INTRODUCTION TO ALGEBRAIC TOPOLOGY TUTORIAL 1

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Exercise 0.1. a) Show that each of the following is a metric for $\mathbb{R}^n$:

$$d(x,y)=(\sum_{i=1}^{n}(x_i-y_i)^2)^{1/2}=||x-y||;\quad d(x,y)=\begin{cases}0 & \text{if} \quad x=y,\\ 1 & \text{if} \quad x\neq y;\end{cases}$$

$$d(x,y)=\sum_{i=1}^{n}|x_i-y_i|;\qquad d(x,y)=\max_{1\leq i\leq n}|x_i-y_i|.$$

b) Show that $d(x,y)=y-x$ does not define a metric on $\mathbb{R}$.
c) Let $n>1$. Show that $d(x,y)=\min_{1\leq i\leq n}|x_i-y_i|$ does not define a metric on $\mathbb{R}^n$.
d) Let d be a metric. Show that d' defined by

$$d'(x,y)=\frac{d(x,y)}{1+d(x,y)}$$

is also a metric.

Exercise 0.2. Show that if $\mathcal{U}$ is the family of open sets arising from a metric space then

i) The empty set $\emptyset$ and the whole set belong to $\mathcal{U}$;
ii) The intersection of two members of $\mathcal{U}$ belongs to $\mathcal{U}$;
iii) The union of any number of members of $\mathcal{U}$ belongs to $\mathcal{U}$.

Exercise 0.3. Let $\mathcal{U}$ be a topology for X. Show that the intersection of a finite number of members of $\mathcal{U}$ is in $\mathcal{U}$. Show by examples that infinite intersection of open sets in a topological space may not be open.

Exercise 0.4. Let X be a metric space and let $x\in X$. Show that $B_\epsilon(x)$ is an open set for any $\epsilon>0$.

Exercise 0.5. Let X be a metric space with metric d. Let d' be the new metric defined in Exercise 0.1. Then (X,d) and (X,d') has the same topology. (Hint: Show that the identity maps $\mathrm{id}_X\colon (X,d)\to(X,d')$ and $\mathrm{id}_X\colon(X,d')\to(X,d)$ are continuous by either using $\epsilon - \delta$-method or showing that the pre-image of open sets are open.)

Exercise 0.6. Prove each of the following statements.

a) If Y is a subset of a topological space X with $Y \subseteq F \subseteq X$ and F is closed then $\bar{Y} \subseteq F$.
b) Y is closed if and only if $Y = \bar{Y}$.
c) $\bar{\bar{Y}} = \bar{Y}$.
d) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
e) $X \setminus \overset{\circ}{Y} = \overline{X \setminus Y}$.
f) $\bar{Y} = Y \cup \partial Y$ where $\partial Y = \bar{Y} \cap \overline{(X \setminus Y)}$ (∂Y is called the boundary of Y).
g) Y is closed if and only if $\partial Y \subseteq Y$.
h) $\partial Y = \emptyset$ if and only if Y is both open and closed.
i) For $a < b \in \mathbb{R}$

$$\partial(a, b) = \partial[a, b] = \{a, b\}.$$

Exercise 0.7. Let $X = \mathbb{R}$ with the usual topology. Find the closure of each of the following subsets of X:

$$A = \{1, 2, 3, \cdots\}, \quad B = \{x | x \text{ is rational}\}, \quad C = \{x | x \text{ is irrational}\}.$$

Exercise 0.8. Show that

1) the subspace (a, b) of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (Hint: Use functions like $x \to \tan(\pi(cx + d))$ for suitable c and d.)
2) the subspaces $(1, \infty), (0, 1)$ of $\mathbb{R}$ are homeomorphic. (Hint: $x \to 1/x$.)
3) $S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology.