. The map

$$q\colon E(\gamma_n^m) \to \mathbb{F}^m \qquad (V,x) \mapsto x$$

is a Gauss map.

Proposition 4.11. *Let ξ be an n-dimensional $\mathbb{F}$-vector bundle.*

1) *If there is a vector bundle morphism*

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\ u\ } & E(\gamma_n^m) \\ \Big\downarrow{\scriptstyle p(\xi)} & & \Big\downarrow{\scriptstyle p(\gamma_n^m)} \\ B(\xi) & \xrightarrow{\ f\ } & G_{n,m}(\mathbb{F}) \end{array}$$

that is an isomorphism when restricted to any fibre of ξ, then $q \circ u\colon E(\xi) \to \mathbb{F}^m$ is a Gauss map.

2) *If there is a Gauss map $g\colon E(\xi) \to \mathbb{F}^m$, then there is a vector bundle morphism $(u,f)\colon \xi \to \gamma_n^m$ such that $qu = g$.*

Proof. (1) is obvious. (2). For each $b \in B(\xi)$, $g(p(\xi)^{-1}(b))$ is an n-dimensional $\mathbb{F}$-subspace of $\mathbb{F}^m$ and so a point in $G_{n,m}(\mathbb{F})$. Define the functions

$$f\colon B(\xi) \to G_{n,m}(\mathbb{F}) \qquad f(b) = g(p(\xi)^{-1}(b)),$$
$$u\colon E(\xi) \to E(\gamma_n^m) \qquad u(z) = (f(p(z)), g(z)).$$

The functions f and u are well-defined. For checking the continuity of f and u, one can look at a local coordinate of ξ and so we may assume that ξ is a trivial bundle, namely, $g\colon B(\xi) \times \mathbb{F}^n \to \mathbb{F}^m$ restricted to each fibre is a linear monomorphism. Let $\{e_1, \ldots, e_n\}$ be the standard $\mathbb{F}$-bases for $\mathbb{F}^n$. Then the map

$$h\colon B \longrightarrow \mathbb{F}^m \times \cdots \times \mathbb{F}^m \qquad b \mapsto (g(b,e_1), g(b,e_2), \ldots, g(b,e_n))$$

is continuous. Since g restricted to each fibre is a monomorphism, the vectors

$$\{g(b,e_1), g(b,e_2), \ldots, g(b,e_n)\}$$

are linearly independent and so

$$(g(b,e_1), g(b,e_2), \ldots, g(b,e_n)) \in \tilde{V}_{n,m}(\mathbb{F})$$

for each b, where $V_{n,m}(\mathbb{F})$ is the open Stiefel manifold over $\mathbb{F}$. Thus

$$h\colon B \longrightarrow \tilde{V}_{n,m}(\mathbb{F}) \qquad b \mapsto (g(b,e_1), g(b,e_2), \ldots, g(b,e_n))$$

is continuous and so the composite

$$f\colon B \xrightarrow{h} \tilde{V}_{n,m}(\mathbb{F}) \xrightarrow{\text{quotient}} G_{n,m}(\mathbb{F})$$

is continuous. The function u is continuous because the composite

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\quad u \quad} & E(\gamma_n^m) \\ \downarrow \Delta & & \downarrow \\ E(\xi)\times E(\xi) & \xrightarrow{(f\circ p(\xi))\times g} & G_{n,m}(\mathbb{F})\times \mathbb{F}^m \end{array}$$

is continuous. This finishes the proof. □

Let ξ be a vector bundle and let $f\colon X \to B(\xi)$ be a (continuous) map. Then the *induced vector* $f^*\xi$ is the pull-back

$$\begin{array}{ccc} E(f^*\xi) = \{(x,y) \in X \times E(\xi) \mid f(x) = p(y)\} & \longrightarrow & E(\xi) \\ \downarrow & & \downarrow p \\ X & \xrightarrow{\quad f \quad} & B(\xi). \end{array}$$

Proposition 4.12. *There exists a Gauss map $g\colon E(\xi) \to \mathbb{F}^m$ $(n \le m \le \infty)$ if and only if*

$$\xi \cong f^*(\gamma_n^m)$$

over $B(\xi)$ for some map $f\colon B(\xi) \to G_{n,m}(\mathbb{F})$.

Proof. $\Longleftarrow$ is obvious.

$\Longrightarrow$ Assume that ξ has a Gauss map g. From Part (2) of Proposition 4.11, there is a commutative diagram

$$\begin{array}{ccc} E(\xi) & \xrightarrow{u} & E(\gamma_n^m) \\ \downarrow p(\xi) & & \downarrow p(\gamma_n^m) \\ B(\xi) & \xrightarrow{f} & G_{n,m}(\mathbb{F}). \end{array}$$

Since $E(f^*\gamma_n^m)$ is defined to be the pull-back, there is commutative diagram

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\tilde{u}} & E(f^*\gamma_n^m) \\ \downarrow p(\xi) & & \downarrow p \\ B(\xi) & = & B(\xi), \end{array}$$

where $\tilde{u}$ restricted to each fibre is a linear isomorphism because both vector-bundle has the same dimension and the Gauss map g restricted to each fibre is a monomorphism. It follows that

$\tilde{u}\colon E(\xi) \to E(f^*\gamma_n^m)$ is one-to-one and onto. Moreover $\tilde{u}$ is a homeomorphism by considering a local coordinate. □

We are going to construct a Gauss map for each vector bundle over a paracompact space. First, we need some preliminary results for bundles over paracompact spaces. (For further information on paracompact spaces, one can see [3, 162-169].

A family of $\mathcal{C} = \{C_\alpha \mid J\}$ of subsets of a space X is called *locally finite* if each $x \in X$ admits a neighborhood W_x such that $W_x \cap C_\alpha \neq \emptyset$ for only finitely many indices $\alpha \in J$. Let $\mathcal{U} = \{U_\alpha\}$ and $\mathcal{V} = \{V_\beta\}$ be two open covers of X. $\mathcal{V}$ is called a *refinement* of $\mathcal{U}$ if for each β, $V_\beta \subseteq U_\alpha$ for some α.

A Hausdorff space X is called *paracompact* if it is regular and if every open cover of X admits a locally finite refinement.

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in J\}$ be an open cover of a space X. A *partition of unity*, subordinate to $\mathcal{U}$, is a collection $\{\lambda_\alpha \mid \alpha \in J\}$ of continuous functions $\lambda_\alpha\colon X \to [0, 1]$ such that

1) The support
$$\mathrm{supp}(\lambda_\alpha) \subseteq U_\alpha$$
for each α, where the support
$$\mathrm{supp}(\lambda_\alpha) = \overline{\{x \in X \mid \lambda_\alpha(x) \neq 0\}}$$
is the closure of the subset of X on which $\lambda_\alpha \neq 0$;
2) for each $x \in X$, there is a neighborhood W_x of x such that $\lambda_{\lambda_\alpha}|_{W_x} \not\equiv 0$ for only finitely many indices $\alpha \in J$. (In other words, the supports of λ_α's are locally finite.)
3) The equation
$$\sum_{\alpha \in J} \lambda_\alpha(x) = 1$$
for all $x \in X$, where the summation is well-defined for each given x because there are only finitely many non-zeros.

We give the following well-known theorem without proof. One may read a proof in [2, pp.17-20].

Theorem 4.13. *If X is a paracompact space and $\mathcal{U} = \{U_\alpha\}$ is an open cover of X, then there exists a partition of unity subordinate to $\mathcal{U}$.* □

Lemma 4.14. *Let ξ be a fibre bundle over a paracompact space B. Then ξ admits an atlas with countable charts.*

Proof. Let $\{(U_\alpha, \phi_\alpha \mid \alpha \in J\}$ be an atlas for ξ. We are going to find another atlas with countable charts.

By Theorem 4.13, there is a partition of unity $\{\lambda_\alpha \mid \alpha \in J\}$ subordinate to $\{U_\alpha \mid \alpha \in J\}$. Let
$$V_\alpha = \lambda_\alpha^{-1}(0, 1] = \{b \in B \mid \lambda_\alpha(b) > 0\}.$$
Then, by the definition of partition of unity, $V_\alpha \subseteq \bar{V}_\alpha \subseteq U_\alpha$. For each $b \in B$, let
$$S(b) = \{\alpha \in J \mid \lambda_\alpha(b) > 0\}.$$
Then, by the definition of partition of unity, $S(b)$ is a finite subset of J.

Now for each finite subset S of J define
$$W(S) = \{b \in B \mid \lambda_\alpha(b) > \lambda_\beta(b) \text{ for each } \alpha \in S \text{ and } \beta \notin S\}$$

$$= \bigcap_{\substack{\alpha \in S \\ \beta \notin S}} (\lambda_\alpha - \lambda_\beta)^{-1}(0,1].$$

Then $W(S)$ is open because for each $b \in W(S)$, by definition of partition of unity, there exists a neighborhood W_b of b such that there are finitely many supports intersect with W_b; and so the above (possibly infinite) intersection of open sets restricted to W_b is only a finite intersection of open sets.

Let S and S' be two subsets of J such that $S \neq S'$ and $|S| = |S'| = m > 0$, where $|S|$ is the number of elements in S. Then there exist $\alpha \in S \setminus S'$ and $\beta \in S' \setminus S$ because $S \neq S'$ but S and S' has the same number elements. We claim that

$$W(S) \cap W(S') = \emptyset.$$

Otherwise there exists $b \in W(S) \cap W(S')$. By definition $W(S)$, $\lambda_\alpha(b) > \lambda_\beta(b)$ because $\alpha \in S$ and $\beta \notin S$. On the other hand, $\lambda_\beta(b) > \lambda_\alpha(b)$ because $b \in W(S')$, $\beta \in S'$ and $\alpha \notin S'$.

Now define

$$W_m = \bigcup_{\substack{b \in B \\ |S| = m}} W(S(b))$$

for each $m \geq 1$. We prove that (1) $\{W_m \mid m = 1, \ldots\}$ is an open cover of B; and (2) ξ restricted to W_m is a trivial bundle for each m. (Then $\{W_m\}$ induces an atlas for ξ.)

To check $\{W_m\}$ is an open cover, note that each W_m is open. For each $b \in B$, $S(b)$ is a finite set and $b \in W(S(b))$ because $\lambda_\beta(b) = 0$ for $\beta \notin S(b)$ and $\lambda_\alpha(b) > 0$ for $\alpha \in S(b)$. Let $m = |S(b)|$, then $b \in W_m$ and so $\{W_m\}$ is an open cover of B.

Now check that ξ restricted to W_m is trivial. From the above, W_m is a disjoint union of $W(S(b))$. It suffices to check that ξ restricted to each $W(S(b))$ is trivial. Fixing $\alpha \in S(b)$, for any $x \in W(S(b))$, then

$$\lambda_\alpha(x) > \lambda_\beta(x)$$

for any $\beta \notin S$. In particular, $\lambda_\alpha(x) > 0$ for any $x \in W(S(b))$. It follows that $W(S(b)) \subseteq V_\alpha \subseteq U_\alpha$. Since ξ restricted to U_α is trivial, ξ restricted to $W(S(b))$ is trivial. This finishes the proof. □

Note. From the proof, if for each $b \in B$ there are at most k sets U_α with $b \in U_\alpha$, then B admits an atlas of finite (at most k) charts. [In this case, check that $W_j = \emptyset$ for $j > k$.]

Theorem 4.15. *Any n-dimensional $\mathbb{F}$-vector bundle ξ over a paracompact space B has a Gauss map $g\colon E(\xi) \to \mathbb{F}^\infty$. Moreover, if ξ has an atlas of k charts, then ξ has a Gauss map $g\colon E(\xi) \to \mathbb{F}^{kn}$.*

Proof. Let $\{(U_i, \phi_i)\}_{1 \leq i \leq k}$ be an atlas of ξ with countable or finite charts, where k is finite or infinite. Let $\{\lambda_i\}$ be the partition of unity subordinate to $\{U_i\}$. For each i, define the map $g_i\colon E(\xi) \to \mathbb{F}^n$ as follows: g_i restricted to $p(\xi)^{-1}(U_i)$ is given by

$$g_i(z) = \lambda_i(z)(p_2 \circ \phi_i^{-1}(z)),$$

where $p_2 \circ \phi_i^{-1}$ is the composite

$$p(\xi)^{-1}(U_i) \xrightarrow{\phi_i^{-1}} U_i \times \mathbb{F}^n \xrightarrow[p_2]{\text{projection}} \mathbb{F}^n;$$

and g_i restricted to the outside of $p(\xi)^{-1}(U_i)$ is 0. Since the closure of $\lambda_i^{-1}(0,1]$ is contained in U_i, g_i is a well-defined (continuous) map. Now define

$$g\colon E(\xi) \to \bigoplus_{i=1}^{k} \mathbb{F}^n = \mathbb{F}^{kn} \qquad g(z) = \sum_{i=1}^{k} g_i(z).$$

This a well-defined (continuous) map because for each z, there is a neighborhood of z such that there are only finitely many g_i are not identically zero on it.

Since each $g_i\colon E(\xi) \to \mathbb{F}^n$ is a monomorphism (actually isomorphism) on the fibres of $E(\xi)$ over b with $\lambda_i(b) > 0$, and since the images of g_i are in complementary subspaces of $\mathbb{F}^{kn}$, the map g is a Gauss map. □

This gives the following classification theorem:

Corollary 4.16. *Every vector bundle over a paracompact space B is isomorphic to an induced vector bundle $f^*(\gamma_n^\infty)$ for some map $f\colon B \to G_{n,\infty}(\mathbb{F})$. Moreover every vector bundle over a paracompact space B with an atlas of finite charts is isomorphic to an induced vector bundle $f^*(\gamma_n^m)$ for some m and some map $f\colon B \to G_{n,m}(\mathbb{F})$.* □

Remarks: It can be proved that $f^*\gamma_n^m \cong g^*\gamma_n^m$ if and only if $f \simeq g\colon B \to G_{n,m}(\mathbb{F})$. From this, one get that the set of isomorphism classes of n-dimensional $\mathbb{F}$-vector bundles over a paracompact space B is isomorphic to the set of homotopy classes $[B, G_{n,\infty}(\mathbb{F})]$.

For instance, if $n = 1$ and $\mathbb{F} = \mathbb{R}$, $G_{1,\infty}(\mathbb{R}) \simeq BO(1) \simeq B\mathbb{Z}/2 \simeq \mathbb{R}\mathrm{P}^\infty$, where BG is so-called the classifying space of the (topological) group G, and $[B, \mathbb{R}\mathrm{P}^\infty] = H^1(B, \mathbb{Z}/2)$, which states that all line bundles are by the first cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. In particular, any real line bundles over a simply connected space is always trivial.

If $n = 1$ and $\mathbb{F} = \mathbb{C}$, then $G_{1,\infty}(\mathbb{C}) \simeq BU(1) \simeq BS^1 \simeq \mathbb{C}\mathrm{P}^\infty$, and $[B, \mathbb{C}\mathrm{P}^\infty] = H^2(B, \mathbb{Z})$, which states that all complex line bundles are by the second integral cohomology.

If $n = 1$ and $\mathbb{F} = \mathbb{H}$, then $G_{1,\infty}(\mathbb{H}) \simeq B\mathrm{Sp}(1) \simeq BS^3 \simeq \mathbb{H}\mathrm{P}^\infty$, and so $[B, G_{1,\infty}(\mathbb{H})] = [B, \mathbb{H}\mathrm{P}^\infty]$. However, the determination of $[B, \mathbb{H}\mathrm{P}^\infty]$ is very hard problem even when B are spheres. If $B = S^n$, then $[B, \mathbb{H}\mathrm{P}^\infty] = \pi_{n-1}(S^3)$ that is only known for n less than 66 or so, by a lot of computations through many papers. Some people even believe that it is impossible to compute the general homotopy groups $\pi_n(S^3)$.

Seminar Topic: Gauss Maps and the Classification of Vector Bundles. (References: for instance [8, pp.26-29,31-33].)

A vector bundle is called *of finite type* if it has an atlas with finite charts. Given two vector bundles ξ and η over B, the *Whitney sum* $\xi \oplus \eta$ is defined to be the pull-back:

$$\begin{array}{ccc}
E(\xi \oplus \eta) & \longrightarrow & E(\xi) \times E(\eta) \\
\big\downarrow & & \big\downarrow {\scriptstyle p(\xi) \times p(\eta)} \\
B & \xrightarrow[\text{diagonal}]{\Delta} & B \times B.
\end{array}$$

Intuitively, $\xi \oplus \eta$ is just the fibrewise direct sum.

Proposition 4.17. *For a vector bundle ξ over a paracompact space B, the following statement are equivalent:*

1) *The bundle ξ is of finite type.*
2) *There exists a map $f\colon B \to G_{n,m}(\mathbb{F})$ such that ξ is isomorphic to $f^*\gamma_n^m$.*
3) *There exists a vector bundle η such that the Whitney sum $\xi \oplus \eta$ is trivial.*

Proof. (1) $\Longrightarrow$ (2) follows from Corollary 4.16. (2) $\Longrightarrow$ (1) It is an exercise to check that γ_n^m is of finite type by using the property that the Grassmann manifold $G_{n,m}(\mathbb{F}) = O(m,\mathbb{F})/O(n,\mathbb{F}) \times O(m-n,\mathbb{F})$ is compact, where $O(n,\mathbb{R}) = O(n)$, $O(n,\mathbb{C}) = U(n)$ and $O(n,\mathbb{H}) = \mathrm{Sp}(n)$. It follows that $\xi \cong f^*\gamma_n^m$ is of finite type.

(2) $\Longrightarrow$ (3). Let $(\gamma_n^m)^*$ be the vector bundle given by

$$E((\gamma_n^m)^*) = \{(V,\vec{v}) \in G_{n,m}(\mathbb{F}) \times \mathbb{F}^m \mid \vec{v} \perp V\}$$

with canonical projection $E((\gamma_n^m)^*) \to G_{n,m}(\mathbb{F})$. Then $\gamma_n^m \oplus (\gamma_n^m)^*$ is an m-dimensional trivial $\mathbb{F}$-vector bundle. It follows that

$$f^*(\gamma_n^m \oplus (\gamma_n^m)^*) = f^*(\gamma_n^m) \oplus f^*((\gamma_n^m)^*)$$

is trivial. Let $\eta = f^*((\gamma_n^m)^*)$. Then $\xi \oplus \eta$ is trivial.

(3) $\Longrightarrow$ (2). The composite

$$E(\xi) \hookrightarrow E(\xi \oplus \eta) = B \times F^m \longrightarrow \mathbb{F}^m$$

is a Gauss map into finite dimensional vector space, where $m = \dim(\xi \oplus \eta)$. By Proposition 4.12, there is a map $f\colon B \to G_{n,m}$ such that $\xi \cong f^*\gamma_n^m$. $\square$

Corollary 4.18. *Let ξ be a $\mathbb{F}$-vector bundle over a compact (Hausdorff) space B. Then there is a $\mathbb{F}$-vector bundle η such that $\xi \oplus \eta$ is trivial.* $\square$

In the view of (stable) K-theory, the Whitney sum is an operation on vector bundles over a (fixed) base-space, where the trivial bundles (of different dimensions) are all regarded as 0. In this sense, the Whitney sum plays as an addition (that is associative and commutative with 0). The bundle ξ with property that $\xi \oplus \eta$ is trivial for some η means that ξ is invertible. Those who are interested in algebra can push notions in algebra to vector bundles by doing constructions fibrewisely. More general situation possibly is the *sheaf theory* (by removing the locally trivial condition) that is pretty useful in algebraic geometry. In algebraic topology, people also study the category whose objects are just continuous maps $f\colon E \to B$ with fixed space B, or even more general category whose objects are *diagrams* over spaces. In the terminology of fibre bundles, a map $f\colon E \to B$ is called a *bundle* (without assuming locally trivial).

5. Tangent Bundles and Vector Fields

5.1. **Tangent Bundles.** Let M be a differentiable manifold of dimension m. As a set, the tangent bundle

$$T(M) = \bigcup_{P\in M} T_P(M),$$

the disjoint union of tangent spaces. We introduce topological and differential structure on $T(M)$ in three stages:

(a) For an open subset $V \subseteq \mathbb{R}^m$, $T(V) \cong V \times \mathbb{R}^m$ using the parallel translation isomorphisms at each point. Take this differential structure on $T(V)$ by regarding $V \times \mathbb{R}^m$ as a subset of $\mathbb{R}^{2m}$. (That is, $T(V)$ is regarded as a differentiable manifold with only one chart $T(V) \cong V \times \mathbb{R}^m$.)

(b) For a chart (U, ϕ) of M, there is a bijection

$$T_\phi \colon T(U) \xrightarrow{\cong} T(\phi(U)) \cong \phi(U) \times \mathbb{R}^m$$

which is a linear isomorphism on each tangent space $T_\phi \colon T_P(M) \longrightarrow T_{\phi(P)}(\phi(U))$, see Subsection 2.4. Take the topological and differential structure on $T(U)$ induced by T_ϕ.

(c) If (V, ψ) is another chart of M, from Diagram 1, there is a commutative diagram

$$\begin{array}{ccc} T_P(M) & \xrightarrow[\cong]{T_\phi} & T_{\phi(P)}(\phi(U \cap V)) \\ \| & & \downarrow T(\psi \circ \phi^{-1}) \\ T_P(M) & \xrightarrow[\cong]{T_\psi} & T_{\psi(P)}(\psi(U \cap V)), \end{array}$$

where $T(\psi \circ \phi^{-1})$ is the linear isomorphism induced by the Jacobian matrix of the differentiable map $\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$. Thus there is a commutative diagram

$$\begin{array}{ccccc} T(U \cap V) & \xrightarrow[\cong]{T_\phi} & T(\phi(U \cap V)) & \xrightarrow{\cong} & \phi(U \cap V) \times \mathbb{R}^m \\ \| & & \downarrow T(\psi \circ \phi^{-1}) & & \downarrow (P, \vec{v}) \mapsto (\psi \circ \phi^{-1}(P), T_P(\psi \circ \phi^{-1})(\vec{v})) \\ T(U \cap V) & \xrightarrow[\cong]{T_\psi} & T(\psi(U \cap V)) & \xrightarrow{\cong} & \psi(U \cap V) \times \mathbb{R}^m, \end{array} \tag{6}$$

where $T_P(\psi \circ \phi^{-1})$ is the Jacobian matrix of the differentiable map $\psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$ at the point P.

In other words, let $\{(U_\alpha, \phi_\alpha)\}$ be a differentiable, then the tangent bundle $T(M)$ is a *differentiable manifold* with a differentiable atlas given by

$$T_{\phi_\alpha} \colon T(U_\alpha) \xrightarrow{\cong} T(\phi_\alpha(U_\alpha)) \cong \phi_\alpha(U_\alpha) \times \mathbb{R}^m$$

and transition functions given as in Diagram 6. (**Note.** As a topological space, $T(M)$ is the quotient space of the disjoint union $\coprod_\alpha T(U_\alpha)$ with equivalence relation given by Diagram 6.)

Proposition 5.1. *The projection $\pi \colon T(M) \to M$, $\vec{v}_P \mapsto P$ is a vector bundle over M. Moreover π is a differentiable submersion.* □

Example. The tangent bundle of spheres are given as follows:

$$T(S^n) = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid |x| = 1,\ y \perp x\}.$$

The projection $\pi \colon T(S^n) \to S^n$ is given by $(x, y) \mapsto x$.

Proposition 5.2. *If $f\colon M^m \to N^n$ is differentiable, then $Tf\colon TM \to TN$ is also differentiable with a morphism of vector bundles*

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & N. \end{array}$$

Proof. By definition, f is differentiable means that there are atlases $\{(U_\alpha, \phi_\alpha)\}$ for M and $\{(V_\beta, \phi_\beta)\}$ for N such that the composites

$$\mathbb{R}^m \supseteq \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \xrightarrow[\cong]{\phi_\alpha^{-1}} f^{-1}(V_\beta) \cap U_\alpha \xrightarrow{f} V_\beta \xrightarrow[\cong]{\psi_\beta} \psi_\beta(V_\beta) \subseteq \mathbb{R}^n$$

are differentiable. There is a commutative diagram

$$\begin{array}{ccc} \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \times \mathbb{R}^m \cong T(\phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) & \xleftarrow{T_{\phi_\alpha}} & T(f^{-1}(V_\beta) \cap U_\alpha) = Tf^{-1}(T(V_\beta)) \cap T(U_\alpha) \\ \downarrow \Theta & & \downarrow Tf \\ \psi_\beta(V_\beta) \times \mathbb{R}^n \cong T(\psi_\beta(V_\beta)) & \xleftarrow{T_{\psi_\beta}} & T(V_\beta), \end{array}$$

where

$$\Theta(P, \vec{v}) = (\psi_\beta \circ f \circ \phi_\alpha^{-1}(P), D(\psi_\beta \circ f \circ \phi_\alpha^{-1}) \mid_P (\vec{v}))$$

and $D(\psi_\beta \circ f \circ \phi_\alpha^{-1}) \mid_P$ is the Jacobian matrix of $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ at P. Thus Tf is differentiable. Since Tf restricted to each fibre is a linear transformation, (Tf, f) is a morphism of tangent bundles and hence the result. □

5.2. **Vector Fields.** Let M be a differentiable manifold. A smooth cross-section X of the bundle projection $\pi\colon TM \to M$ is called a *vector field* on M, that is, $X\colon M \to TM$ is a smooth map such that $\pi \circ X = \mathrm{id}_M$.

Let $f\colon M \to \mathbb{R}$ be a smooth function and let $X, Y\colon M \to TM$ be vector fields. Then the fibrewise addition

$$X + Y\colon M \to TM \quad b \mapsto X(b) + Y(b)$$

and scalar multiplication

$$f \cdot X\colon M \to TM \quad b \mapsto f(b) \cdot X(b)$$

are also vector fields on M.

Let $C^\infty(M)$ denote the set of smooth functions on M. Then $C^\infty(M)$ admits an algebraic structure over $\mathbb{R}$ given by

$$(f + g)(b) = f(b) + g(b) \qquad (fg)(b) = f(b)g(b).$$

Let $\mathbf{VF}(M)$ denote the set of vector fields on M. Then $\mathbf{VF}(M)$ is an abelian group under $X + Y$ with

$$f(gX) = (fg)X \quad f(X + Y) = fX + fY \quad (f + g)X = fX + gX$$

for $f, g \in C^\infty(M)$ and $X, Y \in \mathbf{VF}(M)$. Thus we have the following.

Proposition 5.3. *Let M be a smooth manifold. Then $\mathbf{VF}(M)$ is a module over $C^\infty(M)$.* □

Let X be a vector field and let P be a point in M. Then X admits a local expression in the following sense:

Let (U, ϕ) be a chart around P, that is $\phi\colon U \xrightarrow{\cong} \phi(U) \subseteq \mathbb{R}^m$ such that $\phi(P) = 0$. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xrightarrow{X|_U} & T(U) & \xrightarrow{T_\phi} & T(\phi(U)) \cong \phi(U) \times \mathbb{R}^m \\
\Vert & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{proj.}} \\
U & = & U & \xrightarrow{\phi} & \phi(U).
\end{array}
$$

Thus

$$
T_\phi \circ X|_U(Q) = \left(\phi(Q), \sum_{i=1}^{m} \xi_U^i(Q) \left. \frac{\partial}{\partial x^i} \right|_Q \right) \tag{7}
$$

for $Q \in U$. If (V, ψ) is another chart around P with $\psi(P) = 0$, then the change of coordinates is obtained from the diagram

$$
\begin{array}{ccccc}
\psi(U \cap V) \times \mathbb{R}^m \cong T(\psi(U \cap V)) & \xleftarrow{T_\psi} & T(U \cap V) & \xrightarrow{T_\phi} & T(\phi(U \cap V)) \cong \phi(U \cap V) \times \mathbb{R}^m \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \text{proj.}} \\
\psi(U \cap V) & \xleftarrow{\psi} & U \cap V & \xrightarrow{\phi} & \phi(U \cap V).
\end{array}
$$

Now we are going to describe an action of $\mathbf{VF}(M)$ on $C^\infty(M)$. Let X be a vector field and let f be a smooth function on M. Then Xf is a smooth function on M defined as follows:

Given any $P \in M$, let (U, ϕ) be a chart around P such that $\phi(P) = 0$,

$$
(Xf)(P) = \sum_{i=1}^{m} \xi_U^i(P) \frac{\partial (f \circ \phi^{-1})}{\partial x^i}(0). \tag{8}
$$

The picture is as follows:

$$
\begin{array}{ccccccc}
U & \xrightarrow{X|_U} & TU & \xrightarrow{T\phi} & T(\phi(U)) \cong \phi(U) \times \mathbb{R}^m & \xrightarrow{(x, \vec{v}) \mapsto D_{\vec{v}}(f_U \circ \phi^{-1})(x)} & \mathbb{R} \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
 & & U & \xrightarrow{\phi} & \phi(U) & \xrightarrow{f|_U \circ \phi^{-1}} & \mathbb{R}.
\end{array}
$$

Exercise 5.1. Let X be a vector field on M and let $f\colon M \to \mathbb{R}$ be a smooth function. Prove that Xf is well-defined smooth function on M. [Hint: Since smooth is local property, it suffices to check that Xf is well-defined, that is, to check that if (V, ψ) is another chart around P such $\psi(P) = 0$

and $\psi \circ \phi^{-1}$ smooth on $\phi(U \cap V)$, then

$$\sum_{i=1}^{m} \xi_U^i(P) \frac{\partial(f \circ \phi^{-1})}{\partial x^i}(0) = \sum_{i=1}^{m} \xi_V^i(P) \frac{\partial(f \circ \psi^{-1})}{\partial y^i}(0)$$

by using chain rule. **Note.** From the proof, we need the condition that $\psi \circ \phi^{-1}$ is differentiable. So the definition of Xf depends on the differential structure of M.]

The proof of the following proposition follows from the definition.

Proposition 5.4. *The action of* $\mathbf{VF}(M)$ *on* $C^\infty(M)$ *satisfies the following rules:*

$$X(f+g) = Xf + Xg \qquad X(fg) = Xf \cdot g + f \cdot Xg$$

for $X \in \mathbf{VF}(M)$ *and* $f, g \in C^\infty(M)$*. In other words, for each* $X \in \mathbf{VF}(M)$*, the operation*

$$D_X \colon C^\infty(M) \longrightarrow C^\infty(M) \quad f \mapsto Xf$$

is a derivation on the algebra $C^\infty(M)$*.* □

Let $\mathrm{Der}(C^\infty(M))$ denote the set of derivations on $C^\infty(M)$. Then $\mathrm{Der}(C^\infty(M))$ is a module over $C^\infty(M)$ under the operations: for $D_1, D_2 \in \mathrm{Der}(C^\infty(M))$ and $g \in C^\infty(M)$, the derivations $D_1 + D_2$ and gD_1 are given by

$$(D_1 + D_2)(f) = D_1(f) + D_2(f) \qquad (gD_1)(f) = g \cdot D_1(f).$$

Theorem 5.5. *The function*

$$\Phi \colon \mathbf{VF}(M) \longrightarrow \mathrm{Der}(C^\infty(M)) \quad X \mapsto D_X$$

is an isomorphism of $C^\infty(M)$*-modules.*

We need a lemma for proving Theorem 5.5.

Lemma 5.6. *Let* h *be a smooth function defined on a neighborhood* U *of* P *on* M*. There is a (small) neighborhood* V *of* P *such that* $\bar{V} \subseteq U$*, and a smooth function* g *on* M *such that* $g = h$ *on* V *and* $g = 0$ *on the complement* $M \smallsetminus U$ *of* U

Sketch. First choose a small open neighborhood W of P such that $\bar{W} \subseteq U$. For W, check that there are small ϵ_1- and ϵ_2-neighborhoods $V_{\epsilon_1} \subseteq V_{\epsilon_2} \subseteq W$ of P with $\epsilon_2 > \epsilon_1$, and a smooth function f with $0 \leq f \leq 1$, $f = 1$ on V_{ϵ_1} and $f = 0$ on $M \smallsetminus V_{\epsilon_2}$.

Next define

$$g(Q) = \begin{cases} f(Q)h(Q) & \text{for} \quad Q \in U \\ 0 & \text{for} \quad Q \in M \setminus U. \end{cases}$$

□

Proof of Theorem 5.5. Since

$$\Phi_{X+Y} = \Phi_X + \Phi_Y \qquad \Phi_{gX} = g\Phi_X,$$

the function Φ is a morphism of $C^\infty(M)$-modules.

Step 1. $\mathrm{Ker}(\Phi) = 0$.

Let X be a vector field on M such that $D_X = 0$, that is $Xf = 0$ for all $f \in C^\infty(M)$. For each $P \in M$, let (U, ϕ) be a chart around P with $\phi(P) = 0$, from Equation 8

$$(Xf)(P) = \sum_{i=1}^{m} \xi_U^i(P) \frac{\partial(f \circ \phi^{-1})}{\partial x^i}(0) = 0 \qquad \text{for any} \quad f \in C^\infty(M).$$

Let h_j be smooth function defined on a small neighborhood of P such that $h_j\circ\phi^{-1}=x^j$ on a small neighborhood of 0. By Lemma 5.6, there exists $f_j\in C^\infty(M)$ such that $f_j=h_j$ in a small neighborhood of P. By inputting f_j into the above equation,

$$0=(Xf_j)(P)=\sum_{i=1}^m \xi^i_U(P)\frac{\partial(f_j\circ\phi^{-1})}{\partial x^i}(0)=\sum_{i=1}^m \xi^i_U(P)\frac{\partial x^j}{\partial x^i}(0)=\xi^j(U)(P).$$

for $j=1,\ldots,m$. Thus

$$X(P)=\sum_{i=1}^m \xi^i_U(P)\frac{\partial}{\partial x^i}=0$$

for any given $P\in M$ and so $X=0$, that is $\mathrm{Ker}(\Phi)=0$.

Now we are going to show that Φ is onto, that is, given any derivation D on $C^\infty(M)$, we construct a vector field X such that $D_X=D$, by two steps.

Step 2. If f is a smooth function on M such that $f=0$ at all points of an open set U of M, then $Df=0$ on U.

Let P be any point in U. By Lemma 5.6, there are neighborhoods U_P and W_P of P, and a smooth function $f_P\in C^\infty(M)$ such that

$$\bar W_P\subseteq U_P\subseteq U,\quad f_P|_{W_P}=1\quad f_P|_{M\smallsetminus U_P}=0.$$

Set $g=1-f_P$. Then $gf=f$ because, if $Q\in U_P$, then $f(Q)=0$ and $g(Q)f(Q)=f(Q)=0$ for $Q\in U_P$, and if $Q\notin U_P$, then $g(Q)=1-f_P(Q)=1$, and so $g(Q)f(Q)=f(Q)$ for $Q\notin U_P$.

Now since D is a derivation, $Df=D(gf)=(Dg)f+g(Df)$ and so, for any given $P\in U$,

$$(Df)(P)=(Dg(P))f(P)+g(P)(Df(P))=(Dg(P))\cdot 0+0\cdot(Df(P))=0$$

because $f(P)=0$ and $g(P)=1-f_P(P)=1-1=0$ as $P\in W_P\subseteq U$.

Step 3. Construct the vector field $X\colon M\to TM$ such that $Xf=D(f)$ for any $f\in C^\infty(M)$.

Let P be an arbitrary point in M, and let h be any smooth function defined on a neighborhood V of P. By Lemma 5.6, there a neighborhood U ($\bar U\subseteq V$) and an $f\in C^\infty(M)$ such that $f=h$ on U. If $f'\in C^\infty(M)$ is also equal to h on U, then, by Step 2, $Df=Df'$ on U. Hence, for any $f\in C^\infty(M)$ agreeing h on a neighborhood of P, the value of Df at P is independent on the choice of f.

Now let (U,ϕ) be a chart around P. By Proposition 1.3, there exist real numbers $a^1,a^2,\ldots,a^n\in\mathbb{R}$ such that

$$D(h)(P)=\sum_{i=1}^n a^i\frac{\partial(h\circ\phi^{-1})}{\partial x^i}(P)\tag{9}$$

for any $h\in C^\infty(U)$, where a^i depends on D and P but is independent on h. Define a function $X\colon M\to TM,\quad P\mapsto X(P)$, such that

$$T_\phi\circ X(P)=\sum_{i=1}^m a^i\frac{\partial}{\partial x^i}.$$

Then, for any $f\in C^\infty(M)$, $(Xf)(P)=(Df)(P)$ for any given P and so $Xf=Df$.

To show that X is smooth, it suffices to show that X is smooth in a small neighborhood of P. Let $h_j \in C^\infty(U)$ such that $h_j \circ \phi^{-1} = x^j$. By lemma 5.6, there exists a small neighborhood V_j ($\bar{V}_j \subseteq U$) of P and $f_j \in C^\infty(M)$ such that $f_j = h_j$ on V_j. From Equation 9,

$$D(f_j)(Q) = \sum_{i=1}^{m} a^i(Q) \frac{\partial x^j}{\partial x^i}(Q) = a^j(Q)$$

for $Q \in V_j$. Thus a^j is smooth on V_j and so X is smooth on $\bigcap_{j=1}^{m} V_j$. This finishes the proof. □

One of the important consequences of Theorem 5.5 is to discover the Lie algebra structure on vector fields, namely, given vector fields X and Y, we can construct canonical new vector field $[X, Y]$ called *commutator product* or *bracket* of X and Y.

Let's first look at the structure on $\mathrm{Der}(C^\infty(M))$. Let D_1 and D_2 be two derivations on $C^\infty(M)$. The *commutator product* $[D_1, D_2] \colon C^\infty(M) \to C^\infty(M)$ is linear map defined by

$$[D_1, D_2](f) = D_1(D_2(f)) - D_2(D_1(f)).$$

Lemma 5.7. *Let D_1, D_2 be derivations on $C^\infty(M)$. Then $[D_1, D_2]$ is also a derivation on M. Moreover the following identities hold:*

$$(10) \qquad [D_1, D_2] = -[D_2, D_1] \qquad [D_1, [D_2, D_3]] + [D_2, [D_3, D_1]] + [D_3, [D_1, D_2]] = 0.$$

The latter is called Jacobi identity

Proof.

$$\begin{aligned}
[D_1, D_2](fg) &= D_1(D_2(fg)) - D_2(D_1(fg)) \\
&= D_1(D_2(f)g + fD_2(g)) - D_2(D_1(f)g + gD_1(f)) \\
&= D_1(D_2(f))g + D_2(f)D_1(g) + D_1(f)D_2(g) + fD_1(D_2(g)) \\
&\quad -D_2(D_1(f))g - D_1(f)D_2(g) - D_2(f)D_1(g) - fD_2(D_1(g)) \\
&= ([D_1, D_2](f))g + f([D_1, D_2](g)).
\end{aligned}$$

Check the Jacobi identity by yourself. □

Now given vector fields X and Y, then D_X and D_Y are derivations on $C^\infty(M)$. By Theorem 5.5, there is a unique vector field $[X, Y]$ such that

$$D_{[X,Y]} = [D_X, D_Y]$$

because $[D_X, D_Y]$ is also a derivation. In other words,

$$[X, Y]f = X(Yf) - Y(Xf)$$

for all $f \in C^\infty(M)$. (**Note.** The composition $f \mapsto X(Yf)$ does not define a vector field in general because it is not a derivation.)

Exercise 5.2. *Prove the following identities for the bracket of vector fields:*

(1). $[X + Y, Z] = [X, Z] + [Y, Z]$;
(2). $[X, Y + Z] = [X, Y] + [X, Z]$;
(3). $[X, Y] = -[Y, X]$;
(4). $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$, *for* $f, g \in C^\infty(M)$;
(5). $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Let $\mathbb{F}$ be any field. A vector space V (possibly infinitely dimensional) is called a *Lie algebra* over $\mathbb{F}$ if there is a bi-linear operation: $[X,Y]$ for $X,Y\in V$ such that

$$[X,Y]=-[Y,X]\qquad [X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

for all $X,Y,Z\in V$.

Theorem 5.8. *Let M be a smooth manifold. Then $\mathbf{VF}(M)$ is a Lie algebra over $\mathbb{R}$ and $\Phi\colon \mathbf{VF}(M)\to \mathrm{Der}(C^\infty(M))$ is an isomorphism of Lie algebras.* □

5.3. **Vector Fields on Spheres.** In this subsection, we consider the very classical problem of determining when a sphere S^n has a single unit vector field on it. A *unit vector field* means a vector field $X\colon S^n\to TS^n$ such that $|X(P)|=1$ for each P. Note that

$$TS^n=\{(P,\vec{v})\in\mathbb{R}^{n+1}\oplus\mathbb{R}^{n+1}\mid |P|=1, P\cdot\vec{v}=0\}.$$

Thus a vector field $X\colon S^n\to TS^n$ can be written as a map

$$P\mapsto (P,\phi_X(P)),$$

where $\phi_X\colon S^n\to\mathbb{R}^{n+1}$ is a smooth map such that $P\cdot\phi_X(P)=0$. Conversely, any smooth map $\phi\colon S^n\to\mathbb{R}^{n+1}$ such that $P\cdot\phi(P)=0$ defines a vector field X. If X is a unit vector field, then the smooth map $\phi_X\colon S^n\to\mathbb{R}^{n+1}$ satisfies two equations:

$$|\phi(P)|=1\qquad P\cdot\phi(P)=0$$

for all $P\in S^n$. The first equation tells that ϕ_X is a smooth map from $S^n\to S^n$.

Proposition 5.9. *If S^{n-1} has k orthogonal unit vector fields $X_1,\ldots,X_k$, then S^{nq-1} has k orthogonal unit vector fields $\tilde{X}_1,\ldots,\tilde{X}_k$.*

Proof. By the assumption there are maps $\phi_i\colon S^{n-1}\to\mathbb{R}^n$ such that

$$P\cdot\phi_i(P)=0\qquad \phi_i(P)\cdot\phi_j(P)=\delta_{ij}$$

for all $P\in S^{n-1}$ and $1\le i,j\le k$. Consider S^{nq-1} as the join of q-copies of S^{n-1}, that is,

$$S^{nq-1}=\left\{(t_1P_1,t_2P_2,\ldots,t_qP_q)\in(\mathbb{R}^n)^{\oplus q}=\mathbb{R}^{nq}\mid P_i\in S^{n-1}, 0\le t_i\le 1, \sum_{i=1}^q t_i^2=1\right\}.$$

Define $\tilde{\phi}_i\colon S^{nq-1}\to\mathbb{R}^{nq}$ by

$$\tilde{\phi}_i(t_1P_1,t_2P_2,\ldots,t_qP_q)=(t_1\phi_i(P_1),t_2\phi_i(P_2),\ldots,t_q\phi_i(P_q)).$$

Then $P\cdot\tilde{\phi}_i(P)=0$ and $\tilde{\phi}_i(P)\cdot\tilde{\phi}_j(P)=\delta_{ij}$ for all $P\in S^{nq-1}$. This defines k-orthogonal unit vector fields on S^{nq-1}. □

Corollary 5.10. *Every odd sphere S^{2q-1} has a unit vector field on it.*

Proof. It suffices to show that S^1 has a unit vector field, which is just given by $\phi\colon S^1\to S^2$, $\quad (x_1,x_2)\mapsto(-x_2,x_1)$. □

Proposition 5.11. *If S^n has a unit vector field, then there is a deformation $H\colon S^n\times[0,1]\to S^n$ such that*

$$H(P,0)=P\quad H(P,1)=-P$$

for all $P\in S^n$.

Proof. By the assumption, there is a map $\phi\colon S^n \to \mathbb{R}^{n+1}$ such that $P \cdot \phi(P) = 0$ and $|\phi(P)| = 1$ for all $P \in S^n$. Define $H\colon S^n \times [0,1] \to S^n$ by

$$H(P,t) = \cos(\pi t)P + \sin(\pi t)\phi(P).$$

Then $H(P,0) = P$ and $H(P,1) = -P$ for all $P \in S^n$. [**Note.** This deformation is on the great circle from P to $-P$ in the direction $\phi(P)$.] □

The proof of the following theorem uses a result from algebraic topology that the antipodal map $a\colon S^n \to S^n,\ x \mapsto -x$ is homotopic to the identity map if and only if n is odd.

Theorem 5.12. *The sphere S^n admits a unit vector field on it if and only if n is odd. Thus any even sphere has no unit vector fields.*

Proof of the case $n = 2$ by assuming fundamental groups. We only prove that S^2 has no unit vector fields. Let $SO(n)$ be the subgroup of $O(n)$ consisting of orthogonal matrices of determinant $+1$. There is a principal G-bundle:

$$SO(2) \xrightarrow{\ j\ } SO(3) \xrightarrow{\ \pi\ } S^2,$$

which can be obtained by considering

$$S^2 = \{(1,0,0)\cdot A \mid A \in SO(3)\}$$

the orbit space.

Suppose that S^2 has a (continuous) unit vector field. Then there is a (continuous) map $\phi\colon S^2 \to \mathbb{R}^3$ such that $x \cdot \phi(x) = 0$ and $|\phi(x)| = 1$ for all $x \in S^2$. Define a map $s\colon S^2 \to SO(3)$ by

$$s(x) = \begin{pmatrix} x \\ \phi(x) \\ x \times \phi(x) \end{pmatrix}.$$

Then s is a cross-section to the bundle projection $\pi\colon SO(3) \to S^2$, that is $\pi \circ s = \mathrm{id}_{S^2}$. Define

$$\theta\colon S^2 \times SO(2) \longrightarrow SO(3) \quad (x,g) \mapsto s(x)\cdot g.$$

Then θ is continuous, one-to-one and onto and so θ is a homeomorphism because these are compact spaces. By applying the fundamental groups,

$$\theta_*\colon \pi_1(S^2 \times SO(2)) = \pi_1(S^2) \times \pi_1(SO(2)) \xrightarrow{\ \cong\ } \pi_1(SO(3)).$$

By using the facts that $\pi_1(S^2) = 0$, $SO(2) = S^1$, $\pi_1(S^1) = \mathbb{Z}$, $\pi_1(SO(3)) = \mathbb{Z}/2$, $\theta_*\colon \mathbb{Z} \cong \mathbb{Z}/2$ and hence a contradiction. □

6. Riemann Metric and Cotangent Bundles

6.1. **Riemann and Hermitian Metrics on Vector Bundles.** If $x \in \mathbb{R}$, let $\bar{x} = x$, and if $z = x + iy \in \mathbb{C}$, let $\bar{z} = x - iy$. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Let V be a vector space over $\mathbb{F}$. An *inner product* on V is a function $\beta\colon V \times V \to \mathbb{F}$, the field of scalars, such that

1) $\beta(ax+a'x',y) = a\beta(x,y)+a'\beta(x',y)$, $\beta(x,by+b'y') = \bar{b}\beta(x,y)+\bar{b}'\beta(x,y')$ for $x,x',y,y' \in V$ and $a,a',b,b' \in \mathbb{F}$.
2) $\beta(x,y) = \overline{\beta(y,x)}$ for $x,y \in V$.
3) $\beta(x,x) \geq 0$ in $\mathbb{R}$ and $\beta(x,x) = 0$ if and only if $x = 0$.

With an inner product β on V we can define what it means for x and y to be perpendicular, that is, $\beta(x,y)=0$. On $\mathbb{R}^n$ and $\mathbb{C}^n$ there is a natural inner product, the Euclidean inner product, given by $\beta(x,y)=x\cdot y=\sum\limits_{i=1}^{n} x_i\bar{y}_i$. These formulas hold for $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$.

Definition 6.1. Let ξ be a real or complex vector bundle over B. A *Riemannian* or *Hermitian* matric on ξ is a function $\beta\colon E(\xi\oplus\xi)\to\mathbb{F}$ such that, for each $b\in B$, β restricted to the fibre $p^{-1}(b)\times p^{-1}(b)$ is an inner product on $p^{-1}(b)$. The Riemannian metric refers to $\mathbb{F}=\mathbb{R}$ and the Hermitian metric to $\mathbb{F}=\mathbb{C}$.

For instance, Let ϵ^k be the k-dimensional trivial bundle over B. Then $\beta(b,x,x')=x\cdot x'$ is a Riemannian metric in the real case and Hermitian metric in the complex case.

Theorem 6.2. *Every real or complex vector bundle with a Gauss map has a Riemannian or Hermitian metric.*

Proof. Let $g\colon E(\xi)\to\mathbb{F}^\infty$ be a Gauss map. Define $\beta\colon E(\xi\oplus\xi)\to\mathbb{F}$ by the relation

$$\beta(b,x,x')=g(b,x)\cdot g(b,x')$$

for $x,x'\in p^{-1}(b)$. Since g is continuous and a linear monomorphism on each fibre, β is a Riemannian metric or Hermitian metric. $\square$

Corollary 6.3. *Every vector bundle over a paracompact space has a metric.*

Theorem 6.4. *Let*

$$0\longrightarrow\xi\xrightarrow{\ u\ }\eta\xrightarrow{\ v\ }\zeta\longrightarrow 0$$

be a short exact sequence of vector bundles over B, that is, restricted to each fibre, u is a linear monomorphism, $\mathrm{Im}(u)=\mathrm{Ker}(v)$, v is a linear epimorphism, and u,v are identity on the base-space B. Let β be a metric on η. Then there is a morphism of vector bundles $w\colon\xi\oplus\zeta\longrightarrow\eta$ splitting the above exact sequence in the sense of the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \xi & \xrightarrow{\ u\ } & \eta & \xrightarrow{\ v\ } & \zeta & \longrightarrow & 0\\
 & & \| & & \uparrow{\scriptstyle w} & & \| & & \\
0 & \longrightarrow & \xi & \stackrel{i}{\hookrightarrow} & \xi\oplus\zeta & \stackrel{j}{\twoheadrightarrow} & \zeta & \longrightarrow & 0,
\end{array}$$

where i is the inclusion of the first factor, and j is the projection onto the second factor.

Proof. Let ξ' denote $\mathrm{Im}\,u$, where $E(\xi')\subseteq E(\eta$. Let $E(\zeta')$ be the subset of $x'\in E(\eta)$ such that $\beta(x,x')=0$ for all $x\in E(\xi')$ with $p_\eta(x)=p_\eta(x')$. (That is, ζ' is the fibrewise orthogonal complement of ξ'.)

The composite

$$v|_{\zeta'}\colon\zeta'\subseteq\eta\xrightarrow{\ v\ }\zeta$$

is a bundle isomorphism over B because it is an linear isomorphism on each fibre and identity on the base-space.

Define w to be the composite

$$w\colon\xi\oplus\zeta\xrightarrow{u\oplus(v|_{\zeta'})^{-1}}\xi'\oplus\zeta'\xrightarrow{\text{fibrewise addition}}\eta.$$

Then w is a bundle isomorphism because it is a linear isomorphism on each fibre and identity on the base-space.

The commutativity of the diagram follows from the construction of w. This proves the theorem. □

Corollary 6.5. *Let*

$$\begin{array}{ccc} E(\xi) & \xrightarrow{u} & E(\eta) \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

be a morphism of vector bundles.

(1). *If u is a linear epimorphism on each fibre and ξ has a metric, then*

$$\xi \cong \eta \oplus \xi'$$

for some vector bundle ξ' over B.

(2). *If u is a linear monomorphism on each fibre and η has a metric, then*

$$\eta \cong \xi \oplus \zeta$$

for some vector bundle ζ over B. □

Example 6.6. Let $f\colon M \to N$ be a smooth immersion. Then there is morphism of vector bundles

$$Tf\colon TM \to TN$$

which is a linear monomorphism on each fibre. Consider the commutative diagram

$$\begin{array}{ccccc} TM & \xhookrightarrow{u} & f^*TN & \xrightarrow{v} & TN \\ \downarrow & & \downarrow & \text{pull back} & \downarrow \\ M & = & M & \xrightarrow{f} & N, \end{array}$$

where u is a linear monomorphism on each fibre and $v\colon E(f^*T(N))_b \longrightarrow T_{f(b)}N$ is an isomorphism. If M is paracompact, then every vector bundle over M has a metric and so the vector bundle f^*TN has an orthogonal decomposition

$$f^*TN \cong TM \oplus \nu M.$$

The vector bundle $\nu M \to M$ is called the *normal bundle* of M with respect to the immersion f. (For smooth curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, one can check that νM consists of normal vectors.) If M is a submanifold of N, the normal bundle can be used for constructing so-called *tubular neighborhood* of M. □

Seminar Topic. The exponential map and tubular neighborhoods. [Reference: Serge Lang, *differential manifolds*, pp.95-98.]

6.2. **Constructing New Bundles Out of Old and Cotangent Bundle.** Let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\mathcal{V}$ be the category of all finite dimensional $\mathbb{F}$-vector spaces and all linear isomorphisms. Let

$$T\colon \mathcal{V}\times\mathcal{V}\times\cdots\mathcal{V}\longrightarrow\mathcal{V}$$

be a functor in k variables, that is,

1) to each sequence $(V_1,\ldots,V_k)$ of vector spaces $T(V_1,\ldots,V_k)\in\mathcal{V}$ and
2) to each $f_i\colon V_i\to W_i$ for $1\leq i\leq k$ of linear isomorphisms an isomorphism

$$T(f_1,\ldots,f_k)\colon T(V_1,\ldots,V_k)\longrightarrow T(W_1,\ldots,W_k)$$

so that

3) $T(\mathrm{id}_{V_1},\ldots,\mathrm{id}_{V_k})=\mathrm{id}_{T(V_1,\ldots,V_k)}$ and
4) $T(f_1\circ g_1,\ldots,f_k\circ g_k)=T(f_1,\ldots,f_k)\circ T(g_1,\ldots,g_k)$.

A functor $T\colon\mathcal{V}\times\cdots\times\mathcal{V}\to\mathcal{V}$ is called *continuous* if $T(f_1,\ldots,f_k)$ depends continuously on $f_1,\ldots,f_k$.

Let $T\colon\mathcal{V}\times\cdots\times\mathcal{V}\to\mathcal{V}$ be a continuous functor of k variables, and let $\xi_1,\ldots,\xi_k$ be vector bundles over a common base-space B. Then a new bundle over B is constructed as follows. For each $b\in B$ let

$$F_b=T(F_b(\xi_1),\ldots,F_b(\xi_k)).$$

Let E denote the disjoint union of F_b and define $p\colon E\to B$ by $p(F_b)=b$.

Theorem 6.7. *There exists a canonical topology for E so that $p\colon E\to B$ is a $\mathbb{F}$-vector bundle over B with fibre F_b.*

The bundle is denoted by $T(\xi_1,\ldots,\xi_k)$.

Proof. Let ξ_i be n_i-dimensional $\mathbb{F}$-vector bundle over B. There is a principle $GL_{n_i}(\mathbb{F})$-bundle $\xi_i^{\mathrm{GL}_{n_i}(\mathbb{F})}$ given by $\pi_i\colon E_i\to B$ such that $\xi_i=\xi_i^{\mathrm{GL}_{n_i}(\mathbb{F})}[\mathbb{F}^{n_i}]$ is given by

$$E_i\times_{GL_{n_i}(\mathbb{F})}\mathbb{F}^{n_i}\longrightarrow B.$$

Let $m=\dim_{\mathbb{F}}T(\mathbb{F}^{n_1},\ldots,\mathbb{F}^{n_k})$. Since T is continuous functor, the function

$$T\colon\ \mathrm{GL}_{n_1}(\mathbb{F})\times\cdots\times\mathrm{GL}_{n_k}(\mathbb{F})\longrightarrow\ \mathrm{GL}_m(\mathbb{F})\quad(f_1,\ldots,f_k)\mapsto T(f_1,\ldots,f_k)$$

is continuous group homomorphism.

Let the group $\mathrm{GL}_{n_1}(\mathbb{F})\times\cdots\times\mathrm{GL}_{n_k}(\mathbb{F})$ act on $\mathbb{F}^m$ via T, that is, the action is given by

$$(f_1,\ldots,f_k)\cdot x:=T(f_1,\ldots,f_k)(x)$$

for $x\in\mathbb{F}^m$ and $f_1,\ldots,f_k\in\mathrm{GL}_{n_i}(\mathbb{F})$. Then the vector bundle $p\colon E\to B$ is given by the pull-back

$$\begin{array}{ccc} E & \longrightarrow & (E_1\times\cdots\times E_k)\times_{GL_{n_1}(\mathbb{F})\times\cdots\times GL_{n_k}(\mathbb{F})}\mathbb{F}^m \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi_1\times\cdots\times\pi_k} \\ B & \xrightarrow{\ \Delta\ } & B\times\cdots\times B \end{array}$$

and hence the result. ∎

Remark 1. There is a different proof by looking at local coordinate system. Our proof here is to use the fact that any vector bundle is the induced from a principal G-bundle. Similar results works for general fibre bundles.

Remark 2. From the proof, the new bundle $T(\xi_1, \ldots, \xi_k)$ is obtained from a new bundle over the self Cartesian product of B via the diagonal map.

Proposition 6.8. *Let ξ be a vector bundle over B. Then there is a canonical dual bundle ξ^* over B such that each fibre is the dual vector space of the corresponding fibre of ξ.*

Proof. The left action of the general linear group on $\mathbb{F}^n$ induces a canonical right action on the dual space of $\mathbb{F}^n$ and then switch it to the left action. □

Let $\{g_{\alpha\beta}$ be the transitive functions of ξ, that is, $g_{\alpha\beta}$ is obtained from

$$(U_\alpha \cap U_\beta) \times \mathbb{F}^n \xrightarrow{\phi_\beta} p^{-1}(U_{\alpha\beta}) \xrightarrow{\phi_\alpha^{-1}} (U_\alpha \cap U_\beta) \times \mathbb{F}^n \quad (b, x) \mapsto (b, g_{\alpha\beta}(b, x)).$$

Then the transitive functions for ξ^* are given by $\left(g_{\alpha\beta}^{-1}\right)^*$.

In differential geometry, it is important to construct new bundles out of old. For differentiable manifolds, we can start with the tangent bundles and then construct various new bundles from the tangent bundles. Below we list some examples:

1. The *Whitney sum* $\xi \oplus \eta$ is induced from the functor $T\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}, (V, W) \mapsto V \oplus W$.

2. The *dual bundle* ξ^* is induced from the (contravariant) functor $T(V) = V^* = \mathrm{Hom}(V, \mathbb{F})$.

3. The vector space $\mathrm{Hom}(V, W)$ of linear transformations gives a functor $T\colon \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$, where $\mathcal{V}^{\mathrm{op}}$ is the opposite category of $\mathcal{V}$ by changing the linear isomorphism f to be f^{-1}. [**Note.** $\mathrm{Hom}(V, W)$ is contravariant on V.]

4. T is obtained by the vector space of all symmetric bi-linear transformations from $V \times V$ to W.

5. The tensor product $V \otimes W$. [We will go through tensor products.]

6. The k-th symmetric product of V. [We will go through symmetric products.]

7. The k-th exterior product of V. [We will go through exterior products.]

8. The vector space of all 4-linear transformations $K\colon V \times V \times V \times V \times V \to \mathbb{R}$ satisfying the symmetry relations:

$$K(v_1, v_2, v_3, v_4) = K(v_3, v_4, v_1, v_2) = -K(v_1, v_2, v_4, v_3),$$

$$K(v_1, v_2, v_3, v_4) + K(v_1, v_4, v_2, v_3) + K(v_1, v_3, v_4, v_2) = 0.$$

[This last example is from the theory of Riemann curvature.]

6.3. **Cotangent Bundles and Co-vector Fields.** Let M be a differentiable manifold. The *cotangent bundle* T^*M is defined to be the dual bundle of the tangent bundle of M.

Proposition 6.9. *If a vector bundle ξ has metric, then ξ is isomorphic to its dual bundle ξ^*.*

Proof. Let $\langle -,-\rangle\colon V\times V\to\mathbb{F}$ be any inner product on V. There is a canonical linear isomorphism $\theta\colon V\to V^*$ defined by

$$\theta(y)(x)=\langle x,y\rangle.$$

We just need to show that θ induces a bundle isomorphism. Let $\{g_{\alpha\beta}\}$ be transitive functions for ξ. From the commutative diagram

$$\begin{array}{ccccc}
(U_\alpha\cap U_\beta)\times\mathbb{F}^n\times\mathbb{F}^n & \xrightarrow{\phi_\beta} & p_\xi^{-1}(U_\alpha\cap U_\beta) & \xrightarrow{(b,x,x')\mapsto(b,\beta(x,x'))} & (U_\alpha\cap U_\beta)\times\mathbb{F}\\
\Big\downarrow{\scriptstyle \mathrm{id}\times g_{\alpha\beta}(b,-)\times g_{\alpha\beta}(b,-)} & & \Big\| & & \Big\|\\
(U_\alpha\cap U_\beta)\times\mathbb{F}^n\times\mathbb{F}^n & \xrightarrow{\phi_\alpha} & p_\xi^{-1}(U_\alpha\cap U_\beta) & \xrightarrow{(b,x,x')\mapsto(b,\beta(x,x'))} & (U_\alpha\cap U_\beta)\times\mathbb{F},
\end{array}$$

the transitive functions $g_{\alpha\beta}$ preserves the inner products, that is,

$$\langle g_{\alpha\beta}(b,x),g_{\alpha\beta}(b,y)\rangle=\langle x,y\rangle.$$

By using this, we check that the following diagram

$$\begin{array}{ccc}
(U_\alpha\cap U_\beta)\times\mathbb{F}^n & \xrightarrow{\theta} & (U_\alpha\cap U_\beta)\times\mathrm{Hom}(\mathbb{F}^n,\mathbb{F})\\
\Big\downarrow{\scriptstyle \mathrm{id}\times g_{\alpha\beta}(b,-)} & & \Big\downarrow{\scriptstyle \mathrm{id}\times\left(g_{\alpha\beta}(b,-)^{-1}\right)^*}\\
(U_\alpha\cap U_\beta)\times\mathbb{F}^n & \xrightarrow{\theta} & (U_\alpha\cap U_\beta)\times\mathrm{Hom}(\mathbb{F}^n,\mathbb{F})
\end{array}$$

commutes.

Let $p_2\colon(U_\alpha\cap U_\beta)\times\mathrm{Hom}(\mathbb{F}^n,\mathbb{F})\to\mathrm{Hom}(\mathbb{F}^n,\mathbb{F})$ be the projection to the second coordinate. For any $(b,x)\in(U_\alpha\cap U_\beta)\times\mathbb{F}^n$ and $y\in\mathbb{F}^n$, then

$$\begin{aligned}
p_2\circ\theta\circ(\mathrm{id}\times g_{\alpha\beta}(b,x)(y)&=\langle y,g_{\alpha\beta}(b,x)\rangle\\
&=\langle g_{\alpha\beta}(b,z),g_{\alpha\beta}(b,x)\rangle=\langle z,x\rangle,
\end{aligned}$$

where $z=g_{\alpha\beta}(b,-)^{-1}(y)$, and

$$p_2\circ\left(\mathrm{id}\times\times\left(g_{\alpha\beta}(b,-)^{-1}\right)^*\right)\circ\theta(b,x)(y)=\langle g_{\alpha\beta}(b,-)^{-1}(y),x\rangle=\langle z,x\rangle.$$

Thus the above diagram commutes and hence θ induces a bundle isomorphism. □

Corollary 6.10. *Let M be a paracompact differentiable manifold. Then the cotangent bundle T^*M is isomorphic to the tangent bundle TM.*

Note. The geometric means of tangent bundle and cotangent bundle are different. The local coordinate system for tangent bundle is given by $\{\frac{\partial}{\partial x_i}\}$, while the local coordinate system for the cotangent bundle is $\{dx_i\}$, where dx_i is the differential of the function $(x_1,\ldots,x_n)\to x_i$ which is the dual of $\partial_i=\frac{\partial}{\partial x_i}$.

Let M be a differentiable manifold. A (smooth) cross-section of the cotangent bundle $p\colon T^*M \to M$ is called a *covector field* or 1-*form*. In other words, a 1-form is a smooth map $\omega\colon M \to T^*M$ such that $p \circ \omega = \mathrm{id}_M$.

For a (smooth) function f on an open set U, in $\mathbb{R}^n$ or indeed on a manifold, and a tangent vector v_p at $p \in U$, all the following are equal:

i) $v_p(f)$, the value of the directional derivative v_p on the local function f;
ii) $df(p)(v_p)$, the value of the differential of f at p on the vector v_p, which, as v varies, expresses $df(p)$ as a covector at p;
iii) $f_*(v_p)$, the image of v_p under the derivative of f;
iv) $[f \circ \gamma]_{f(p)}$, the equivalence class of curves representing $f_*(v_p)$ when the class $[\gamma]_p$ represents v_p;
iv) $(f \circ \gamma)'(0)$, the classical notation for the derivative of the real-valued function $f \circ \gamma$ of a real variable.

Strictly speaking, (iii) and (iv) should be thought of as tangent vector to $\mathbb{R}$ at $f(p)$ and the other three as real numbers, but modulo parallel translation they are the same. We urge the reader not to become overwhelmed by this plethora of definitions. It is precisely the fact that there are so many different ways of looking at essentially the same object that leads to the beauty and power of differential geometry and analysis.

7. Tensor Bundles, Tensor Fields and Differential Forms

7.1. **Tensor Product.** The construction of tensor product is actually an algebraic question. Let R be a ring. If A and B are right and left R-module, respectively, a *middle linear map* from $A \times B$ to an abelian group C is a function $f\colon A \times B \to C$ such that

$$f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b), \tag{11}$$

$$f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2), \tag{12}$$

$$f(ar, b) = f(a, rb) \tag{13}$$

for $a_1, a_2, a \in A$, $b_1, b_2, b \in B$ and $r \in \mathbb{R}$.

The tensor product, denoted by $A \otimes_R B$ (or simply $A \otimes B$ is the ground ring R is clear), is *universal object* with respect to middle linear maps in the follows sense:

> There is a middle linear map $i\colon A \times B \to A \otimes_R B$ with the universal property that, for any middle linear map $f\colon A \times B \to C$, there is a **unique** linear map $\tilde{f}\colon A \otimes_R B \to C$ such that $f = \tilde{f} \circ i\colon A \times B \to C$, that is, there is a commutative diagram
>
> $$\begin{array}{ccc} A \times B & \xrightarrow{\ i\ } & A \otimes_R B \\ \Big\| & & \Big\downarrow \tilde{f} \\ A \times B & \xrightarrow{\ f\ } & C. \end{array}$$

By using this universal property, the tensor product $A \otimes_R B$ is unique up to isomorphism (if it exists). The explicit construction of $A \otimes_R B$ is given as follows:

$A \otimes_R B$ is the quotient of the free abelian group generated by $A \times B$ by the subgroup generated by:

i) $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$;
ii) $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$;
iii) $(ar, b) - (a, rb)$.

The image of (a, b) in $A \otimes_R B$ is denoted by $a \otimes b$. By the construction, the elements in $A \otimes_R B$ can be given by the finite linear combination $\sum_i n_i a_i \otimes b_i$ for $n_i \in \mathbb{Z}, a_i \in A$ and $b_i \in B$.

Recall that the rules for left R-modules are given by

i) $r(a + b) = ra + rb$;
ii) $(r + s)a = ra + sa$;
iii) $r(sa) = (rs)a$; [This one is changed to be $(as)r = a(sr)$ for right R-modules.]
iv) For unitary modules, $1a = a$.

For a **commutative** ring R, a left R-module can be regarded as a right R-module, where the right action is given by

$$ar = ra.$$

[**Note.** One need to be careful about left and right R-modules in case that R is non-commutative. Namely if we want to switch a left R-module to a right R-module by the above formula, we need to redefine the multiplication structure in R given by $r * s = sr$.]

Similarly, general speaking, $A \otimes_R B$ is only an abelian group. But for a commutative ring R, $A \otimes_R B$ admit an R-module structure by the following proposition.

Proposition 7.1. *Let R be a commutative ring. Then $A \otimes_R B$ is an R-module with the R-action given by*

$$r \cdot \sum_i n_i a_i \otimes b_i = \sum_i n_i (a_i r) \otimes b_i.$$

Proof. Consider the function:

$$\phi \colon R \times \mathbb{Z}(A \times B) \longrightarrow A \otimes B \quad (s, \sum_i n_i (a_i, b_i) = \sum_i a_i s \otimes b_i,$$

where $\mathbb{Z}(A \otimes B)$ is the free abelian group generated by $A \times B$. Since

$$(a_1 + a_2)s \otimes b = a_1 s \otimes b + a_2 s \otimes b$$
$$as \otimes (b_1 + b_2) = (as) \otimes b_1 + (as) \otimes b_2$$
$$(ar)s \otimes b = a(rs) \otimes b = a \otimes (rs)b = a \otimes (sr)b = a \otimes s(rb) = as \otimes rb,$$

the map ϕ factors through the quotient $R \times A \otimes_R B$, that is, there is a commutative diagram

$$\begin{CD} R \times \mathbb{Z}(A \times B) @>{\phi}>> A \otimes_R B \\ @VVV @| \\ R \times A \otimes_R B @>{\mu}>> A \otimes_R B. \end{CD}$$

Now it is straight forward to check that the rules for R-modules hold. □

Proposition 7.2. *The tensor product has the following basic properties:*

1) *If R is a ring with identity, then*

$$A \otimes_R R \cong A \quad \text{and } R \otimes_R B \cong B$$

for right unitary R-module A and left unitary R-module B.

2) *Associativity:*

$$(A \otimes_R B) \otimes_S C \cong A \otimes_R (B \otimes_S C)$$

for rings R and S, right R-module A, left R-module and right S-module B, and left S-module C.

3) *There are group isomorphisms*

$$\left(\bigoplus_{i \in I} A_i\right) \otimes_R B \cong \bigoplus A_i \otimes_R B$$

$$A \otimes_R \left(\bigoplus_{j \in J} B_j\right) \cong \bigoplus_{j \in J} A \otimes B_j$$

for right R-modules A_i, A and left R-modules B_j, B.

4) *If R is commutative, then there is an isomorphism of $\mathbb{R}$-modules*

$$A \otimes_R B \cong B \otimes_R A.$$

5) *Adjoint associativity: For rings R and S, right R-module A, left R-module and right S-module B, left S-module C, there is an isomorphism of abelian groups*

$$\alpha\colon \operatorname{Hom}_S(A \otimes_R B, C) \cong \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C))$$

defined for each $f\colon A \otimes_R B \to C$ by

$$[\alpha(f)(a)](b) = f(a \otimes b).$$

Sketch. We ask the reader to finish the proof as an exercise:

(1). The isomorphisms are given by

$$a \mapsto a \otimes 1 \qquad b \mapsto 1 \otimes b$$

for $a \in A$ and $b \in B$.

(2). Check that

$$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$$

induces a linear isomorphism.

(3). Check that

$$\left(\sum_i a_i\right) \otimes b \mapsto \sum_i a_i \otimes b$$

$$a \otimes \left(\sum_j b_j\right) \mapsto \sum_j a \otimes b_j$$

induces an isomorphism.

(4). Check that $a \otimes b \mapsto b \otimes a$ induces an isomorphism.

(5). Construct the inverse $\beta\colon \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C)) \to \operatorname{Hom}_S(A \otimes_R B, C)$ by

$$[\beta(g)](a \otimes b) = [g(a)](b).$$

□

Corollary 7.3. *Let V and W be vector spaces over a field. Then*

1) $\dim V\otimes W=(\dim V)(\dim W)$ *if V and W are finitely dimensional;*
2) $\theta_{V,W}\colon V^*\otimes W\cong \mathrm{Hom}(V,W)$ *given by*

$$[\theta_{V,W}(f\otimes y)](x)=f(x)y.$$

3) $(V\otimes W)^*\cong V^*\otimes W^*$*;*

Proof. Let $\mathbb{F}$ be the ground field. (1). Let $\dim V=n$ and $\dim W=m$. Then

$$V\otimes W\cong \mathbb{F}^n\otimes\mathbb{F}^m\cong \bigoplus_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \mathbb{F}\otimes_{\mathbb{F}}\mathbb{F}\cong\mathbb{F}^{nm}.$$

(2). Let $W=\bigoplus_{j\in J}\mathbb{F}$. Then

$$V^*\otimes W\cong\bigoplus_{j\in J}V^*\otimes\mathbb{F}\xrightarrow[\cong]{\oplus\,\theta_{V,\mathbb{F}}}\bigoplus_{j\in J}\mathrm{Hom}(V,\mathbb{F})=\bigoplus\mathrm{Hom}(V,W).$$

[**Note.** One can directly show that θ is an isomorphism.

(3).

$$(V\otimes W)^*=\mathrm{Hom}(V\otimes W,\mathbb{F})\cong\mathrm{Hom}(V,\mathrm{Hom}(W,\mathbb{F}))=\mathrm{Hom}(V,W^*)\cong V^*\otimes W^*.$$

□

Corollary 7.4. *Let ξ and η be vector bundles over a space B. Then the vector bundle* $\mathrm{Hom}(\xi,\eta)$ *isomorphic to $\xi^*\otimes\eta$.*

Note. One need to be careful for tensor product over which ring (if the ground ring is unclear). For instance, $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}\cong\mathbb{C}$ is a 2-dimensional real space, while $\mathbb{C}\otimes_{R}\mathbb{C}$ is a 4-dimensional real space.

7.2. **Tensor Algebras.** Let R be a commutative ring with identity. A unitary R-module A is called an (associative) *algebra* if there is multiplication

$$\mu\colon A\otimes_R A\to A$$

[**Note.** Any middle linear map $A\times A\to A$ induces a unique linear map $A\otimes_R A\to A$.] and a unit

$$\eta\colon R\longrightarrow A.$$

such that

1) μ and η are morphisms of R-bi-modules;
2) unitary property: there is a commutative diagram

$$\begin{array}{ccccc} A\otimes\mathbb{F} & \xrightarrow{\mathrm{id}_A\otimes\eta} & A\otimes A & \xleftarrow{\eta\otimes\mathrm{id}_A} & \mathbb{F}\otimes A\\ \| & & \downarrow{\scriptstyle\mu} & & \|\\ A & = & A & = & A;\end{array}$$

3) associative law: there is a commutative diagram

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\mu\otimes \mathrm{id}_A} & A\otimes A \\ \Big\downarrow{\scriptstyle \mathrm{id}\otimes\mu} & & \Big\downarrow{\scriptstyle \mu} \\ A\otimes A & \xrightarrow{\mu} & A. \end{array}$$

Note. Write $a\cdot b$ for $\mu(a\otimes b)$. From (1), $\eta(r)=\eta(r\cdot 1)=r\eta(1)$. Let $e=\eta(1)\in A$, then by (2) we have

$$a\cdot e=a\cdot\eta(1)=a=e\cdot a.$$

The map $\eta\colon R\to A$ is multiplicative because

$$\eta(r)\cdot\eta(s)=[r\eta(1)]\cdot[s\cdot\eta(1)]=rs[\eta(1)\cdot\eta(1)]=rs\eta(1)=\eta(rs).$$

Exercise 7.1. *Let A and B be algebra over R. Show that $A\otimes_R B$ is also algebra over R under the multiplication*

$$(a\otimes b)\cdot(a'\otimes b')=(a\cdot a')\otimes(b\cdot b'.$$

Let V be an R-module. The *tensor algebra* $T(V)$ is defined by

$$T(V)=\bigoplus_{i=0}^{\infty}V^{\otimes n}$$

as an R-module, where $V^{\otimes 0}=\mathbb{R}$ and

$$V^{\otimes_R n}=V\otimes_R V\otimes_R\cdots\otimes_R V$$

is the n-fold self tensor product of V over R. The elements in $T(V)$ can be written as (non-commutative) polynomials

$$f=\sum_{i=0}^{\infty}f_i$$

with only finitely many $f_i\neq 0$, where $f_i\in V^{\otimes i}$ called the i-homogeneous component of f.

Let $\eta\colon R\to T(V)$ be the inclusion and let

$$\mu\colon T(V)\otimes_R T(V)\longrightarrow T(V)$$

be induced from the formal product:

$$\left(V^{\otimes a}\right)\otimes_R\left(V^{\otimes b}\right) = V^{\otimes a+b},$$

that is, if $f=\sum\limits_{i=0}^{\infty}f_i$ and $g=\sum\limits_{i=0}^{\infty}g_i$ with $f_i,g_i\in V^{\otimes i}$, then

$$f\cdot g=\mu(f,g)=\sum_{k=0}^{\infty}\sum_{i+j=k}f_ig_j.$$

The above multiplication and unit make $T(V)$ to be an algebra over R called the *tensor algebra* generated by V.

Proposition 7.5. *The tensor algebra $T(V)$ has the following universal property:*

Let A be any algebra and let $f\colon V \to A$ be any R-linear map, then there is a ***unique*** *algebraic map $\tilde{f}\colon T(V) \to A$ such that $\tilde{f}|_V = f$.*

Proof. Let $\tilde{f}\colon T(V) \to A$ be the map such that f restricted to $V^{\otimes n}$ is given by

$$\tilde{f}(a_1 \otimes \cdots \otimes a_n) = f(a_1)\cdot f(a_2)\cdots f(a_n)$$

for $a_i \in V$. Then $\tilde{f}$ is an algebraic map such that $\tilde{f}|_V = f$.

Let $\phi\colon T(V) \to A$ any algebraic map such that $\phi|_V = f$. Then

$$\phi(a_1 \otimes \cdots \otimes a_n) = f(a_1)\cdot f(a_2)\cdots f(a_n)$$

for $a_i \in V$ and so $\phi = \tilde{f}$. □

Note. Let the ground ring R be a field and let V be an m-dimensional vector space. Let $\{e_1, \ldots, e_m\}$ be a basis for V. Then

$$e_{i_1} \otimes \cdots e_{i_k},$$

$1 \leq i_1, \ldots, i_k \leq m$, is a basis for $V^{\otimes k}$.

7.3. **Graded Modules, Graded Commutative Algebras and Exterior Algebras.** Let R be a commutative ring with identity. A *graded module* M means a direct sum

$$M = \bigoplus_{n=-\infty}^{\infty} M_n.$$

A *graded map* $f\colon M \to N$ means

$$f = \bigoplus_{n=-\infty}^{\infty} f_n\colon M = \bigoplus_{n=-\infty}^{\infty} M_n \longrightarrow N = \bigoplus_{n=1}^{\infty} N_n$$

for $f_n\colon M_n \to N_n$. Let M and N be graded modules. Then $M \otimes_R N$ is a graded module in the sense that

$$M \otimes_R N = \bigoplus_{n=-\infty}^{\infty} \left(\bigoplus_{i+j=n} M_i \otimes N_j \right),$$

in other words,

$$(M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes N_j.$$

For graded modules M and N, let

$$T\colon M \otimes_R N \xrightarrow{\cong} N \otimes_R M$$

be the graded map such that T_n is given by

$$T_n\colon (M \otimes_R N)_n = \bigoplus_{i+j=n} M_i \otimes_R N_j \xrightarrow{\bigoplus_{i+j=n}(-1)^{ij}\tau} \bigoplus_{j+i=n} N_j \otimes_R M_i,$$

where $\tau(a \otimes b) = b \otimes a$. In other words

$$T(a \otimes b) = (-1)^{|a||b|} b \otimes a$$

for $a \in M_{|a|}$ and $b \in N_{|b|}$.

A graded R-module A is called a *graded algebra* if A admits a graded multiplication $\mu\colon A\otimes_R A \to A$ and a graded unit $\eta\colon R \to A$, where R is regarded as a graded ring in the sense that $(R)_0 = R$ and $(R)_n = 0$ for $n \neq 0$. A graded algebra A is called *commutative* if the diagram

$$\begin{array}{ccc} A\otimes_R A & \xrightarrow{T} & A\otimes_R A \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ A & = & A \end{array}$$

commutes, in other words,

$$a\cdot b = (-1)^{|a||b|} b\cdot a$$

for $a \in A_{|a|}$ and $b \in A_{|b|}$.

For graded algebras A and B, the graded module $A\otimes_R B$ has the multiplication given by the following exercise.

Exercise 7.2. *Let A and B be graded algebras over R. Show that $A\otimes_R B$ is a graded algebra under the multiplication:*

$$A\otimes_R B\otimes_R A\otimes_R B \xrightarrow{\mathrm{id}_A\otimes T\otimes \mathrm{id}_B} A\otimes_R A\otimes_R B\otimes_R B \xrightarrow{\mu_A\otimes\mu_B} A\otimes_R B.$$

Proposition 7.6. *Let R be a commutative ring with identity and let V be a graded R-module. Then $T(V)$ is a graded R-algebra.* □

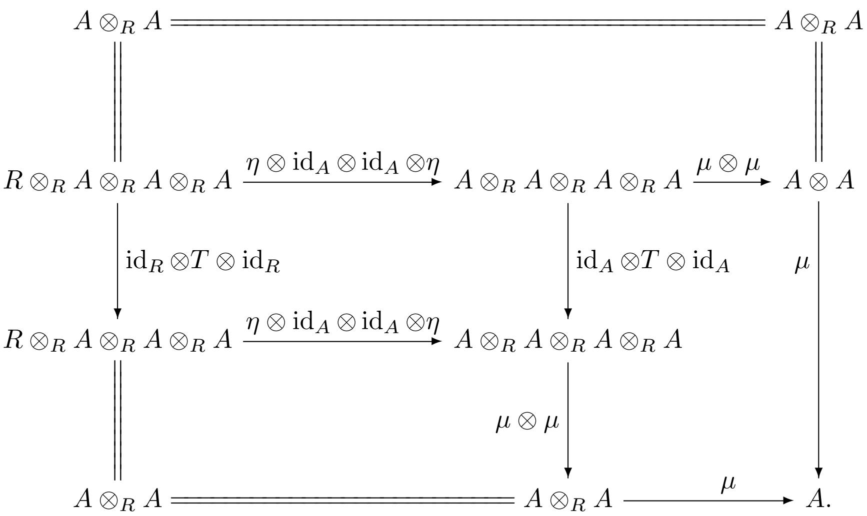
Proposition 7.7. *A graded algebra A is commutative if and only if the multiplication $\mu\colon A\otimes_R A \to A$ is an algebraic map.*

Proof. Suppose that A is commutative. Then there is a commutative diagram

$$\begin{array}{ccccc} A\otimes_R A\otimes_R A\otimes_R A & \xleftarrow{\mathrm{id}_A\otimes T\otimes\mathrm{id}_A} & A\otimes_R A\otimes_R A\otimes_R A & \xrightarrow{\mu\otimes\mu} & A\otimes_R A \\ \| & & \downarrow{\scriptstyle \mathrm{id}_A\otimes\mu\otimes\mathrm{id}_A} & & \downarrow{\scriptstyle\mu} \\ A\otimes_R A\otimes_R A\otimes_R A & \xrightarrow{\mathrm{id}\otimes\mu\otimes\mathrm{id}} & A\otimes_R A\otimes_R A & & \\ \downarrow{\scriptstyle\mu\otimes\mu} & & \downarrow{\scriptstyle\mu\circ(\mu\otimes\mathrm{id}_A)} & & \\ A\otimes_R A & \xrightarrow{\mu} & A & = & A \end{array}$$

Thus μ is an algebraic map.

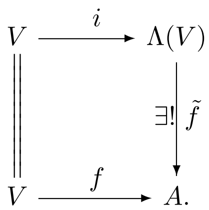
Conversely suppose that μ is an algebraic map. Then there is a commutative diagram



Thus A is graded commutative. $\square$

Let V be a graded R-module. The *free commutative graded algebra* $\Lambda(V)$, with a morphism of R-modules $i\colon V \to \Lambda(V)$, is defined by the following universal property:

> Let A be any commutative graded algebra and let $f\colon V \to A$ be any morphism of graded R-modules. Then there is a **unique** morphism of graded R-algebras $\tilde{f}\colon \Lambda(V) \to A$ such that $f = \tilde{f} \circ i$, that is there is a commutative diagram



By the universal property, $\Lambda(V)$ is unique to isomorphism of algebras. The existence is given by the construction that $\Lambda(V)$ is the quotient algebra of $T(V)$ modulo the two sided ideal generated by the graded commutators

$$[a, b] = ab - (-1)^{|a||b|}ba$$

for $a \in T(V)_{|a|}$ and $b \in T(V)_{|b|}$.

Proposition 7.8. *Let R be a field and let V and W be graded vector spaces. Then*

$$\Lambda(V \oplus W) \cong \Lambda(V) \otimes \Lambda(W).$$

Proof. We use the universal property to prove this statement. Let $i\colon V \otimes W \to \Lambda(V) \otimes \Lambda(W)$ be the map

$$V \oplus W \xrightarrow{(v,w)\mapsto v\otimes e + e\otimes w} \Lambda(V) \otimes \Lambda(W).$$

Let A be any commutative graded R-algebra and let $f\colon V\otimes W\to A$ be any morphism of R-modules. Let f_1 and f_2 be given by

$$f_1\colon V \xhookrightarrow{v\mapsto(v,0)} V\oplus W \xrightarrow{f} A$$

$$f_2\colon W \xhookrightarrow{w\mapsto(0,w)} V\oplus W \xrightarrow{f} A.$$

Then there are unique morphisms of algebras

$$\tilde f_1\colon \Lambda(V)\to A$$

$$\tilde f_2\colon \Lambda(W)\to A$$

such that $\tilde f_1\circ i_V = f_1$ and $\tilde f_2\circ i_W = f_2$. Let $\tilde f$ be the composite

$$\Lambda(V)\otimes\Lambda(W) \xrightarrow{\tilde f_1\otimes\tilde f_2} A\otimes A \xrightarrow{\mu_A} A.$$

then $\tilde f$ is an algebraic map such that $\tilde f\circ i = f$.

Let $g\colon \Lambda(V)\otimes\Lambda(W)\to A$ be any algebraic map such that $g\circ i = f$. Then the composite

$$T(V\oplus W) \xrightarrow{\tilde i} \Lambda(V)\otimes\Lambda(W) \xrightarrow{g} A$$

is a (unique) algebraic map such that $g\circ\tilde i|_{V\oplus W} = \tilde f\circ\tilde i|_{V\oplus W} = f$. Since $\tilde i$ is onto, $g=\tilde f$ and so $\Lambda(V)\otimes\Lambda(W)\cong\Lambda(V\oplus W)$ by the universal property. □

Now we consider the special cases of $\Lambda(V)$. Let ground ring be a field $\mathbb{F}$ of characteristic 0 and let V be a finite dimensional vector space with a basis $\{x_1,\ldots,x_n\}$.

Case I. Consider V as a graded module by $(V)_2 = V$ and $(V)_n = 0$ for $n\neq 2$. Then $\Lambda(V)$ is the *polynomial algebra* $\mathbb{F}[V]$ generated by $x_1,\ldots,x_n$ because the commutators

$$[x_i,x_j] = x_ix_j - (-1)^{|x_i||x_j|}x_jx_i = x_ix_j - x_jx_i.$$

Case II. Consider V as a graded module by $(V)_1 = V$ and $(V)_n = 0$ for $n\neq 1$. Then $\Lambda(V)$ is the *exterior algebra* generated by $x_1,\ldots,x_n$, that is, $\Lambda(V)$ is generated by $x_1,\ldots,x_n$ subject to the relations

$$0 = [x_i,x_j] = x_ix_j - (-1)^{|x_i||x_j|}x_jx_i = x_ix_j + x_jx_i$$

or $x_ix_j = -x_jx_i$. In particular $2x_i^2 = 0$ and so $x_i^2=0$ since $\mathbb{F}$ is of characteristic 0.

Case III. Consider V as a graded module by $V = V_1\oplus V_2$ where $\dim V_1 = s$ and $\dim V_2 = t$ with $s+t=n$. Then

$$\Lambda(V)\cong\Lambda(V_1)\otimes\Lambda(V_2) = \Lambda(V_2)\otimes\mathbb{F}[V_2]$$

is the tensor product of the polynomial algebra and the exterior algebra.

7.4. **Tensor Bundles, Tensor Fields.** Now let $\mathbb{F} = \mathbb{R}$. Let V be a finite dimensional vector spaces. Let

$$T^r_s(V) = V^{\otimes r} \otimes (V^*)^{\otimes s} \cong \mathrm{Hom}(V^{\otimes s}, V^{\otimes r}).$$

Let ξ be a vector bundle over B. Then $T^r_s(\xi)$ is a vector bundle over B, called the *tensor bundle* of type (r,s) on ξ, because T is (covariant on the first r factors and contravariant on the rest s factors) functor on V.

Let M be a differentiable manifold and let ξ be the tangent bundle over M. Then

$$T^r_s(M) := T^r_s(\xi) = T^{\otimes r} \otimes (T^*)^{\otimes s}(M)$$

is called the *tensor bundle* of type (r,s) of M. A (smooth) cross-section of the tensor bundle $T^r_s(M)$ is called a *tensor field* of type (r,s).

Note that $T^1_0(M) = T(M)$ and $T^0_1(M) = T^*(M)$. Thus a tensor field of type $(1,0)$ is a vector field and a tensor field of type $(0,1)$ is a covector field.

Recall that

$$V^* \otimes V^* \cong (V \otimes V)^* = \mathrm{Hom}(V \otimes V, \mathbb{R}) \cong \{f\colon V \oplus V \to \mathbb{R} \mid f \text{ bi-linear}\}.$$

The Riemann metric on a differentiable manifold can be described as a tensor field of type $(0,2)$. Let M be a differentiable manifold. A *Riemann metric* is a tensor field $g \in T^0_2(M)$ such that for each m, g_m is an inner product, that is, positive definite symmetric and bilinear.

7.5. **Differential Forms.** Let $\mathbb{F} = \mathbb{R}$ and let V be a vector space of dimension n. Recall that the exterior algebra $\Lambda(V^*)$ can be considered as skew symmetric algebra generated by V^*, that is modulo the relations

$$xy = -yx$$

for $x, y \in V^*$. [Or free graded commutative algebra by considering V^* as a graded module by setting $(V^*)_1 = V^*$ and $(V^*)_q = 0$ for $q \neq 1$.] Then there is decomposition

$$\Lambda(V^*) = \bigoplus_{k=0}^{\infty} \Lambda^k(V^*),$$

where $\Lambda^k(V^*)$ consists of homogeneous elements of degree k. Each Λ^k is a contravariant functor on V and so, for any vector bundle ξ over B, there are constructions $\Lambda(\xi^*)$ and $\Lambda^k(\xi^*)$.

Let M be a differentiable manifold. A *differential form* of order k [or simply k-*form*] is defined to be a (smooth) cross-section of $\Lambda^k T^*(M)$. We work out a local basis for $\Lambda^k T^*(M)$.

The multiplication in $\Lambda(V^*)$ is denoted by $x \wedge y$. Let $\{\partial_1, \ldots, \partial_n\}$ be a basis for V and let $\{dx_1, \ldots, dx_n\}$ be the dual basis for V^*. By the definition, $\Lambda(V^*)$ is generated by $dx_1, \ldots, dx_n$ subject to the relations

$$dx_i \wedge dx_j = -dx_j \wedge dx_i$$

for any $1 \le i, j \le n$. In particular, $dx_i \wedge dx_i = 0$ for $1 \le i \le n$. It follows that $\Lambda^k(V^*)$ is spanned by the elements

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n$.

Proposition 7.9. *A basis for* $\Lambda^k(V^*)$ *is given by*

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n$. *In particular,* $\dim \Lambda^k(V^*) = \binom{n}{k}$.

Proof. Let $A = \bigoplus_{k=0}^{\infty} A_k$ be the graded vector spaces with A_k is the vector space spanned by the letters

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n$. Note that $A_i = V^*$. Define a formal multiplication on A by

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \cdot (dx_{j_1} \wedge \cdots dx_{j_t}) = \begin{cases} 0 & \text{if} \quad \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_t\} \neq \emptyset \\ \pm dx_{l_1} \wedge \cdots \wedge dx_{l_{k+t}} & \text{if} \quad \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_t\} = \emptyset, \end{cases}$$

where $l_1 < l_2 < \ldots < l_{k+t}$, $\{l_1, \ldots, l_{k+t}\} = \{i_1, \ldots, i_k, j_1, \ldots, j_t\}$ and the sign $\pm$ is obtained by reorganizing $(i_1, \ldots, i_k, j_1, \ldots, j_t)$ into $(l_1, \ldots, l_{k+t})$. For instance,

$$(dx_1 \wedge dx_3 \wedge dx_5) \cdot (dx_2 \wedge dx_4 \wedge dx_6) = (-1)^{2+1} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6.$$

Then it is straightforward to show that A is the free graded commutative algebra generated by V^* by checking that A is an algebra with the universal property. Thus $A \cong \Lambda(V^*)$ as graded algebras and so

$$A_k \cong \Lambda^k(V^*).$$

In particular,

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n$ give a basis for $\Lambda^k(V^*)$. □

Let ω be a k-form on an n-dimensional manifold M. Then ω admits local coordinates

$$\sum_{1 \le i_1 < \cdots < i_k \le n} a^{i_1 \cdots i_k}(x_1, \ldots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Note. $dx_i \wedge dx_j = -dx_j \wedge dx_i$ can be observed from: $d(x_i x_j) = x_i dx_j + x_j dx_i$. Assume that we want $ddx_i = ddx_j = dd(x_i x_j) = 0$. Then

$$0 = dd(x_i x_j) = d(x_i dx_j + x_j dx_i) = (dx_i \wedge dx_j + x_i ddx_j) + (dx_j \wedge dx_i + x_j ddx_i) = dx_i \wedge dx_j + dx_j \wedge dx_i.$$

8. Orientation and Integration

8.1. **Alternating Multi-linear Functions and Forms.** Let V be a vector space. An k-linear map

$$\mu \colon V \times \cdots \times V \to \mathbb{R}$$

is called *alternating* if it is zero whenever two coordinates are equal, that is,

$$\mu(x_1, \ldots, x_k) = 0$$

if $x_p = x_q$ for some $1 \le p < q \le k$.

Proposition 8.1. *Let μ be a k-linear map. Then μ is alternating is and only if*

$$\mu(\cdots x_i \cdots x_j \cdots) = -\mu(\cdots x_j \cdots x_i \cdots).$$

for $1 \le i < j \le k$.

Proof. Suppose that μ is alternating. Then

$$\begin{aligned} 0 &= \mu(\cdots x_i + x_j \cdots x_i + x_j \cdots) \\ &= \mu(\cdots x_i \cdots x_i \cdots) + \mu(\cdots x_i \cdots x_j \cdots) + \mu(\cdots x_j \cdots x_i \cdots) + \mu(\cdots x_j \cdots x_j \cdots) \\ &= \mu(\cdots x_i \cdots x_j \cdots) + \mu(\cdots x_j \cdots x_i \cdots). \end{aligned}$$

Conversely assuming the assumption in the statement holds, then

$$\mu(\cdots x_i \cdots x_i \cdots) = -\mu(\cdots x_i \cdots x_i \cdots)$$

and so

$$2\mu(\cdots x_i \cdots x_i \cdots) = 0 \quad \Rightarrow \quad \mu(\cdots x_i \cdots x_i \cdots) = 0.$$

□

Let $\tilde{\Lambda}^k(V^*)$ denote the set of alternating k-linear maps from $V^{\oplus k}$ to $\mathbb{R}$. Then $\tilde{\Lambda}^k(V^*)$ is a vector space.

Proposition 8.2. *There is a canonical isomorphism of vector spaces:*

$$\rho_V\colon\ \left(\Lambda^k(V)\right)^* \longrightarrow \tilde{\Lambda}^k(V^*).$$

Proof. Let B_k be the vector space of k-linear maps from $V^{\oplus k}$ to $\mathbb{R}$. Then the map

$$\rho\colon\ \left(V^{\otimes k}\right)^* \longrightarrow B_k \quad [\rho(f)](x_1, \ldots, x_k) = f(x_1 \otimes \cdots x_k)$$

is an isomorphism by the universal property of tensor product that any multi-linear map induces a unique linear map from tensor product. Let I_k be the sub vector space of $V^{\otimes k}$ spanned by the k-fold tensors

$$\cdots \otimes \overset{i}{\alpha} \otimes \cdots \otimes \overset{j}{\alpha} \otimes \cdots$$

for some $i < j$ and $\alpha \in V$. Then $\rho(f)$ is alternating if and only if $f|_{I_k} = 0$. It follows that there is a commutative diagram

$$\begin{array}{ccccc} \left(V^{\otimes k}/I_k\right)^* & \hookrightarrow & \left(V^{\otimes k}\right)^* & \twoheadrightarrow & I_k^* \\ \cong\downarrow \rho & & \downarrow \rho & & \\ \tilde{\Lambda}^k(V^*) & \hookrightarrow & B_k, & & \end{array}$$

that is, ρ restricted to $\left(V^{\otimes k}/I_k\right)^*$ gives an isomorphism

$$\rho\colon\ \left(V^{\otimes k}/I_k\right)^* \longrightarrow \tilde{\Lambda}^k(V^*).$$

Now we use the graded arguments to show that $V^{\otimes k}/I_k \cong \Lambda^k(V)$, where V is regarded as a graded module by setting $(V)_1 = V$ and $(V)_n = 0$ for $n \neq 1$. Let

$$I = \bigoplus_{k=1}^{\infty} I_k \subseteq T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

Then I is a graded two-sided ideal of $T(V)$ and so the graded quotient

$$A = \bigoplus_{k=0} V^{\otimes k}/I_k,$$

where $I_0 = 0$, is a graded quotient algebra. Check that A is graded commutative. Then the quotient $T(V) \to A$ factors through $\Lambda(V)$ by the universal property of $\Lambda(V)$. On other hand, since $x_ix_j + x_jx_i \in I$ by considering $(x_i + x_j)^{\otimes 2} = x_i^2 + x_ix_j + x_jx_i + x_j^2$, the quotient $T(V) \to \Lambda(V)$ factors through A. Thus $A = \Lambda(V)$ and hence the result. □

Proposition 8.3. *Let V be a finite dimensional vector space. There is a canonical isomorphism*

$$\theta_V : \Lambda^k(V^*) \cong \left(\Lambda^k(V)\right)^*$$

Proof. We consider V as a graded module by setting $(V)_1 = V$ and $(V)_n = 0$ for $n \neq 1$. The tensor algebra $T(V)$ is then a graded algebra and so is $T(V) \otimes T(V)$, where the multiplication on $T(V) \otimes T(V)$ is given by

$$T(V) \otimes T(V) \otimes T(V) \otimes T(V) \xrightarrow{\mathrm{id} \otimes T \otimes} T(V) \otimes T(V) \otimes T(V) \otimes T(V) \xrightarrow{\mu \otimes \mu} T(V) \otimes T(V).$$

The linear map

$$\Delta \colon V \longrightarrow T(V) \otimes T(V) \quad x \mapsto x \otimes 1 + 1 \otimes x$$

induces a unique morphism of algebras

$$\psi \colon T(V) \longrightarrow T(V) \otimes T(V)$$

with a commutative diagram

$$\begin{array}{ccc} T(V) & \xrightarrow{\psi} & T(V) \otimes T(V) \\ \downarrow q & & \downarrow q \otimes q \\ \Lambda(V) & \xrightarrow{\psi} & \Lambda(V) \otimes \Lambda(V). \end{array}$$

By taking the graded dual [Let $A = \oplus A_n$ be a graded module. The graded dual $A^* = \oplus A_n^*$.], then there is a commutative diagram

$$\begin{array}{ccc} T(V)^* & \xleftarrow{\psi^*} & T(V)^* \otimes T(V)^* \\ \uparrow q^* & & \uparrow q^* \otimes q^* \\ \Lambda(V)^* & \xleftarrow{\psi^*} & \Lambda(V)^* \otimes \Lambda(V)^*. \end{array}$$

It follows that $\Lambda(V)^*$ is a graded commutative algebra under ψ^*. [Check the associativity and commutativity.] The inclusion $V^* \to \Lambda(V)^*$ induces a unique morphism of graded algebras

$$\theta_V : \Lambda(V^*) \longrightarrow \Lambda(V)^*$$

such that θ_V restricted to V^* is the identity. Now one can prove that θ_V is an isomorphism by induction on the dimension of V and by using the fact that

1) θ_V is an isomorphism if $\dim V = 1$.

2) For any finite dimensional vector spaces V and W, there is a commutative diagram

$$\begin{array}{ccc}
\Lambda(V^* \oplus W^*) & \xrightarrow{\theta_{V\oplus W}} & \Lambda(V\oplus W)^* \\
\uparrow \cong & & \downarrow \cong \\
\Lambda(V^*)\otimes\Lambda(W^*) & \xrightarrow{\theta_V\otimes\theta_W} & \Lambda(V)^*\otimes\Lambda(W)^*.
\end{array}$$

□

The map

$$\Lambda(V^*)\cong\Lambda(V)^* \xrightarrow{q^*} T(V)^*$$

is a (faithful) representation of $\Lambda(V^*)$ into $T(V)^*$ via the multiplication

$$\psi^*\colon T(V)^*\otimes T(V)^* \longrightarrow T(V)^* \quad \alpha\otimes\beta\mapsto\alpha*\beta.$$

That is for $\phi_1,\ldots,\phi_k\in V^*$

$$q_n^*(\phi_1\wedge\cdots\wedge\phi_k)=\phi_1*\cdots*\phi_k.$$

The multiplication ψ^* can be described as follows:

Let $\langle f,y\rangle=f(y)$ for $f\in W^*$ and $y\in W$. Let $\phi_1\in T(V)^*_k=(V^{\otimes k})^*$, $\phi_2\in T(V)^*_l=(V^{\otimes l})^*$, $x_1,\ldots,x_{k+l}\in V$. Then

$$\begin{aligned}
\langle\phi_1*\phi_2,x_1\cdots x_{k+1}\rangle &= \langle\psi^*(\phi_1\otimes\phi_2),x_1\cdots x_{k+l}\rangle\\
&=\langle\phi_1\otimes\phi_2,\psi(x_1\cdots x_{k+l}\rangle\\
&=\langle\phi_1\otimes\phi_2,\psi(x_1)\cdot\cdots\cdot\psi(x_{k+l})\rangle\\
&=\langle\phi_1\otimes\phi_2,(x_1\otimes1+1\otimes x_1)\cdot\cdots\cdot(x_{k+1}\otimes1+1\otimes x_{k+l}\rangle\\
&=\sum_{(k,l)-\text{shuffles}}\epsilon_\sigma\langle\phi_1,x_{\sigma(1)}\cdots x_{\sigma(k)}\rangle\langle\phi_2,x_{\sigma(k+1)}\cdots x_{\sigma(k+l)}\rangle,
\end{aligned}$$

where ϵ_σ is the sign of σ and a permutation σ is a (k,l)-shuffle if

$$\sigma(1)<\cdots<\sigma(k) \qquad \sigma(k+1)<\cdots<\sigma(k+l).$$

The above formula describes the *exterior product* on alternating k-linear maps using signed shuffle product.

Proposition 8.4. *For $\phi_1,\ldots,\phi_k\in V^*$ and $v_1,\ldots,v_k\in V$, there is formula*

$$(\phi_1\wedge\cdots\wedge\phi_k)(v_1,\ldots,v_k)=\det(\phi_i(v_j)).$$

Proof. The proof is given by induction on k. The result is trivial for $k=1$. Assume that the statement holds for $k-1$. Then

$$\begin{aligned}
&(\phi_1\wedge(\phi_2\wedge\cdots\wedge\phi_k))(v_1,\ldots,v_k)=\\
&=\sum_{(1,k-1)-\text{shuffles}}\epsilon_\sigma\phi_1(v_{\sigma(1)})(\phi_2\wedge\cdots\wedge\phi_k))(v_{\sigma(2)},\ldots,v_{\sigma(k)})\\
&=\sum_{j=1}^k(-1)^{j-1}\phi_1(v_j)\det(\phi_l(v_m))_{1j}=\det(\phi_i(v_j)),
\end{aligned}$$

where A_{ij} denote the sub-matrix of A by deleting i-row and j-column. □

Remark. The graded dual $T(V)^*$ of the tensor algebra with the multiplication ψ^* is the cohomology of the loop space $\Omega\Sigma X$. Since $T(V)^*$ is a contravariant functor on V, there is a construction $T(\xi)^*$ for any vector bundle ξ while ψ^* gives certain algebraic structure on the bundle $T(\xi)^*$. If $\xi = TM$, then $T(\xi)^* = \bigoplus_{s=0}^{\infty} T_s^0 M$. If the ground field is of characteristic 0 (for instance in that case $\mathbb{F} = \mathbb{R}, \mathbb{C}$), there is an functorial algebraic decomposition $T(V)^* \cong \Lambda(V^*) \otimes B(V^*)$ for certain commutative algebra $B(V^*)$. The factor $\Lambda(V^*)$ gives differential forms while the Riemann metric as a tensor form of type $(0,2)$ comes from the factor $B(V^*)$.

Let N be a differentiable manifold and let ω be a k-form on N. Then, for each $q \in N$, ω_q is an alternating k-linear map on T_qN. Globally ω is a smooth map $\omega\colon T^{\oplus k}N \to \mathbb{R}$ such that ω is fibrewise alternating k-linear. Let $f\colon M \to N$ be a smooth map. Then the composite

$$T^{\oplus k}M \xrightarrow{T^{\oplus k}f} T^{\oplus k}N \xrightarrow{\omega} \mathbb{R}$$

is a k-form on M called the *pull-back* of ω by f, denoted by $f^*\omega$. In detail $f^*(\omega)$ is given by the formula

$$f^*(\omega)(p)(v_1, \ldots, v_k) = \omega(f(p))(Tf(v_1), \ldots, Tf(v_k)).$$

For 0-forms, that is smooth functions $g \in C^\infty(N)$, we define $f^*(g) = g \circ f \in C^\infty(M)$.

Proposition 8.5. *The following properties hold for the pull-backs of forms:*

1) $f^*(\lambda_1\omega_1 + \lambda_2\omega_2) = \lambda_1 f^*(\omega_1) + \lambda_2 f^*(\omega_2)$ *for real numbers* λ_i *and* k*-forms* ω_i*;*
2) $f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)$. *In particular, when* g *is a smooth function on* N*, then* $f^*(g\omega) = f^*(g)f^*(\omega) = (g \circ f)f^*(\omega)$.
3) $(g \circ f)^*(\omega) = f^*(g^*(\omega))$ *for a smooth map* $g\colon N \to P$ *and a* k*-form* ω *on* P.

Proof. The proof can be given by showing that the formulas hold on each fibre. (1) and (3) are obvious. (2) follows from the fact that any linear map $f\colon V \to W$ induces a linear map $f^*\colon W^* \to V^*$ and so an algebraic map $\Lambda(f^*)\colon \Lambda(W^*) \to \Lambda(V^*)$. ☐

Note. A smooth map $f\colon M \to N$ may not send a vector field on M to N in general. Note that covector field is a 1-form and so any covector field on N can be pull-back to M by f.

Proposition 8.6. *Let* V *and* W *be open in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *with coordinates* (x_i) *and* (y_j) *respectively. Let* $\theta\colon V \to W$ *be a smooth map. Then the following holds:*

1) $\theta^*(dy^i) = \sum_{j=1}^n \frac{\partial\theta_i}{\partial x_j} dx^j$.
2) *If* θ *is a diffeomorphism (so that* $m = n$*) and* $\omega = f dy^1 \wedge \cdots dy^m$ *is a (general)* m*-form on* W*, then*

$$\theta^*(\omega) = (f \circ \theta)\left[\det\left(\frac{\partial\theta_i}{\partial x_j}\right)\right] dx^1 \wedge dx^2 \wedge \cdots dx^m.$$

Proof. (1). If $\theta^*(dy^i) = \sum_{j=1}^{n} f_j dx^j$, then $f_j = (\theta^*(dy^i))\left(\frac{\partial}{\partial x_j}\right)$. Now

$$\theta^*(dy^i)\left(\frac{\partial}{\partial x_j}\right) = dy^i\left(\theta_*\frac{\partial}{\partial x_j}\right) = dy^i\left(\sum_{k=1}^{m}\frac{\partial\theta_k}{\partial x_j}\frac{\partial}{\partial y_k}\right) = \frac{\partial\theta_i}{\partial x_j}.$$

(2).

$$\theta^*(\omega) = \theta^*(f)\theta^*(dy^1) \wedge \theta^*(dy^2) \wedge \theta^*(dy^m).$$

$$= (f \circ \theta)\theta^*(dy^1) \wedge \theta^*(dy^2) \wedge \theta^*(dy^m).$$

From Part (1), $\theta^*(dy^i) = \frac{\partial \theta_i}{\partial x_j} dx^j$, that is

$$\theta^*(dy^i)\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial \theta_i}{\partial x_j},$$

the assertion follows from Proposition 8.4 because

$$\theta^*(\omega)\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right) = (f \circ \theta)\det\left(\frac{\partial \theta_i}{\partial x_j}\right)$$

and $dx^1 \wedge \cdots \wedge dx^m$ is only basis for $\Lambda^m(\mathbb{R}^m)$. □

8.2. **Orientation of Manifolds.** An atlas $\{U_\alpha, \phi_\alpha\}$ of a real vector bundle ξ is called *oriented* if its transitive functions, as elements in $\mathrm{GL}_n(\mathbb{R})$, have positive determinants on its domain $U_\alpha \cap U_\beta$. A real vector bundle is called *oriented* if it has an oriented atlas.

Let ξ be a real n-dimensional vector bundle over B. Recall that there is a principal $\mathrm{GL}_n(\mathbb{R})$-bundle $\xi^{\mathrm{GL}_n(\mathbb{R})}$ such that $\xi \cong \xi^{\mathrm{GL}_n(\mathbb{R})}[\mathbb{R}^n]$. Let

$$\mathrm{GL}_n^+(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) > 0\}.$$

Then $\mathrm{GL}_n^+(\mathbb{R})$ is a (normal) subgroup of $\mathrm{GL}_n(\mathbb{R})$ with cokernel $\mathbb{Z}/2 = \{-1, 1\}$. If ξ is oriented, then the transitive functions map into $\mathrm{GL}_n^+(\mathbb{R})$ and so there is a principal $\mathrm{GL}_n^+(\mathbb{R})$-bundle $\xi^{\mathrm{GL}_n^+(\mathbb{R})}$ such that $\xi \cong \xi^{\mathrm{GL}_n^+(\mathbb{R})}[\mathbb{R}^n]$. Conversely, if there is a principal $\mathrm{GL}_n^+(\mathbb{R})$-bundle $\tilde{\xi}$ such that $\xi \cong \tilde{\xi}[\mathbb{R}^n]$, then ξ is oriented because its transitive functions obtained from an atlas of $\tilde{\xi}$ map into $\mathrm{GL}_n^+(\mathbb{R})$. This gives the following:

Proposition 8.7. *An n-dimensional real vector bundle ξ is oriented if and only if there exists a principal $\mathrm{GL}_n^+(\mathbb{R})$-bundle $\xi^{\mathrm{GL}_n^+(\mathbb{R})}$ such that $\xi \cong \xi^{\mathrm{GL}_n^+(\mathbb{R})}[\mathbb{R}^n]$.* □

Let ξ be **any** principal $\mathrm{GL}_n(\mathbb{R})$-bundle given by $E \to B$. Then there is a commutative diagram of principal G-bundles

$$\begin{array}{ccccc}
\mathrm{GL}_n^+(\mathbb{R}) & \hookrightarrow & \mathrm{GL}_n(\mathbb{R}) & \twoheadrightarrow & \mathbb{Z}/2 \\
\| & & \downarrow & & \downarrow \\
\mathrm{GL}_n^+(\mathbb{R}) & \hookrightarrow & E & \longrightarrow & \tilde{B}^\xi \\
& & \downarrow & & \downarrow \\
& & B & = & B,
\end{array}$$

where $\tilde{B}^\xi = E/\operatorname{GL}_n^+(\mathbb{R})$. We have two principal G-bundles $E \to \tilde{B}^\xi$ with $G = \operatorname{GL}_n^+(\mathbb{R})$ and $\tilde{B}^\xi \to B$ with $G = \mathbb{Z}/2$. Moreover there is a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathbb{R}^n & = = = & \mathbb{R}^n \\
 & & \downarrow & & \downarrow \\
\mathbb{Z}/2 & \hookrightarrow & E \times_{\operatorname{GL}_n^+(\mathbb{R})} \mathbb{R}^n & \longrightarrow & E \times_{\operatorname{GL}_n(\mathbb{R})} \mathbb{R}^n = E(\xi[\mathbb{R}^n]) \\
\| & & \downarrow & \text{pull-back} & \downarrow \\
\mathbb{Z}/2 & \hookrightarrow & \tilde{B}^\xi & \longrightarrow & B,
\end{array}
\tag{14}
$$

where the right two columns are induced vector bundles.

