INTRODUCTION TO ALGEBRAIC TOPOLOGY
TUTORIAL 1

JIE WU

Exercise 0.1. a) Show that each of the following is a metric for $\mathbb{R}^n$:

$$d(x, y) = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = ||x - y||; \quad d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y; \end{cases}$$

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|; \quad d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$ 

b) Show that $d(x, y) = y - x$ does not define a metric on $\mathbb{R}$.

c) Let $n > 1$. Show that $d(x, y) = \min_{1 \leq i \leq n} |x_i - y_i|$ does not define a metric on $\mathbb{R}^n$.

d) Let $d$ be a metric. Show that $d'$ defined by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric.

Exercise 0.2. Show that if $U$ is the family of open sets arising from a metric space then

i) The empty set $\emptyset$ and the whole set belong to $U$;

ii) The intersection of two members of $U$ belongs to $U$;

iii) The union of any number of members of $U$ belongs to $U$.

Exercise 0.3. Let $U$ be a topology for $X$. Show that the intersection of a finite number of members of $U$ is in $U$. Show by examples that infinite intersection of open sets in a topological space may not be open.

Exercise 0.4. Let $X$ be a metric space and let $x \in X$. Show that $B_{\epsilon}(x)$ is an open set for any $\epsilon > 0$.

Exercise 0.5. Let $X$ be a metric space with metric $d$. Let $d'$ be the new metric defined in Exercise 0.1. Then $(X, d)$ and $(X, d')$ has the same topology. (Hint: Show that the identity maps $id_X: (X, d) \rightarrow (X, d')$ and $id_X: (X, d') \rightarrow (X, d)$ are
continuous by either using $\epsilon - \delta$-method or showing that the pre-image of open sets are open.)

**Exercise 0.6.** Prove each of the following statements.

a) If $Y$ is a subset of a topological space $X$ with $Y \subseteq F \subseteq X$ and $F$ is closed then $\bar{Y} \subseteq F$.

b) $\bar{Y}$ is closed if and only if $Y = \bar{Y}$.

c) $\bar{Y} \subseteq Y$.

d) $\bar{A} \cup \bar{B} = \bar{A} \cup \bar{B}$.

e) $X \setminus \bar{Y} = \overline{X \setminus Y}$.

f) $\bar{Y} = Y \cup \partial Y$ where $\partial Y = \bar{Y} \cap (\overline{X \setminus Y})$ ($\partial Y$ is called the boundary of $Y$).

g) $Y$ is closed if and only if $\partial Y \subseteq Y$.

h) $\partial Y = \emptyset$ if and only if $Y$ is both open and closed.

i) For $a < b \in \mathbb{R}$
\[ \partial(a, b) = \partial[a, b] = \{a, b\}. \]

**Exercise 0.7.** Let $X = \mathbb{R}$ with the usual topology. Find the closure of each of the following subsets of $X$:

$A = \{1, 2, 3, \ldots\}, \quad B = \{x | x \text{ is rational}\}, \quad C = \{x | x \text{ is irrational}\}$.

**Exercise 0.8.** Show that

1) the subspace $(a, b)$ of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (Hint: Use functions like $x \to \tan(\pi(cx + d))$ for suitable $c$ and $d$.)

2) the subspaces $(1, \infty), (0, 1)$ of $\mathbb{R}$ are homeomorphic. (Hint: $x \to 1/x$.)

3) $S^n \setminus \{(0, 0, \ldots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology.