

# LECTURE NOTES ON DIFFERENTIABLE MANIFOLDS

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## 1. TANGENT SPACES, VECTOR FIELDS IN $\mathbb{R}^n$ AND THE INVERSE MAPPING THEOREM

1.1. **Tangent Space to a Level Surface.** Let  $\gamma$  be a curve in  $\mathbb{R}^n$ :  $\gamma: t \mapsto (\gamma^1(t), \gamma^2(t), \dots, \gamma^n(t))$ . (A curve can be described as a vector-valued function. Converse a vector-valued function gives a curve in  $\mathbb{R}^n$ .) The *tangent line* at the point  $\gamma(t_0)$  is given with the direction

$$\frac{d\gamma}{dt}(t_0) = \left( \frac{d\gamma^1}{dt}(t_0), \dots, \frac{d\gamma^n}{dt}(t_0) \right).$$

(Certainly we need to assume that the derivatives exist. We may talk about *smooth curves*, that is, the curves with all continuous higher derivatives.)

Consider the level surface  $f(x^1, x^2, \dots, x^n) = c$  of a differentiable function  $f$ , where  $x^i$  refers to  $i$ -th coordinate. The *gradient vector* of  $f$  at a point  $P = (x^1(P), x^2(P), \dots, x^n(P))$  is

$$\nabla f = \left( \frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n} \right).$$

Given a vector  $\vec{u} = (u^1, \dots, u^n)$ , the *directional derivative* is

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \frac{\partial f}{\partial x^1}u^1 + \dots + \frac{\partial f}{\partial x^n}u^n.$$

The *tangent space* at the point  $P$  on the level surface  $f(x^1, \dots, x^n) = c$  is the  $(n-1)$ -dimensional (if  $\nabla f \neq 0$ ) space through  $P$  normal to the gradient  $\nabla f$ . In other words, the tangent space is given by the equation

$$\frac{\partial f}{\partial x^1}(P)(x^1 - x^1(P)) + \dots + \frac{\partial f}{\partial x^n}(P)(x^n - x^n(P)) = 0.$$

From the geometric views, the tangent space *should* consist of all tangents to the smooth curves **on the level surface** through the point  $P$ . Assume that  $\gamma$  is a curve through  $P$  (when  $t = t_0$ ) that lies in the level surface  $f(x^1, \dots, x^n) = c$ , that is

$$f(\gamma^1(t), \gamma^2(t), \dots, \gamma^n(t)) = c.$$

By taking derivatives on both sides,

$$\frac{\partial f}{\partial x^1}(P)(\gamma^1)'(t_0) + \dots + \frac{\partial f}{\partial x^n}(P)(\gamma^n)'(t_0) = 0$$

and so the tangent line of  $\gamma$  is really normal (orthogonal) to  $\nabla f$ . When  $\gamma$  runs over all possible curves on the level surface through the point  $P$ , then we obtain the tangent space at the point  $P$ .

Roughly speaking, *a tangent space is a vector space attached to a point in the surface.*

How to obtain the tangent space: *take all tangent lines of smooth curve through this point on the surface.*

**1.2. Tangent Space and Vectors Fields on  $\mathbb{R}^n$ .** Now consider the tangent space of  $\mathbb{R}^n$ . According to the ideas in the previous subsection, first we assume a given point  $P \in \mathbb{R}^n$ . Then we consider all smooth curves passes through  $P$  and then take the tangent lines from the smooth curves. The obtained vector space at the point  $P$  is the  $n$ -dimensional space. But we can look at in a little detail.

Let  $\gamma$  be a smooth curve through  $P$ . We may assume that  $\gamma(0) = P$ . Let  $\omega$  be another smooth curve with  $\omega(0) = P$ .  $\gamma$  is called to be *equivalent* to  $\omega$  if the directives  $\gamma'(0) = \omega'(0)$ . The tangent space of  $\mathbb{R}^n$  at  $P$ , denoted by  $T_P(\mathbb{R}^n)$ , is then the set of equivalence class of all smooth curves through  $P$ .