Proposition 8.8. *Let $\xi[\mathbb{R}^n]$ be a vector bundle over B, where ξ is a principal $\operatorname{GL}_n(\mathbb{R})$-bundle over B. Then the following are equivalent each other:*

1) *$\xi[\mathbb{R}^n]$ is oriented.*
2) *The two covering $B^\xi \to B$ has a cross-section.*
3) *The two covering $B^\xi \to B$ is a trivial bundle.*

Proof. (2)$\Longleftrightarrow$(3) is obvious.

(1) $\Longrightarrow$ (2). Suppose that $\xi[\mathbb{R}^n]$ is oriented. Then there is a principal $\operatorname{GL}_n^+(\mathbb{R})$-bundle ξ' such that $\phi\colon \xi'[\mathbb{R}^n] \cong \xi[\mathbb{R}^n]$. The bundle isomorphism $\phi\colon \xi'[\mathbb{R}^n] \to \xi[\mathbb{R}^n]$ induces a morphism of principal $\operatorname{GL}_n^+(\mathbb{R})$-bundles

$$
\begin{array}{ccc}
E(\xi') & \xrightarrow{\tilde{\phi}} & E(\xi) \\
\downarrow & & \downarrow \\
B & = = = & B,
\end{array}
$$

where $\tilde{\phi}$ is the unique map such that the following diagram commutes

$$
\begin{array}{ccccccc}
E(\xi'[\mathbb{R}^n]^{\oplus n}) & = & E(\xi') \times_{\operatorname{GL}_n^+(\mathbb{R})} (\mathbb{R}^n)^{\oplus n} & \xrightarrow{\phi^{\oplus n}} & E(\xi) \times_{\operatorname{GL}_n(\mathbb{R})} (\mathbb{R}^n)^{\oplus n} & = & E(\xi[\mathbb{R}^n]^{\oplus n}) \\
 & & \uparrow & & \uparrow & & \\
E(\xi') & = = = & E(\xi') \times \{e_1, \ldots, e_n\} & \xrightarrow{\tilde{\phi}} & E(\xi) \times \{e_1, \ldots, e_n\} & = = = & E(\xi)
\end{array}
$$

because in each fibre

$$\phi^{\oplus n}(b \cdot g, e_1, e_2, \ldots, e_n) = \phi^{\oplus n}(b, g(e_1), g(e_2), \ldots, g(e_n))$$

$$= (b, \phi_b \circ g(e_1), \phi_b \circ g(e_2), \ldots, \phi_b \circ g(e_n)) = (b \cdot \phi_b \circ g, e_1, \ldots, e_n) \in E(\xi).$$

[**Note.** For each point $b\cdot g\in E(\xi')$, g can be regarded as a linear isomorphism of the fibre $p(\xi'[\mathbb{R}^n])^{-1}(b)\to p(\xi'[\mathbb{R}^n])^{-1}(b)$ and $\phi\circ g$ defines an element in $E(\xi)$.] Now from the commutative diagram

$$\begin{array}{ccccc}
\mathrm{GL}_n^+(\mathbb{R}) & \hookrightarrow & E(\xi) & \longrightarrow & \tilde{B}^{\xi}\\
\| & & \uparrow & & \uparrow\\
\mathrm{GL}_n^+(\mathbb{R}) & \hookrightarrow & E(\xi') & \longrightarrow & B,
\end{array}$$

there is a cross-section $B\to\tilde{B}^{\xi}$.

(2) $\Longrightarrow$ (1). Suppose that $B^{\xi}\to B$ has a cross-section $s\colon B\to B^{\xi}$. Let ξ' be the pull-back

$$\begin{array}{ccccc}
\mathrm{GL}_n^+(\mathbb{R}^n) & \hookrightarrow & E(\xi) & \longrightarrow & \tilde{B}^{\xi}\\
\| & & \uparrow & \text{pull-back} & \uparrow s\\
\mathrm{GL}_n^+(\mathbb{R}^n) & \hookrightarrow & E(\xi') & \longrightarrow & B.
\end{array}$$

Then $\xi'[\mathbb{R}^n]\cong\xi[\mathbb{R}^n]$ and so ξ is oriented. □

Note. From the proof, for a vector bundle $\xi[\mathbb{R}^n]$, the two covering $\tilde{B}^{\xi}\to B$ is independent on the choice of principal G-bundle representation ξ, that is, if $\xi'[\mathbb{R}^n]\cong\xi[\mathbb{R}^n]$, then there is a bundle isomorphism from $\tilde{B}^{\xi'}\to B$ to $\tilde{B}^{\xi}\to B$.

Exercise 8.1. *Let ξ be an oriented vector bundle over B and let $f\colon X\to B$ be a map. Prove that $f^*\xi$ is oriented.*

Exercise 8.2. *Let ξ and η be oriented vector bundles. Show that $\xi\oplus\eta$ is also oriented.*

Exercise 8.3. *Let ξ and η be vector bundles . Prove that $\xi\times\eta$ given by $E(\xi)\times E(\eta)\to B(\xi)\times B(\eta)$ is oriented if and only if both ξ and η are oriented.*

.

Lemma 8.9. *Let ξ be a real n-dimensional vector bundle over a paracompact space B. Then there is a principal $O(n)$-bundle $\xi^{O(n)}$ such that $\xi\cong\xi^{O(n)}[\mathbb{R}^n]$. Moreover ξ is oriented if and only if there is a principal $SO(n)$-bundle $\xi^{SO(n)}$ such that $\xi\cong\xi^{SO(n)}[\mathbb{R}^n]$.*

Proof. By Corollary 4.16, there is a map $f\colon B\to G_{n,\infty}(\mathbb{R})$ such that $\xi\cong f^*(\gamma_n^{\infty})$. Note that γ_n^{∞} is induced by the canonical principal $O(n)$-bundle $V_{n,\infty}(\mathbb{R})\to G_{n,\infty}(\mathbb{R})$. Let $E(\xi^{O(n)})$ be the pull-back

$$\begin{array}{ccc}
E(\xi^{O(n)}) & \longrightarrow & V_{n,\infty}(\mathbb{R})\\
\downarrow & & \downarrow\\
B & \xrightarrow{f} & G_{n,\infty}(\mathbb{R}).
\end{array}$$

Then $\xi^{O(n)}[\mathbb{R}^n]\cong f^*(\gamma_n^{\infty})\cong\xi$.

Suppose that $\xi \cong \xi^{SO(n)}[\mathbb{R}^n]$. Then ξ is oriented because the transitive functions of $\xi^{SO(n)}$ has determinant 1.

Conversely assume that ξ is oriented. By the proof of the above proposition, the bundle isomorphism $\xi^{O(n)}[\mathbb{R}^n] \cong \xi^{\mathrm{GL}_n(\mathbb{R})}$ induces a morphism of principal $O(n)$-bundles $\phi\colon \xi^{O(n)} \to \xi^{\mathrm{GL}_n(\mathbb{R})}$. Since $SO(n) = O(n) \cap \mathrm{GL}_n^+(\mathbb{R})$, there is a commutative diagram of principal G-bundles

$$\begin{array}{ccccc} \mathrm{GL}_n^+(\mathbb{R}) & \hookrightarrow & E(\xi^{\mathrm{GL}_n(\mathbb{R})}) & \longrightarrow & \tilde{B}^{\xi} \\ \Vert & & \uparrow\cong & & \uparrow\cong \\ SO(n) & \hookrightarrow & E(\xi^{O(n)}) & \longrightarrow & \tilde{B}^{\xi}. \end{array}$$

Since ξ is oriented, there is a cross-section $s\colon B \to B^{\xi}$. Let ξ' be the pull-back

$$\begin{array}{ccc} E(\xi') & \longrightarrow & E(\xi^{O(n)}) \\ \downarrow & & \downarrow \\ B & \xrightarrow{s} & \tilde{B}^{\xi}. \end{array}$$

. Then ξ' is a principal $SO(n)$-bundle such that $\xi \cong \xi'[\mathbb{R}^n]$. □

Theorem 8.10. *Let ξ be a real n-dimensional vector bundle over a paracompact space B. Then the following statements are equivalent:*

1) *ξ is oriented.*
2) *The n-fold exterior product bundle $\Lambda^n(\xi)$ is a trivial line bundle.*
3) *There is a nowhere zero cross-section to the n-fold exterior bundle $\Lambda^n(\xi)$.*
4) *The dual bundle ξ^* is oriented.*
5) *There is a nowhere zero cross-section to the n-fold exterior bundle $\Lambda^n(\xi)$.*
6) *The n-fold exterior product bundle $\Lambda^n(\xi^*)$ is a trivial line bundle.*

Proof. Note that a real line bundle is trivial if and only if it has a nowhere zero-cross-section. Thus (2) ⇔ (3) and (5) ⇔ (6). It obvious that (1) ⇔ (4).

(1) $\Longrightarrow$ (2). Suppose that ξ is oriented. By the previous lemma, there is an $SO(n)$-bundle $\xi^{SO(n)}$ such that $\xi \cong \xi^{SO(n)}[\mathbb{R}^n]$. Thus

$$E(\Lambda^n(\xi)) = E(\xi^{SO(n)}) \times_{SO(n)} \Lambda^n(\mathbb{R}^n).$$

Let $g \in SO(n)$, that is, $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism. By Proposition 8.4,

$$\Lambda^n(g)\colon \Lambda^n(\mathbb{R}) = \mathbb{R} \longrightarrow \Lambda^n(\mathbb{R}^n) = \mathbb{R}$$

is multiple by $\det(g) = 1$. Thus the action of $SO(n)$ on $\Lambda^n(\mathbb{R}) = \mathbb{R}$ is trivial and $\Lambda^n(\xi)$ is a trivial bundle. Similarly (4) $\Longrightarrow$ (5).

(3) $\Longrightarrow$ (1). Suppose that $\Lambda^n\xi$ is a trivial bundle. There is a principal $O(n)$-bundle $\xi^{O(n)}$ such that $\xi \cong \xi^{O(n)}[\mathbb{R}^n]$. Since $SO(n)$ acts trivially on $\Lambda^n(\mathbb{R}^n)$,

$$\begin{aligned} E(\Lambda^n(\xi)) = E(\xi^{O(n)}) \times_{O(n)} \Lambda^n(\mathbb{R}^n) &= \left[E(\xi^{O(n)})/SO(n)\right] \times_{O(n)/SO(n)} \Lambda^n(\mathbb{R}^n) \\ &= \tilde{B}^{\xi} \times_{\mathbb{Z}/2} \Lambda^n(\mathbb{R}^n). \end{aligned}$$

By the assumption, the line bundle $\tilde{B}^\xi \times_{\mathbb{Z}/2} \Lambda^n(\mathbb{R}^n)$ is trivial and so the principal $\mathbb{Z}/2$-bundle $\tilde{B}^\xi \to B$ is trivial. Thus there is a cross-section $B \to \tilde{B}^\xi$ and so ξ is oriented. □

A differentiable manifold M is called *oriented* if its tangent bundle is oriented. In other words, M has an atlas which Jacobians of coordinate transformations are always positive. An *orientation* of M is a *maximal* oriented atlas. A nowhere zero m-form is called a *volume form*, where $m = \dim M$.

Corollary 8.11. 1) *A paracompact manifold M is oriented if and only if there exists a volume form on M.*
2) *For any differentiable manifold M, there is an oriented differentiable manifold $\tilde{M}$ with principal $\mathbb{Z}/2$-bundle $\tilde{M} \to M$.* □

8.3. **Integration of m-forms on Oriented m-Manifolds.** Let M be a differentiable m-manifold and let ω be an m-form on M such that

$$\operatorname{supp}(\omega) = \overline{\{b \mid \omega(b) \neq 0\}}$$

is contained in the coordinate neighborhood U_α of a chart (U_α, ϕ_α), where $\phi_\alpha\colon U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$. Let

$$(\phi_\alpha^{-1})^*(\omega) = a_\alpha(x) dx^1 \wedge \cdots \wedge dx^m.$$

Then we define

$$\int_M \omega = \int_{U_\alpha} \omega = \int_{\phi_\alpha} a_\alpha(x) dx^1 dx^2 \cdots dx^n$$

to be the Riemann integral over $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^m$.

We have to check that this definition is independent on local coordinate system and **we will need the condition that M is oriented**.

If the support of ω is also contained in the coordinate neighborhood of another chart (U_β, ϕ_β). Let $\phi_\alpha = \theta_{\alpha\beta} \circ \phi_\beta$, that is $\theta_{\alpha\beta}$ is given by

$$\phi_\beta(U_\alpha \cap U_\beta) \xrightarrow{\phi_\beta^{-1}} U_\alpha \cap U_\beta \xrightarrow{\phi_\alpha} \phi_\alpha(U_\alpha \cap U_\beta).$$

Then

$$\begin{aligned} (\phi_\alpha^{-1})^*(\omega) &= (\phi_\beta^{-1} \circ \theta_{\alpha\beta}^{-1})^*(\omega) \\ &= (\theta_{\alpha\beta}^{-1})^* \circ (\phi_\beta^{-1})^*(\omega) = \theta_{\beta\alpha}^* \circ (\phi_\beta^{-1})^*(\omega) \\ &= (\theta_{\beta\alpha})^*(a_\beta dy^1 \wedge \cdots \wedge dy^m) \\ &= (a_\beta \circ \theta_{\beta\alpha} \det(J(\theta_{\beta\alpha})) dx^1 \wedge \cdots \wedge dx^m, \end{aligned}$$

where $J(\theta_{\beta\alpha})$ is the Jacobian matrix of $\theta_{\beta\alpha}$. Thus

$$a_\alpha(x) = (a_\beta \circ \theta_{\beta\alpha})(x) \det(J(\theta_{\beta\alpha}))(x)$$

and, writing $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$,

$$\begin{aligned} \int_{U_\alpha} \omega = \int_{U_{\alpha\beta}} \omega &= \int_{\phi_\alpha(U_{\alpha\beta})} a_\alpha(x) dx^1 dx^2 \cdots dx^m \\ &= \int_{\theta_{\alpha\beta}(\phi_\beta(U_{\alpha\beta}))} a_\alpha(x) dx^1 dx^2 \cdots dx^m \\ &= \int_{\phi_\beta(U_{\alpha\beta})} a_\alpha \theta_{\alpha\beta} |\det(J(\theta_{\alpha\beta}))| dy^1 dy^2 \cdots dy^m \end{aligned}$$

by changing of variable formula for multiple integrals in $\mathbb{R}^m$. Assume that M is oriented and both charts belong to the same orientation, then $\det(J(\theta_{\alpha\beta})) > 0$. This precisely why we need our manifolds to be oriented. Under this assumption,

$$\int_{U_\alpha} \omega = \int_{\phi_\beta(U_{\alpha\beta})} a_\beta(y) dy^1 dy^2 \cdots dy^m = \int_M \omega$$

independent on oriented charts.

Definition 8.12. Let ω be an m-form on an oriented compact manifold and let $\{g_i \mid i = 1, \ldots, k\}$ be a partition of unitary subordinate to a open covering $\{U_i\}$ from the orientation of M. Then we define

$$\int_M \omega = \sum_{i=1}^{k} \int_M g_i \omega,$$

where $\int_M g_\alpha \omega$ is defined as above, the summation is well defined because there are only finitely many non-zero terms.

Lemma 8.13. *$\int_M \omega$ is well-defined.*

Proof. Let $\{h_j \mid j = 1, \ldots, l\}$ be another partition of unitary subordinate to $\{V_j\}$ on M. Then $\{g_i h_j\}$ is a partition of unitary subordinate to $\{U_i \cap V_j\}$ on M. Now

$$\sum_{i=1}^{k} \int_M g_i \omega = \sum_{i=1}^{k} \int_M \sum_{j=1}^{l} h_j g_i \omega = \sum_{i=1}^{k} \sum_{j=1}^{l} \int_M h_j g_i \omega$$

$$= \sum_{j=1}^{l} \int_M \sum_{i=1}^{k} h_j g_i \omega = \sum_{j=1}^{l} \int_M \left(\sum_{i=1}^{k} g_i \omega \right) = \sum_{j=1}^{l} \int_M h_j \omega.$$

□

9. The Exterior Derivative and the Stokes Theorem

9.1. **Exterior Derivative on $\mathbb{R}^m$.** For a sequence $I = (i_1, i_2, \ldots, i_k)$, write dx^I for $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

Definition. For a k-form $\omega = \sum_I a_I dx^I$ on an open subset U of $\mathbb{R}^m$, define $d\omega$ by

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \sum_{i=1}^{m} \frac{\partial a_I}{\partial x_i} dx^i \wedge dx^I.$$

From the definition, $d(dx^I) = d(1 \cdot dx^I) = 0 \wedge dx^I = 0$.

For instance, let $m = 4$, $da(x) = \sum_{i=1}^{m} \frac{\partial a(x)}{\partial x_i} dx_i = \sum_{i=1}^{4} \frac{\partial a(x)}{\partial x_i} dx_i$,

$$d(a(x)dx^2 \wedge dx^4) = \sum_{i=1}^{4} \frac{\partial a(x)}{\partial x_i} dx_i \wedge dx^2 \wedge dx^4$$

$$= \frac{\partial a(x)}{\partial x_1} dx_1 \wedge dx^2 \wedge dx^4 + \frac{\partial a(x)}{\partial x_3} dx_3 \wedge dx^2 \wedge dx^4 = \frac{\partial a(x)}{\partial x_1} dx_1 \wedge dx^2 \wedge dx^4 - \frac{\partial a(x)}{\partial x_3} dx_2 \wedge dx^3 \wedge dx^4.$$

Proposition 9.1. *The following properties hold for the operation d:*

1) *d is R-linear:*

$$d(\lambda_1\omega + \lambda_2\omega_2) = \lambda_1 d\omega_1 + \lambda_2 d\omega_2$$

for real constants λ_i *and* k*-forms* ω_i.

