INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWERS TO TUTORIAL 1

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Exercise 0.1 d). Since

$$d'(x,y) = 0 \Longleftrightarrow d(x,y) = 0 \Longleftrightarrow x = y,$$

condition (1) in the definition of metric spaces holds. Let a and b be positive numbers. Then

$$\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{a+b}{1+a+b}$$

because

$$(1+a+b)a(1+b)+(1+a+b)b(1+a) = (1+a+b)(2+a+b)(a+b) \geq (1+a)(1+b)(a+b).$$

Observe that $f(t) = \frac{t}{1+t}$ is an increasing function because

$$f'(t)\frac{1}{(1+t)^2}.$$

It follows that

$$d'(x,y)+d'(x,z) = \frac{d(x,y)}{1+d(x,y)}+\frac{d(x,z)}{1+d(x,z)} \geq \frac{d(x,y)+d(x,z)}{1+d(x,y)+d(x,z)} \geq \frac{d(y,z)}{1+d(y,z)} = d'(y,z)$$

and hence the result.

Exercise 0.2 ii) Let U_1 and U_2 be two open sets and let $x \in U_1 \cap U_2$. There exist positive numbers ϵ_1 and ϵ_2 such that $B_{\epsilon_1}(x) \subseteq U_1$ and $B_{\epsilon_2}(x) \subseteq U_2$. Let $\epsilon = \min \epsilon_1, \epsilon_2$. Then

$$B_\epsilon(x) \subseteq U_1 \cap U_2$$

and hence $U_1 \cap U_2$ is open.

iii) Let $\{U_\alpha|_{\alpha\in I}\}$ be a collection of open sets and let $x \in \bigcup\limits_{\alpha\in I} U_\alpha$. Then there exists $\alpha_0 \in I$ such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there exists a positive number $\epsilon > 0$ such that

$$B_\epsilon(x) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in I} U_\alpha$$

and hence $\bigcup_\alpha U_\alpha$ is open.

Exercise 0.3 The first assertion can be proved by induction. An example for the second assertion is:

Let $U_n = (0, 1 + \frac{1}{n})$ as subspace of $\mathbb{R}$ for $\neq 1$.

Exercise 0.4 Let $x' \in B_\epsilon(x)$ and let

$$\epsilon' = \frac{\epsilon - d(x', x)}{2}.$$

Then

$$B_{\epsilon'}(x') \subseteq B_\epsilon(x)$$

because, for $y \in B_{\epsilon'}(x')$,

$$d(y, x) \leq d(x', y) + d(x', x) \leq \epsilon' + d(x', x) = \frac{\epsilon}{2} + \frac{d(x, x')}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Exercise 0.5 1) *the identity map* $\mathrm{id}_X \colon (X, d) \to (X, d')$ *is continuous.*

Let $x \in X$ and let $\epsilon > 0$. Let $\delta = \epsilon$. When $d(x, x') < \epsilon$, then

$$d'(x, x') = \frac{d(x, x')}{1 + d(x, x')} \leq \frac{\epsilon}{1 + \epsilon} < \epsilon.$$

2) *the identity map* $\mathrm{id}_X \colon (X, d') \to (X, d)$ *is continuous.* Let $x \in X$ and let $\epsilon > 0$. Let

$$\delta = \frac{\epsilon}{1 + \epsilon} \Longleftrightarrow \epsilon = \frac{\delta}{1 - \delta}.$$

When $d'(x, x') < \delta$, then

$$d(x, x') = \frac{d'(x, x')}{1 - d'(x, x')} < \frac{\delta}{1 - \delta} = \epsilon,$$

where we use the fact that $f(t) = t/(1-t)$ is an increasing function over $[0, 1)$ because $f'(t) = 1/(1-t)^2 > 0$.

Thus (X, d) is homoemorphic to (X, d') under the identity map and hence they have the same topology.

Exercise 0.6 f) Since

$$X \setminus (Y \cup \partial Y) = (X \setminus Y) \cap (X \setminus \partial Y) = (X \setminus Y) \cap (X \setminus (\bar{Y} \cap \overline{X \setminus Y}))$$
$$= (X \setminus Y) \cap ((X \setminus \bar{Y}) \cup (X \setminus \overline{X \setminus Y})) = ((X \setminus Y) \cap (X \setminus \bar{Y})) \cup ((X \setminus Y) \cap (X \setminus \overline{X \setminus Y}))$$
$$= (X \setminus \bar{Y}) \cup \emptyset = X \setminus \bar{Y},$$

$Y \cup \partial Y = \bar{Y}$.

Exercise 0.7. $\bar{A} = A$, $\bar{B} = \bar{C} = \mathbb{R}$.

Exercise 0.8. 1) First the map

$$f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} \quad x \mapsto \tan(x)$$

is a homoemorphism, where $f^{-1}(x) = \arctan(x)$. Let

$$g\colon (a, b) \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

be a map defined by

$$g(x) = \frac{\pi}{2} - \frac{\pi(b-x)}{b-a}.$$

Then g is a homeomorphism with

$$g^{-1}(x) = b - \frac{b-a}{\pi}(\frac{\pi}{2} - y).$$

Thus (a, b) is homeomorphic to $\mathbb{R}$ under the composite of f and g.

2) $f(x) = \frac{1}{x}$ is a homeomorphism between these two intervals.

3) Let

$$\phi\colon S^n \setminus \{(0, 0, \cdots, 0, 1)\} \longrightarrow \mathbb{R}^n$$

be a map defined by

$$\phi(x_1, x_2, \cdots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}} \cdots, \frac{x_n}{1 - x_{n+1}}).$$

Then ϕ is a homeomorphism with

$$\phi^{-1}\colon \mathbb{R}^n \to S^n \setminus \{(0, 0, \cdots, 0, 1)\}$$

given by

$$\phi^{-1}(x_1, \cdots, x_n) = \frac{1}{1 + \|x\|^2}(2x_1, 2x_2, \cdots, 2x_n, \|x\|^2 - 1).$$