Let  $T(\mathbb{R}^n) = \bigcup_{P \in \mathbb{R}^n} T_P(\mathbb{R}^n)$ , called the tangent bundle of  $\mathbb{R}^n$ . If  $S$  is a region of  $\mathbb{R}^n$ , let  $T(S) = \bigcup_{P \in S} T_P(S)$ , called the tangent bundle of  $S$ .

**Note.** Each  $T_P(\mathbb{R}^n)$  is an  $n$ -dimensional vector space, but  $T(S)$  is *not* a vector space. In other words,  $T(S)$  is obtained by attaching a vector space  $T_P(\mathbb{R}^n)$  to each point  $P$  in  $S$ . Also  $S$  is assumed to be a region of  $\mathbb{R}^n$ , otherwise the tangent space of  $S$  (for instance  $S$  is a level surface) could be a proper subspace of  $T_P(\mathbb{R}^n)$ .

If  $\gamma$  is a smooth curve from  $P$  to  $Q$  in  $\mathbb{R}^n$ , then the tangent space  $T_P(\mathbb{R}^n)$  moves along  $\gamma$  to  $T_Q(\mathbb{R}^n)$ . The direction for this moving is given  $\gamma'(t)$ , which introduces the following important concept.

**Definition 1.1.** A *vector field*  $V$  on a region  $S$  of  $\mathbb{R}^n$  is a smooth map (also called  $C^\infty$ -map)

$$V: S \rightarrow T(S) \quad P \mapsto \vec{v}(P).$$

Let  $V: P \mapsto \vec{v}(P)$  and  $W: P \mapsto \vec{w}(P)$  be two vector fields and let  $f: S \rightarrow \mathbb{R}$  be a smooth function. Then  $V + W: P \mapsto \vec{v}(P) + \vec{w}(P)$  and  $fV: P \mapsto f(P)\vec{v}(P)$  give (pointwise) addition and scalar multiplication structure on vector fields.

**1.3. Operator Representations of Vector Fields.** Let  $J$  be an open interval containing 0 and let  $\gamma: J \rightarrow \mathbb{R}^n$  be a smooth curve with  $\gamma(0) = P$ . Let  $f = f(x^1, \dots, x^n)$  be a smooth function defined on a neighborhood of  $P$ . Assume that the range of  $\gamma$  is contained in the domain of  $f$ . By applying the chain rule to the composite  $T = f \circ \gamma: J \rightarrow \mathbb{R}$ ,

$$D_\gamma(f) := \frac{dT}{dt} = \sum_{i=1}^n \frac{d\gamma^i(t)}{dt} \frac{\partial f}{\partial x^i} \Big|_{x^i=\gamma^i(t)}$$

**Proposition 1.2.**

$$D_\gamma(af + bg) = aD_\gamma(f) + bD_\gamma(g), \quad \text{where } a, b \text{ are constant.}$$

$$D_\gamma(fg) = D_\gamma(f)g + fD_\gamma(g).$$

Let  $C^\infty(\mathbb{R}^n)$  denote the set of smooth functions on  $\mathbb{R}^n$ . An operation  $D$  on  $C^\infty(\mathbb{R}^n)$  is called a *derivation* if  $D$  maps  $C^\infty(\mathbb{R}^n)$  to  $C^\infty(\mathbb{R}^n)$  and satisfies the conditions

$$D(af + bg) = aD(f) + bD(g), \quad \text{where } a, b \text{ are constant.}$$

$$D(fg) = D(f)g + fD(g).$$

**Example:** For  $1 \leq i \leq n$ ,

$$\partial_i: f \mapsto \frac{\partial f}{\partial x^i}$$

is a derivation.

**Proposition 1.3.** Let  $D$  be any derivation on  $C^\infty(\mathbb{R}^n)$ . Given any point  $P$  in  $\mathbb{R}^n$ . Then there exist real numbers  $a^1, a^2, \dots, a^n \in \mathbb{R}$  such that

$$D(f)(P) = \sum_{i=1}^n a^i \partial_i(f)(P)$$

for any  $f \in C^\infty(\mathbb{R}^n)$ , where  $a^i$  depends on  $D$  and  $P$  but is independent on  $f$ .

