INTRODUCTION TO ALGEBRAIC TOPOLOGY

ANSWER OF THE SELECTED PROBLEMS IN TUTORIAL 3

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Problem 1. Let $X$ and $Y$ be pointed spaces. Show that $\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$ and $\Omega^{n+m}(X) \cong \Omega^m(\Omega^n(X))$.

Proof. Since $S^n$ is Hausdorff, we have

$$\text{Map}_*(S^n, X \times Y) \cong \text{Map}_*(S^n, X) \times \text{Map}_*(S^n, Y)$$

(by Theorem 2.9.6) and hence

$$\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$$

(by the definition).

Since $S^n$ and $S^m$ are compact Hausdorff, we have

$$\text{Map}_*(S^m, \text{Map}_*(S^n, X)) \cong \text{Map}_*(S^n \land S^m, X) \cong \text{Map}_*(S^{n+m}, X)$$

(by Theorem 2.9.9). Assertion 2 follows. $\square$

Problem 3. Show that $\mathbb{CP}^n / \mathbb{CP}^{n-1} \cong S^{2n}$.

Proof. Let

$$S^2_+ = \{ z = (z_1, \ldots, z_n, r) \in \mathbb{C}^n \times \mathbb{R} \subseteq \mathbb{C}^{n+1} \mid \| z \| = 1, r \geq 0 \}$$

Then $S^2_+ \cong D^{2n}$. Let $\phi: S^2_+ \rightarrow \mathbb{CP}^n / \mathbb{CP}^{n-1}$ be the composite

$$S^2_+ \subseteq \mathbb{C}^{n+1} \setminus \{ 0 \} \xrightarrow{\text{proj}} \mathbb{CP}^n \xrightarrow{\text{proj}} \mathbb{CP}^n / \mathbb{CP}^{n-1}.$$ 

Then the map $\phi$ factors through the quotient

$$S^2_+ / \partial S^2_+ \cong D^{2n} / S^{2n-1} \cong S^{2n}.$$ 

Let $\tilde{\phi} : S^2_+ / \partial S^2_+ \rightarrow \mathbb{CP}^n / \mathbb{CP}^{n-1}$ be the map induced by $\phi$. We show that $\tilde{\phi}$ is a homeomorphism. Since $S^2_+ / \partial S^2_+$ is compact and $\mathbb{CP}^n / \mathbb{CP}^{n-1}$ is Hausdorff, it suffices to show that $\tilde{\phi}$ is one-to-one and onto.

To show that $\tilde{\phi}$ is onto. Let $l \in \mathbb{CP}^n \setminus \mathbb{CP}^{n-1}$ and let $z = (z_1, \ldots, z_{n+1})$ be a unit vector representing $l$. Since $l \not\in \mathbb{CP}^{n-1}$, $z_{n+1} \neq 0$ and

$$z \sim \frac{z_{n+1}}{|z_{n+1}|} \cdot \left( \frac{1}{|z_{n+1}|} z_1 \tilde{z}_{n+1}, \ldots, \frac{1}{|z_{n+1}|} z_n \tilde{z}_{n+1}, |z_{n+1}| \right).$$
Thus $l$ lies in the image of $\tilde{\phi}$ or $\tilde{\phi}$ is onto.

To show that $\tilde{\phi}$ is one-to-one. Let $(z_1, \ldots, z_n, r) \neq (z'_1, \ldots, z'_n, r')$ be two points in $S^2_+ \setminus \partial S^2_+$. Then $r, r' > 0$. We claim that $z' = (z'_1, \ldots, z'_n, r')$ does not lie in the complex line determined by $z = (z_1, \ldots, z_n, r)$. Otherwise there is a complex number $\alpha$ such that $z' = \alpha z$. In particular, $r' = \alpha r$. Since $z, z'$ are unit vectors, the length $|\alpha| = 1$. It follows that $\alpha = 1$ or $z = z'$ because $r, r' > 0$. This contradicts to $z \neq z'$. Thus $\tilde{\phi}$ is one-to-one and hence the result.

**Problem 4.** Show that and the complex projective space $\mathbb{C}P^n$ is a $2n$-manifold.

**Proof.** Let $l$ be a complex line in $\mathbb{C}^{n+1}$ passing the origin and let $l^\perp$ be the $n$-dimensional complex vector space perpendicular to $l$. $l^\perp$ is isomorphic to $\mathbb{C}^n$. This gives an embedding $\iota: \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$. Since $l$ does not lie in the image of $\mathbb{C}P^{n-1}$ under the embedding $\iota$ and $\mathbb{C}P^n/\iota(\mathbb{C}P^{n-1}) \cong \mathbb{C}P^n/\mathbb{C}P^{n-1} \cong S^{2n}$, there is an open neighborhood of $l$ in $\mathbb{C}P^n$ that is homeomorphic to an open set in $\mathbb{R}^{2n}$.

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