2) *d is a graded derivation:*

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$$

for a k*-form* ω_1 *and any form* ω_2.

3) *d is a differential:*

$$d^2\omega = d(d(\omega)) = 0$$

for any form ω.

4) *d is natural:*

$$d(f^*\omega) = f^*(d\omega),$$

where ω *is a* k*-form on* $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ *and* $f\colon V \to U$ *is differentiable.*

Proof. (1) is obvious.

(2). Let $\omega_1 = \sum_I a_I dx^I$ and let $\omega_2 = \sum_J b_J dx^J$. Then

$$d(\omega_1 \wedge \omega_2) = d\left(\sum_{I,J} a_I b_J dx^I \wedge dx^J\right)$$

$$= \sum_{I,J} d(a_I b_J) \wedge dx^I \wedge dx^J$$

$$= \sum_{I,J} b_J da_I \wedge dx^I \wedge dx^J + a_I db_J \wedge dx^I \wedge dx^J$$

$$= \sum_J b_J d\omega_1 \wedge dx^J + (-1)^k \sum_{I,J} a_I dx^I \wedge db_J \wedge dx^J = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2.$$

(3) For 0-forms (or functions) $a(x)$,

$$d^2(a(x)) = d\left(\sum_{i=1}^m \frac{\partial a}{\partial x_i} dx^i\right) = \sum_{i=1}^m d\left(\frac{\partial a}{\partial x_i} dx^i\right) = \sum_{i,j=1}^m \frac{\partial^2 a}{\partial x_i \partial x_j} dx^j \wedge dx^i = 0$$

because $\frac{\partial^2 a}{\partial x_i \partial x_j} = \frac{\partial^2 a}{\partial x_j \partial x_i}$, $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$. For general case $\omega = \sum_I a_I dx^I$,

$$d^2\omega = d\left(\sum_I da_I \wedge dx^I\right) = \sum_I d(da_I \wedge dx^I) = \sum_I d^2(a_I) \wedge dx^I - da_I \wedge d(dx^I) = 0$$

(4). If $\omega = g$ is a 0-form on $\mathbb{R}^m$, then

$$f^*(dg) = f^*\left(\sum_{i=1}^m \frac{\partial g}{\partial x_i} dx^i\right) = \sum_{i=1}^m \left(\frac{\partial g}{\partial x_i} \circ f\right) f^*(dx^i) = \sum_{i=1}^m \left(\frac{\partial g}{\partial x_i} \circ f\right) \sum_{j=1}^n \frac{\partial f_i}{\partial y_j} dy^j$$

$$= \sum_{j=1}^n \frac{\partial g \circ f}{\partial y_j} dy^j = d(g \circ f) = d(f^*(g)).$$

If $\omega = dx^i$, then

$$d(f^*(dx^i)) = d\left(\sum_{j=1}^n \frac{\partial f_i}{\partial y_j} dy_j\right) = \sum_{j=1}^n d\left(\frac{\partial f_i}{\partial y_j} dy_j\right) = \sum_{j,k=1}^n \frac{\partial^2 f_i}{\partial y_j \partial y_k} dy^k \wedge dy^j = 0 = f^*(d(dx^i)).$$

If $\omega = dx^I = dx^{i_1} \wedge \cdots \wedge dx^k$, then

$$d(f^* dx^I) = d(f^*(dx^{i_1}) \wedge \cdots \wedge f^*(dx^{i_k}))$$

$$= \sum_{s=1}^m (-1)^{s-1} f^*(dx^{i_1}) \wedge \cdots \wedge f^*(dx^{i_{s-1}}) \wedge d(f^*(dx^{i_s})) \wedge f^*(dx^{i_{s+1}}) \wedge \cdots f^*(dx^{i_k}) = 0 = f^*(d(dx^I)).$$

For a general k-form $\omega = \sum_I a_I dx^I$,

$$d(f^*\omega) = \sum_I d(f^*(a_I) f^*(dx^I)) = \sum_I d(f^*(a_I)) \wedge f^*(dx^I) + f^*(a_I) d(f^*(dx^I))$$

$$= \sum_I f^*(da_I)) \wedge f^*(dx^I) = f^*\left(\sum_I da_I \wedge dx^I\right) = f^*(d\omega).$$

We finish the proof. □

9.2. **Exterior Derivative on Manifolds.** Let M be a differentiable n-manifold with a k-form ω on M. Let (U_α, ϕ_α) be a chart, that is, $\phi_\alpha \colon U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$. Then

$$\omega_\alpha = (\phi_\alpha^{-1})^*(\omega)$$

is a k-form on $\phi_\alpha(U_\alpha)$.

Definition. For a k-form ω on M, the differential $d\omega$ is the $(k+1)$-form η such that for each chart (U_α, ϕ_α),

$$\eta \mid_{U_\alpha} = \phi_\alpha^*(d(\phi_\alpha^{-1})^*(\omega)).$$

Proposition 9.2. *Let M be a differentiable manifold and let ω be a k-form on M. Then $d\omega$ is well-defined.*

Proof. Let (U_β, ϕ_β) be another chart and let $\theta_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$. Then, restricting to $U_\alpha \cap U_\beta$,

$$\phi_\alpha^*(d(\phi_\alpha^{-1})^*(\omega)) = (\theta_{\alpha\beta}\phi_\beta)^* d(\phi_\alpha^{-1})^*(\omega)$$

$$= (\phi_\beta)^*(\theta_{\alpha\beta}^* d(\phi_\alpha^{-1})^*(\omega)) = (\phi_\beta)^*\left(d\left(\theta_{\alpha\beta}^*(\phi_\alpha^{-1})^*(\omega)\right)\right) = (\phi_\beta)^*\left(d\left((\phi_\alpha^{-1} \circ \theta_{\alpha\beta})^*(\omega)\right)\right)$$

$$= (\phi_\beta)^*\left(d\left((\phi_\beta^{-1})^*(\omega)\right)\right)$$

and hence the result. □

Note. We do not assume that M is oriented.

Exercise 9.1. *Let ω is a k-form on M with $k \geq 1$. Prove that*

$$d\omega(X_1, \ldots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}))$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),$$

where $\hat{X}_i$ means 'omit X_i'.

9.3. **Stokes' Theorem.** Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. Recall that a (topological) n-manifold means a Hausdorff space that is locally homeomorphic to open subsets of $\mathbb{R}^n_+$. Let M be a differentiable manifold and let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for M. Recall that

$$\partial M = \bigcup_\alpha \phi_\alpha^{-1} \left(\phi_\alpha(U_\alpha) \cap \partial R^n\right),$$

where $\partial R^n = \mathbb{R}^{n-1}$, with an atlas given by

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M}) \mid U_\alpha \cap \partial M \neq \emptyset\}.$$

We now examine the coordinate transformation from an oriented atlas on M at a point $p \in \partial M$. Note that the transition functions $\theta_{\alpha\beta}$ map $\phi_\beta(U_\alpha \cap U_\beta) \cap \partial\mathbb{R}^n$ onto $\phi_\alpha(U_\alpha \cap U_\beta) \cap \partial\mathbb{R}^n$. Let $\bar{\theta}_{\alpha\beta}$ be the restriction $\theta_{\alpha\beta}|_{\mathbb{R}^{n-1}}$ of $\theta_{\alpha\beta}$, mapping (the open subset of) $\mathbb{R}^{n-1}$ into $\mathbb{R}^{n-1}$. Then the Jacobian $J\theta_{\alpha\beta}$ has the block decomposition

$$J\theta_{\alpha\beta} = \begin{pmatrix} J\bar{\theta}_{\alpha\beta} & 0 \\ * & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

with $\frac{\partial x_n}{\partial y_n} > 0$ because $J\theta_{\alpha\beta}$ is a linear transformation mapping $\mathbb{R}^{n-1}$ into $\mathbb{R}^{n-1}$ and the last row $(\partial x_n \partial y_1, \ldots, \partial x_n \partial y_n)$ has positive projection on the last coordinate. Since $\det(J\theta_{\alpha\beta}) = \det(J\bar{\theta}_{\alpha\beta}) \cdot \frac{\partial x_n}{\partial y_n}$, we have the following.

Proposition 9.3. *If M is an oriented, then ∂M is also oriented.* □

Let $\iota\colon \partial M \to M$ be the inclusion. Let (U, ϕ) be a chart of M. Then there is a commutative diagram

$$\begin{array}{ccccccc}
\partial M & \supseteq W = U \cap \partial M & \overset{\iota}{\hookrightarrow} & U & \subseteq & M \\
& \Big\downarrow \phi|_W & & \Big\downarrow \phi & & \\
& \mathbb{R}^{n-1} & \overset{\kappa}{\hookrightarrow} & \mathbb{R}^n_+, & &
\end{array}$$

where κ is the canonical inclusion.

Lemma 9.4. *$\iota^* \circ \phi^*(dx^n) = 0$ and $\iota^* \circ \phi^*(dx^i) = (\phi|_W)^*(dx^i)$ for $1 \leq i \leq n-1$.*

Proof. Note that $\kappa_*(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$ for $1 \leq i \leq n-1$. Then

$$(\kappa^*(dx^i)) \left(\frac{\partial}{\partial x_j}\right) = dx^i \left(\kappa_* \left(\frac{\partial}{\partial x_j}\right)\right) = \delta_{ij}$$

and so $\kappa^*(dx^i) = dx^i$ for $1 \le i \le n-1$ and $\kappa^*(dx^n) = 0$. It follows that

$$\iota^* \circ \phi^*(dx^i) = (\phi|_W)^* \circ \kappa^*(dx^i) = (\phi|_W)^*(dx^i) \qquad \text{for } 1 \le i \le n-1,$$

$$\iota^* \circ \phi^*(dx^n) = (\phi|_W)^* \circ \kappa^*(dx^n) = (\phi|_W)^*(dx^n) = 0.$$

□

From the arguments of Proposition 9.3, if $(x^1, \ldots, x^n)$ is a *positive local coordinate system* for M, then we can choose a positive orientation on ∂M such that $(x^1, \ldots, x^{n-1})$ is a positive local coordinate system. However, it is more convenient to choose the positive orientation of ∂M such that if *n is even, then $(x^1, \ldots, x^{n-1})$ is positive, while if n is odd, $(x^1, \ldots, x^{n-1})$ is negative.* (In other words, the modification of the orientation of ∂M is modified by $(-1)^n$.) Such an orientation of ∂M is called an *orientation of ∂M compatible with the orientation of M.*

Remark 9.5. If M is a closed smooth compact oriented region the boundary of $\mathbb{R}^n$, then the compatible orientation on ∂M is given such that the normal vector to ∂M is *outgoing.* For instance, if the locally coordinate system of M is given by $(x_1, \ldots, x_n)$ with $x_j \ge 0$ and $-\epsilon_i < x_i < \epsilon_i$ for $i \ne j$, then the outgoing normal vector is given by $N_p = -dx^j$ and

$$(-dx^j)\wedge(-1)^j dx^1\wedge\cdots\wedge dx^{j-1}\wedge dx^{j+1}\wedge\cdots\wedge dx^n = (-1)^{j+1}(-1)^{j-1}dx^1\wedge\cdots\wedge dx^n = dx^1\wedge\cdots\wedge dx^n,$$

that is, the orientation $(-1)^j dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$ is given such that it $\wedge$-product with the outgoing normal vector (from left) is the usual orientation of $\mathbb{R}^n$. See [12, pp.119-121] and [9, p.282] for detailed explanations of the compatible orientations.

A generalization of the fundamental theorem of the integral calculus: $\int_a^b f'(x)dx = f(b) - f(a)$ is as follows:

Theorem 9.6 (Stokes)**.** *Let M^n be a compact oriented manifold with boundary, let ∂M have the compatible orientation, $\iota\colon \partial M \to M$ be the inclusion and $\omega \in \Omega^{n-1}(M)$ be an $(n-1)$-form on M. Then*

$$\int_{\partial M} \iota^*(\omega) = \int_M d\omega.$$

Proof. Suppose that the result valid for a form ω such that the support of ω lies in a coordinate neighborhood U. Then, for a general $(m-1)$-form ω, let $\{f_k \mid k = 1, \ldots, m\}$ be a partition of unity subordinate to a covering of M by coordinate neighborhoods $\{U_k\}$ from the oriented atlas on M. Then, for $\omega_k = f_k\omega$, we have $\omega = \sum_{k=1}^{m} \omega_k$ and

$$\int_{\partial M} \iota^*(\omega) = \sum_{k=1}^{n} \int_{\partial M} \iota^*(\omega_k) = \sum_{k=1}^{m} \int_M d\omega_k = \int_M d\sum_{k=1}^{m} \omega_k = \int_M \omega.$$

Thus it remains to prove the special case of the theorem.

So we suppose that the support of ω is contained in a coordinate neighborhood U for a chart (U, ϕ), and that on $V = \phi(U)$

$$(\phi^{-1})^*(\omega) = \sum_{j=1}^{n} a_j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n,$$

where $\widehat{dx^j}$ indicates that the factor dx^j is omitted. Then

$$(\phi^{-1})^*(d\omega) = d((\phi^{-1})^*(\omega)) = \sum_{j=1}^{n} d(a_j) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

$$= \sum_{j=1}^{n} \frac{\partial a_j}{\partial x_j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial a_j}{\partial x_j} dx^1 \wedge \cdots \wedge dx^n.$$

Let V be contained in the product $Q = [c_1, d_1] \times \cdots \times [c_n, d_n]$ with $c_n = 0$. Extend a_j to Q by setting that $a_j(x) = 0$ for $x \notin V$. Write Q_j for the product $[c_1, d_1] \times \cdots \times \widehat{[c_j, d_j]} \times \cdots \times [c_n, d_n]$ of all the intervals except the j-th, but continue to write x_i for the coordinate in $[c_i, d_i]$ whether it occurs before or after the deleted factor. Then

$$\int_M d\omega = \int_U d\omega = \int_V (\phi^{-1})^*(d\omega)$$

$$= \int_V \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial a_j}{\partial x_j} dx^1 \cdots dx^n$$

$$= \sum_{j=1}^{n} \int_Q (-1)^{j-1} \frac{\partial a_j}{\partial x_j} dx^1 \cdots dx^n$$

$$= \sum_{j=1}^{n} \int_{Q_j} (-1)^{j-1} \left[a_j(x_1, \ldots, x_{j-1}, d_j, x_{j+1}, \ldots, x_n)\right.$$

$$\left. - a_j(x_1, \ldots, x_{j-1}, c_j, x_{j+1}, \ldots, x_n)\right] dx^1 \cdots \widehat{dx^j} \cdots dx^n \qquad \text{(by the Fubini Theorem)}$$

Since the points

$$(x_1, \ldots, x_{j-1}, d_j, x_{j+1}, \ldots, x_n), (x_1, \ldots, x_{j-1}, c_j, x_{j+1}, \ldots, x_n), (x_1, \ldots, x_{n-1}, d_n) \notin V$$

for $j < n$,

$$a_j(x_1, \ldots, x_{j-1}, d_j, x_{j+1}, \ldots, x_n) = 0 \qquad j < n$$

$$a_j(x_1, \ldots, x_{j-1}, c_j, x_{j+1}, \ldots, x_n) \qquad j < n$$

$$a_n(x_1, \ldots, x_{n-1}, d_n) = 0.$$

Thus

$$\int_M d\omega = (-1)^n \int_{Q_n} a_n(x_1, \ldots, x_{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

On the other hand,

$$({\phi|_W}^{-1})^*(\iota^*\omega) = \kappa^*((\phi^{-1})^*(\omega)) = a_n \circ \kappa dx^1 \wedge \cdots \wedge dx^{n-1}$$

$$= a_n(x_1, \ldots, x_{n-1}, 0) dx^1 \wedge \cdots \wedge dx^{n-1}$$

and so

$$\int_{\partial M} \iota^*\omega = \int_{U \cap \partial M} \iota^*\omega = \int_{V \cap \mathbb{R}^{n-1}} ({\phi|_W}^{-1})^*(\iota^*\omega)$$

$$= (-1)^n \int_{Q_n} a_n(x_1, \ldots, x_{n-1}, 0) dx^1 \cdots dx^{n-1} \qquad \text{(where } (-1)^n \text{ is used)}$$

$$= \int_M d\omega.$$

□

Corollary 9.7. *If M^m is a compact oriented differentiable manifold without boundary and $\omega \in \Omega^{m-1}M$ is an $(m-1)$-form on M, then $\int_M d\omega = 0$.* □

Proposition 9.8. *If M is a compact oriented differentiable manifold with boundary, then there is no smooth map $f\colon M \to \partial M$ such that $f \circ \iota = \mathrm{id}_{\partial M}$.*

Proof. Suppose that there were such a map f. Let ω be a volume $(m-1)$-form on ∂M arising from the compatible orientation. Then

$$df^*(\omega) = f^*(d\omega) = 0$$

because $d\omega$ in $\Omega^m(\partial M)$ which is 0. Thus

$$0 = \int_M df^*(\omega) = \int_{\partial M} \iota^* f^*(\omega) = \int_{\partial M} \omega.$$

But, as ω is a volume form, it will have everywhere positive coefficients in every charts from the orientation atlas on ∂M. Then if $\{g_i\}$ is a subordinate partition of unity, $\partial_{\partial M} g_i \omega > 0$ and so

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} g_i \omega > 0.$$

We have the contradiction and hence the result. □

Corollary 9.9 (Brouwer Fixed Theorem)**.** *Every differentiable map $g\colon D^n \to D^n$ of the closed unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Sketch. Suppose that g has no fixed points. Let $f(p)$ be the intersection of the directional line $\overrightarrow{g(p)p}$ with boundary of ∂D^n. Then one can check that f is smooth with $\iota \circ f = \mathrm{id}_{\partial D^n}$. □

Example. **[Green's Theorem]** Let D be a domain in the plane bounded by a (piecewise) smooth closed curve C. Let

$$\omega = Pdx + Qdy,$$

then

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

and so, by the Stokes' theorem, we have

$$\int_C Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Example.[Divergence Theorem] Let D be a bounded domain in $\mathbb{R}^3$ with a smooth boundary, and let (x, y, z) be a positive coordinate system in $\mathbb{R}^3$. Set

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy.$$

Then

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz$$

and so, by the Stokes' theorem,

$$\int\int\int_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dxdydz=\int_{\partial D}\iota^*\omega.$$

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