*Proof.* Write  $x$  for  $(x^1, \dots, x^n)$ . Define

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(t(x - P) + P) dt.$$

Then

$$\begin{aligned} f(x) - f(P) &= \int_0^1 \frac{d}{dt} f(t(x - P) + P) dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x^i}(t(x - P) + P) \cdot (x^i - x^i(P)) dt \\ &= \sum_{i=1}^n (x^i - x^i(P)) \int_0^1 \frac{\partial f}{\partial x^i}(t(x - P) + P) dt = \sum_{i=1}^n (x^i - x^i(P)) g_i(x). \end{aligned}$$

Since  $D$  is a derivation,  $D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1)$  and so  $D(1) = 0$ . It follows that  $D(c) = 0$  for any constant  $c$ . By applying  $D$  to the above equations,

$$\begin{aligned} D(f(x)) &= D(f(x) - f(P)) = \sum_{i=1}^n D(x^i - x^i(P))g_i(x) + (x^i - x^i(P))D(g_i(x)) \\ &= \sum_{i=1}^n D(x^i)g_i(x) + (x^i - x^i(P))D(g_i(x)) \end{aligned}$$

because  $D(f(P)) = D(x^i(P)) = 0$ . Let  $a^i = D(x^i)(P)$  which only depends on  $D$  and  $P$ . By evaluating at  $P$ ,

$$D(f)(P) = \sum_{i=1}^n D(x^i)(P)g_i(P) + 0 = \sum_{i=1}^n a^i g_i(P).$$

Since

$$\begin{aligned} g_i(P) &= \int_0^1 \frac{\partial f}{\partial x^i}(t(P - P) + P)dt = \int_0^1 \frac{\partial f}{\partial x^i}(P)dt = \frac{\partial f}{\partial x^i}(P) = \partial_i(f)(P), \\ D(f)(P) &= \sum_{i=1}^n a^i \partial_i(f)(P), \end{aligned}$$

which is the conclusion. □

From this proposition, we can give a new way to looking at vector fields:  
Given a vector fields  $P \mapsto \vec{v}(P) = (v^1(P), v^2(P), \dots, v^n(P))$ , a derivation

$$D_{\vec{v}} = \sum_{i=1}^n v^i(P) \cdot \partial_i$$

on  $C^\infty(\mathbb{R}^n)$  is called an *operator representation* of the vector field  $P \mapsto \vec{v}(P)$ .

**Note.** The operation  $v^i(x)\partial_i$  is given as follows: for any  $f \in C^\infty(\mathbb{R}^n)$ ,

$$D_{\vec{v}}(f)(P) = \sum_{i=1}^n v^i(P) \cdot \partial_i(f)(P)$$

for any  $P$ .

From this new view, the tangent spaces  $T(\mathbb{R}^n)$  admits a basis  $\{\partial_1, \partial_2, \dots, \partial_n\}$ .

**1.4. Integral Curves.** Let  $V: \mathbf{x} \mapsto \vec{v}(\mathbf{x})$  be a (smooth) vector field on an neighborhood  $U$  of  $P$ . An *integral curve* to  $V$  is a smooth curve  $\mathbf{s}: (-\delta, \epsilon) \rightarrow U$ , defined for suitable  $\delta, \epsilon > 0$ , such that

$$\mathbf{s}'(t) = \vec{v}(\mathbf{s}(t))$$

for  $-\delta < t < \epsilon$ .

**Theorem 1.4.** *Let  $V: \mathbf{x} \mapsto \vec{v}(\mathbf{x})$  be a (smooth) vector field on an neighborhood  $U$  of  $P$ . Then there exists an integral curve to  $V$  through  $P$ . Any two such curves agree on their common domain.*

*Proof.* The proof is given by assuming the fundamental existence and uniqueness theorem for systems of first order differential equations.

The requirement for a curve  $\mathbf{s}(t) = (s^1(t), \dots, s^n(t))$  to be an integral curve is:

$$\begin{cases} \frac{ds^1(t)}{dt} = v^1(s^1(t), s^2(t), \dots, s^n(t)) \\ \frac{ds^2(t)}{dt} = v^2(s^1(t), s^2(t), \dots, s^n(t)) \\ \dots\dots\dots \\ \frac{ds^n(t)}{dt} = v^n(s^1(t), s^2(t), \dots, s^n(t)) \end{cases}$$

with the initial conditions

$$\begin{aligned} \mathbf{s}(0) = P \quad (s^1(0), s^2(0), \dots, s^n(0)) &= (x^1(P), x^2(P), \dots, x^n(P)) \\ \mathbf{s}'(0) = \vec{v}(P) \quad \left( \frac{ds^1}{dt}(0), \dots, \frac{ds^n}{dt}(0) \right) &= (v^1(P), \dots, v^n(P)). \end{aligned}$$

Thus the statement follows from the fundamental theorem of first order ODE.  $\square$

**Example 1.5.** Let  $n = 2$  and let  $V: P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$ , where  $v^1(x, y) = x$  and  $v^2(x, y) = y$ . Given a point  $P = (a^1, a^2)$ , the equation for the integral curve  $\mathbf{s}(t) = (x(t), y(t))$  is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = y(t) \end{cases}$$

with initial conditions  $(x(0), y(0)) = (a^1, a^2)$  and  $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, a^2)$ . Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^t).$$

**Example 1.6.** Let  $n = 2$  and let  $V: P \mapsto \vec{v}(P) = (v^1(P), v^2(P))$ , where  $v^1(x, y) = x$  and  $v^2(x, y) = -y$ . Given a point  $P = (a^1, a^2)$ , the equation for the integral curve  $\mathbf{s}(t) = (x(t), y(t))$  is

$$\begin{cases} x'(t) = v^1(\mathbf{s}(t)) = x(t) \\ y'(t) = v^2(\mathbf{s}(t)) = -y(t) \end{cases}$$

with initial conditions  $(x(0), y(0)) = (a^1, a^2)$  and  $(x'(0), y'(0)) = \vec{v}(a^1, a^2) = (a^1, -a^2)$ . Thus the solution is

$$\mathbf{s}(t) = (a^1 e^t, a^2 e^{-t}).$$

### 1.5. Implicit- and Inverse-Mapping Theorems.

**Theorem 1.7.** Let  $D$  be an open region in  $\mathbb{R}^{n+1}$  and let  $F$  be a function well-defined on  $D$  with continuous partial derivatives. Let  $(x_0^1, x_0^2, \dots, x_0^n, z_0)$  be a point in  $D$  where

$$F(x_0^1, x_0^2, \dots, x_0^n, z_0) = 0 \quad \frac{\partial F}{\partial z}(x_0^1, x_0^2, \dots, x_0^n, z_0) \neq 0.$$

Then there is a neighborhood  $N_\epsilon(z_0) \subseteq \mathbb{R}$ , a neighborhood  $N_\delta(x_0^1, \dots, x_0^n) \subseteq \mathbb{R}^n$ , and a **unique** function  $z = g(x^1, x^2, \dots, x^n)$  defined for  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$  with values  $z \in N_\epsilon(z_0)$  such that

1)  $z_0 = g(x_0^1, x_0^2, \dots, x_0^n)$  and

$$F(x^1, x^2, \dots, x^n, g(x^1, \dots, x^n)) = 0$$

for all  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$ .

2)  $g$  has continuous partial derivatives with

$$\frac{\partial g}{\partial x^i}(x^1, \dots, x^n) = -\frac{F_{x^i}(x^1, \dots, x^n, z)}{F_z(x^1, \dots, x^n, z)}$$

for all  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$  where  $z = g(x^1, \dots, x^n)$ .

3) If  $F$  is smooth on  $D$ , then  $z = g(x^1, \dots, x^n)$  is smooth on  $N_\delta(x_0^1, \dots, x_0^n)$ .

*Proof. Step 1.* We may assume that  $\frac{\partial F}{\partial z}(x_0^1, x_0^2, \dots, x_0^n, z_0) > 0$ . Since  $F_z$  is continuous, there exists a neighborhood  $N_\epsilon(x_0^1, x_0^2, \dots, x_0^n, z_0)$  in which  $F_z$  is continuous and positive. Thus for fixed  $(x^1, \dots, x^n)$ ,  $F$  is strictly increasing on  $z$  in this neighborhood. It follows that there exists  $c > 0$  such that

$$F(x_0^1, x_0^2, \dots, x_0^n, z_0 - c) < 0 \quad F(x_0^1, x_0^2, \dots, x_0^n, z_0 + c) > 0$$

with

$$(x_0^1, x_0^2, \dots, x_0^n, z_0 - c), (x_0^1, x_0^2, \dots, x_0^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \dots, x_0^n, z_0).$$

**Step 2.** By the continuity of  $F$ , there exists a small  $\delta > 0$  such that

$$F(x^1, x^2, \dots, x^n, z_0 - c) < 0 \quad F(x^1, x^2, \dots, x^n, z_0 + c) > 0$$

with

$$(x^1, x^2, \dots, x^n, z_0 - c), (x^1, x^2, \dots, x^n, z_0 + c) \in N_\epsilon(x_0^1, x_0^2, \dots, x_0^n, z_0)$$

for  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$ .

**Step 3.** Fixed  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$ ,  $F$  is continuous and strictly increasing on  $z$ . There is a **unique**  $z$ ,  $z_0 - c < z < z_0 + c$ , such that

$$F(x^1, \dots, x^n, z) = 0.$$

This defines a function  $z = g(x^1, \dots, x^n)$  for  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$  with values  $z \in (z_0 - c, z_0 + c)$ .

**Step 4.** Prove that  $z = g(x^1, \dots, x^n)$  is continuous. Let  $(x_1^1, \dots, x_1^n) \in N_\delta(x_0^1, \dots, x_0^n)$ . Let  $(x_1^1(k), \dots, x_1^n(k))$  be any sequence in  $N_\delta(x_0^1, \dots, x_0^n)$  converging to  $(x_1^1, \dots, x_1^n)$ . Let  $A$  be any subsequential limit of  $\{z_k = g(x_1^1(k), \dots, x_1^n(k))\}$ , that is  $A = \lim_{s \rightarrow \infty} z_{k_s}$ . Then, by the continuity of  $F$ ,

$$\begin{aligned} 0 &= \lim_{s \rightarrow \infty} F(x_1^1(k_s), \dots, x_1^n(k_s), z_{k_s}) \\ &= F(\lim_{s \rightarrow \infty} x_1^1(k_s), \dots, \lim_{s \rightarrow \infty} x_1^n(k_s), \lim_{s \rightarrow \infty} z_{k_s}) \\ &= F(x_1^1, \dots, x_1^n, A). \end{aligned}$$

By the unique solution of the equation,  $A = g(x_1^1, \dots, x_1^n)$ . Thus  $\{z_k\}$  converges  $g(x_1^1, \dots, x_1^n)$  and so  $g$  is continuous.

**Step 5.** Compute the partial derivatives  $\frac{\partial z}{\partial x^i}$ . Let  $h$  be small enough. Let

$$z + k = g(x^1, \dots, x^{i-1}, x^i + h, x^{i+1}, \dots, x^n),$$

that is

$$F(x^1, \dots, x^i + h, \dots, x^n, z + k) = 0$$

with  $z_0 - c < z + k < z_0 + c$ . Then

$$\begin{aligned} 0 &= F(x^1, \dots, x^i + h, \dots, x^n, z + k) - F(x^1, \dots, x^n, z) \\ &= F_{x^i}(x^1, \dots, \tilde{x}^i, \dots, x^n, \tilde{z})h + F_z(x^1, \dots, \tilde{x}^i, \dots, x^n, \tilde{z})k \end{aligned}$$

by the mean value theorem (Consider the function

$$\phi(t) = F(x^1, \dots, x^i + th, \dots, x^n, z + tk)$$

for  $0 \leq t \leq 1$ . Then  $\phi(1) - \phi(0) = \phi'(\xi)(1 - 0)$ , where  $\tilde{x}^i$  is between  $x^i$  and  $x^i + h$ , and  $\tilde{z}$  is between  $z$  and  $z + k$ . Now

$$\begin{aligned} \frac{\partial g}{\partial x_i} &= \lim_{h \rightarrow 0} \frac{g(x^1, \dots, x^{i-1}, x^i + h, x^{i+1}, \dots, x^n) - z}{h} = \lim_{h \rightarrow 0} \frac{k}{h} \\ &= - \lim_{h \rightarrow 0} \frac{F_{x_i}(x^1, \dots, \tilde{x}^i, \dots, x^n, \tilde{z})}{F_z(x^1, \dots, \tilde{x}^i, \dots, x^n, \tilde{z})} = - \frac{F_{x_i}}{F_z}, \end{aligned}$$

where  $\tilde{z} \rightarrow z$  as  $h \rightarrow 0$  because  $g$  is continuous (and so  $k \rightarrow 0$  as  $h \rightarrow 0$ ).

**Step 6.** Since  $F_z$  is not zero in this small neighborhood,  $g_{x_i}$  is continuous for each  $i$ . If  $F$  is smooth, then all higher derivatives of  $g$  are continuous and so  $g$  is also smooth.  $\square$

**Theorem 1.8** (Implicit Function Theorem). *Let  $D$  be an open region in  $\mathbb{R}^{2n}$  and let  $F_1, F_2, \dots, F_n$  be functions well-defined on  $D$  with continuous partial derivatives. Let  $(x_0^1, x_0^2, \dots, x_0^n, u_0^1, u_0^2, \dots, u_0^n)$  be a point in  $D$  where*

$$\begin{cases} F_1(x_0^1, x_0^2, \dots, x_0^n, u_0^1, u_0^2, \dots, u_0^n) = 0 \\ F_2(x_0^1, x_0^2, \dots, x_0^n, u_0^1, u_0^2, \dots, u_0^n) = 0 \\ \dots \dots \dots \\ F_n(x_0^1, x_0^2, \dots, x_0^n, u_0^1, u_0^2, \dots, u_0^n) = 0 \end{cases}$$

and the Jacobian

$$J = \frac{\partial(F_1, F_2, \dots, F_n)}{\partial(u^1, u^2, \dots, u^n)} = \det \left( \frac{\partial F_i}{\partial u^j} \right) \neq 0$$

at the point  $(x_0^1, x_0^2, \dots, x_0^n, u_0^1, u_0^2, \dots, u_0^n)$ . Then there are neighborhoods  $N_\delta(x_0^1, \dots, x_0^n)$ ,  $N_{\epsilon_1}(u_0^1)$ ,  $N_{\epsilon_2}(u_0^2)$ ,  $\dots$ ,  $N_{\epsilon_n}(u_0^n)$ , and **unique** functions

$$\begin{cases} u^1 = g_1(x^1, x^2, \dots, x^n) \\ u^2 = g_2(x^1, x^2, \dots, x^n) \\ \dots \dots \dots \\ u^n = g_n(x^1, x^2, \dots, x^n) \end{cases}$$

defined for  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$  with values  $u^1 \in N_{\epsilon_1}(u_0^1), \dots, u^n \in N_{\epsilon_n}(u_0^n)$  such that

- 1)  $u_0^i = g_i(x_0^1, x_0^2, \dots, x_0^n)$  and

$$F_i(x^1, x^2, \dots, x^n, g_i(x^1, \dots, x^n)) = 0$$

for all  $1 \leq i \leq n$  and all  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$ .

- 2) Each  $g_i$  has continuous partial derivatives with

$$\frac{\partial g_i}{\partial x^j}(x^1, \dots, x^n) = - \frac{1}{J} \cdot \frac{\partial(F_1, \dots, F_n)}{\partial(u^1, u^2, \dots, u^{j-1}, x^j, u^{j+1}, \dots, u^n)}$$

for all  $(x^1, \dots, x^n) \in N_\delta(x_0^1, \dots, x_0^n)$  where  $u^i = g_i(x^1, \dots, x^n)$ .

- 3) If each  $F_i$  is smooth on  $D$ , then each  $u^i = g_i(x^1, \dots, x^n)$  is smooth on  $N_\delta(x_0^1, \dots, x_0^n)$ .

*Sketch of Proof.* The proof is given by induction on  $n$ . Assume that the statement holds for  $n - 1$  with  $n > 1$ . (We already prove that the statement holds for  $n = 1$ .) Since the matrix

$$\left( \frac{\partial F_i}{\partial u^j} \right)$$

is invertible at the point  $P = (x_0^1, x_0^2, \dots, x_0^n, u_0^1, u_0^2, \dots, u_0^n)$  (because the determinant is not zero), we may assume that

$$\frac{\partial F_n}{\partial u^n}(P) \neq 0.$$

(The entries in the last column can not be all 0 and so, if  $\frac{\partial F_i}{\partial u^n}(P) \neq 0$ , we can interchange  $F_i$  and  $F_n$ .)

