INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWERS TO TUTORIAL 2

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Problem 1. Show that any map from a compact space to a Hausdorff space is a closed map.

Proof. Let $f\colon X \to Y$ be a map from a compact space X to a Hausdorff space Y. Let A be closed in the compact space X. Then A is compact (Theorem 2.8.3) and so $f(A)$ is compact in the Hausdorff space Y (Theorem 2.8.2). Thus $f(A)$ is closed (Theorem 2.8.5). $\square$

Problem 3. Show that $X \wedge (Y \wedge Z)$ is homeomorphic to $X \wedge (Y \wedge Z)$ if X and Z are locally compact and Hausdorff.

Proof. By Theorem 2.8.9, the the maps $X \times Y \times Z \to (X \wedge Y) \times Z$ and $X \times Y \times Z \to X \times (Y \wedge Z)$ are quotient maps. The composite

$$X \times Y \times Z \longrightarrow X \times (Y \wedge Z) \longrightarrow X \wedge (Y \wedge Z)$$

factors through the quotient space $(X\wedge Y)\times Z$ and so the resulting map $(X\wedge Y)\times Z \to X\wedge(Y\wedge Z)$ is continuous. Furthermore this resulting map factors through the quotient $(X \wedge Y) \wedge Z$. Thus there is a continuous map $f\colon (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)$ such that the diagram

$$\begin{array}{ccc}
X \times Y \times Z & = & X \times Y \times Z \\
\downarrow & & \downarrow \\
(X \wedge Y) \wedge Z & \xrightarrow{f} & X \wedge (Y \wedge Z).
\end{array}$$

Similarly there is a continuous map $g\colon X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$ such that the diagram

$$\begin{array}{ccc} X \times Y \times Z & = & X \times Y \times Z \\ \downarrow & & \downarrow \\ X \wedge (Y \wedge Z) & \xrightarrow{g} & (X \wedge Y) \wedge Z. \end{array}$$

Clearly $g = f^{-1}$ as a function. The assertion follows. □

Problem 6. Show that (1). $\mathbb{R}P^n$ is Hausdorff and (2). $S^n/(\mathbb{Z}/2) \cong \mathbb{R}P^n$.

Proof. 1) $\mathbb{R}P^n$ is Hausdorff. To prove this, let l_1 and l_2 be two elements in $\mathbb{R}P^n$, that is two lines in $\mathbb{R}^{n+1}$ passing the origin. Let $q\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ be the quotient map and let $x, y \in \mathbb{R}^{n+1}$ with $\|x\| = \|y\| = 1$, $x \in l_1$ and $y \in l_2$. Let ϵ be a positive number such that $\epsilon < \min\{\|x+y\|, \|x-y\|\}$. (Such an ϵ exists because x and y are linearly independent vectors.) Consider the open balls $B_{\epsilon/2}(x)$ and $B_{\epsilon/2}(y)$. Show that $q^{-1}(q(B_{\epsilon/2}(x)))$ and $q^{-1}(q(B_{\epsilon/2}(y)))$ are disjoint open sets in $\mathbb{R}^{n+1} \setminus \{0\}$. By this, you get that $q(B_{\epsilon/2}(x))$ and $q(B_{\epsilon/2}(y))$ are disjoint open neighborhoods of x and y, respectively. (So $\mathbb{R}P^n$ is Hausdorff by the definition.)

2) Let $\pi\colon S^n \to \mathbb{R}P^n$ be the composite $S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$, that is $\pi(x)$ is the line passing x and the origin. You find that $\pi(x) = \pi(-x)$. By using this, show that π induces a well-defined function $\bar{\pi}\colon S^n/(\mathbb{Z}/2) \to \mathbb{R}P^n$. Now

i) $\bar{\pi}$ is a map by the definition of quotient topology,
ii) $\bar{\pi}$ is onto and
iii) $\bar{\pi}$ is one-to-one.

Because S^n is compact, the quotient $S^n/(\mathbb{Z}/2)$ is compact. By Problem 1 above, $\bar{\pi}$ is a closed bijective continuous function and so $\bar{\pi}$ is a homeomorphism. □