INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWERS TO TUTORIAL 2

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Problem 1. Show that any map from a compact space to a Hausdorff space is a closed map.

Proof. Let $f : X \to Y$ be a map from a compact space $X$ to a Hausdorff space $Y$. Let $A$ be closed in the compact space $X$. Then $A$ is compact (Theorem 2.8.3) and so $f(A)$ is compact in the Hausdorff space $Y$ (Theorem 2.8.2). Thus $f(A)$ is closed (Theorem 2.8.5).

Problem 3. Show that $X \wedge (Y \wedge Z)$ is homeomorphic to $X \wedge (Y \wedge Z)$ if $X$ and $Z$ are locally compact and Hausdorff.

Proof. By Theorem 2.8.9, the maps $X \times Y \times Z \to (X \wedge Y) \times Z$ and $X \times Y \times Z \to X \times (Y \wedge Z)$ are quotient maps. The composite

$$
X \times Y \times Z \twoheadrightarrow X \times (Y \wedge Z) \twoheadrightarrow X \wedge (Y \wedge Z)
$$

factors through the quotient space $(X \wedge Y) \times Z$ and so the resulting map $(X \wedge Y) \times Z \to X \wedge (Y \wedge Z)$ is continuous. Furthermore this resulting map factors through the quotient $(X \wedge Y) \wedge Z$. Thus there is a continuous map $f : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$ such that the diagram

$$
\begin{array}{c}
X \times Y \times Z \twoheadrightarrow X \times Y \times Z \\
\downarrow \\
(X \wedge Y) \wedge Z \xrightarrow{f} X \wedge (Y \wedge Z).
\end{array}
$$
Similarly there is a continuous map $g: X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$ such that the diagram

\[
\begin{array}{ccc}
X \times Y \times Z & \xrightarrow{g} & X \times Y \times Z \\
\downarrow & & \downarrow \\
X \wedge (Y \wedge Z) & \xrightarrow{g} & (X \wedge Y) \wedge Z
\end{array}
\]

Clearly $g = f^{-1}$ as a function. The assertion follows.

**Problem 6.** Show that (1) $\mathbb{R}P^n$ is Hausdorff and (2) $S^n / (\mathbb{Z}/2) \cong \mathbb{R}P^n$.

**Proof.** 1) $\mathbb{R}P^n$ is Hausdorff. To prove this, let $l_1$ and $l_2$ be two elements in $\mathbb{R}P^n$, that is two lines in $\mathbb{R}^{n+1}$ passing the origin. Let $q: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the quotient map and let $x, y \in \mathbb{R}^{n+1}$ with $\|x\| = \|y\| = 1$, $x \in l_1$ and $y \in l_2$. Let $\epsilon$ be a positive number such that $\epsilon < \min\{\|x + y\|, \|x - y\|\}$. (Such an $\epsilon$ exists because $x$ and $y$ are linearly independent vectors.) Consider the open balls $B_{\epsilon/2}(x)$ and $B_{\epsilon/2}(y)$. Show that $q^{-1}(q(B_{\epsilon/2}(x)))$ and $q^{-1}(q(B_{\epsilon/2}(y)))$ are disjoint open sets in $\mathbb{R}^{n+1} \setminus \{0\}$. By this, you get that $q(B_{\epsilon/2}(x))$ and $q(B_{\epsilon/2}(y))$ are disjoint open neighborhoods of $x$ and $y$, respectively. (So $\mathbb{R}P^n$ is Hausdorff by the definition.)

2) Let $\pi: S^n \to \mathbb{R}P^n$ be the composite $S^n \subseteq \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$, that is $\pi(x)$ is the line passing $x$ and the origin. You find that $\pi(x) = \pi(-x)$. By using this, show that $\pi$ induces a well-defined function $\bar{\pi}: S^n / (\mathbb{Z}/2) \to \mathbb{R}P^n$. Now

i) $\bar{\pi}$ is a map by the definition of quotient topology,

ii) $\bar{\pi}$ is onto and

iii) $\bar{\pi}$ is one-to-one.

Because $S^n$ is compact, the quotient $S^n / (\mathbb{Z}/2)$ is compact. By Problem 1 above, $\bar{\pi}$ is a closed bijective continuous function and so $\bar{\pi}$ is a homeomorphism.