From the previous theorem, there is a solution

$$u^n = g_n(x^1, \dots, x^n, u^1, \dots, u^{n-1})$$

to the last equation. Consider

$$\begin{cases} G_1 = F_1(x^1, \dots, x^n, u^1, \dots, u^{n-1}, g_n) \\ G_2 = F_2(x^1, \dots, x^n, u^1, \dots, u^{n-1}, g_n) \\ \dots\dots\dots \\ G_{n-1} = F_{n-1}(x^1, \dots, x^n, u^1, \dots, u^{n-1}, g_n). \end{cases}$$

Then

$$\frac{\partial G_i}{\partial u^j} = \frac{\partial F_i}{\partial u^j} + \frac{\partial F_i}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j}$$

for  $1 \leq i, j \leq n-1$ , where

$$\frac{\partial F_n}{\partial u^j} + \frac{\partial F_n}{\partial u^n} \cdot \frac{\partial g_n}{\partial u^j} = 0.$$

Let

$$B = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \frac{\partial g_n}{\partial u^1} & \frac{\partial g_n}{\partial u^2} & \frac{\partial g_n}{\partial u^3} & \dots & \frac{\partial g_n}{\partial u^{n-1}} & 1 \end{pmatrix}$$

Then

$$\left( \frac{\partial F_i}{\partial u^j} \right) \cdot B = \begin{pmatrix} \left( \frac{\partial G_i}{\partial u^j} \right)_{n-1, n-1} & * \\ 0 & \frac{\partial F_n}{\partial u^n} \end{pmatrix}.$$

By taking the determinant,

$$J = \frac{\partial(F_1, \dots, F_n)}{\partial(u^1, \dots, u^n)} = \frac{\partial F_n}{\partial u^n} \cdot \frac{\partial(G_1, \dots, G_{n-1})}{\partial(u^1, \dots, u^{n-1})}.$$

Thus  $\frac{\partial(G_1, \dots, G_{n-1})}{\partial(u^1, \dots, u^{n-1})} \neq 0$  at  $P$  and, by induction, there are solutions

$$u^i = g_i(x^1, \dots, x^n)$$

for  $1 \leq i \leq n-1$ . □

**Theorem 1.9** (Inverse Mapping Theorem). *Let  $D$  be an open region in  $\mathbb{R}^n$ . Let*

$$\begin{cases} x^1 = f_1(u^1, \dots, u^n) \\ x^2 = f_2(u^1, \dots, u^n) \\ \dots\dots\dots \\ x^n = f_n(u^1, \dots, u^n) \end{cases}$$

be functions defined on  $D$  with continuous partial derivatives. Let  $(u_0^1, \dots, u_0^n) \in D$  satisfy  $x_0^i = f_i(u_0^1, \dots, u_0^n)$  and the Jacobian

$$\frac{\partial(x^1, \dots, x^n)}{\partial(u^1, \dots, u^n)} \neq 0 \quad \text{at} \quad (u_0^1, \dots, u_0^n).$$

Then there are neighborhood  $N_\delta(x_0^1, \dots, x_0^n)$  and  $N_\epsilon(u_0^1, \dots, u_0^n)$  such that

$$\begin{cases} u^1 = f_1^{-1}(x^1, \dots, x^n) \\ u^2 = f_2^{-1}(x^1, \dots, x^n) \\ \dots\dots\dots \\ u^n = f_n^{-1}(x^1, \dots, x^n) \end{cases}$$

is well-defined and has continuous partial derivatives on  $N_\delta(x_0^1, \dots, x_0^n)$  with values in  $N_\epsilon(u_0^1, \dots, u_0^n)$ . Moreover if each  $f_i$  is smooth, then each  $f_i^{-1}$  is smooth.

*Proof.* Let  $F_i = f_i(u^1, \dots, u^n) - x_i$ . The assertion follows from the Implicit Function Theorem.  $\square$

## 2. TOPOLOGICAL AND DIFFERENTIABLE MANIFOLDS

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