INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWER OF THE SELECTED PROBLEMS IN TUTORIAL 3

JIE WU

Problem 1. Let X and Y be pointed spaces. Show that $\Omega^n(X\times Y)\cong\Omega^n(X)\times\Omega^n(Y)$ and $\Omega^{n+m}(X)\cong\Omega^m(\Omega^n(X))$.

Proof. Since S^n is Hausdorff, we have

$$\mathrm{Map}_*(S^n, X\times Y)\cong \mathrm{Map}_*(S^n, X)\times\mathrm{Map}_*(S^n, Y)$$

(by Theorem 2.9.6) and hence

$$\Omega^n(X\times Y)\cong\Omega^nX\times\Omega^nY$$

(by the definition).

Since S^n and S^m are compact Hausdorff, we have

$$\mathrm{Map}_*(S^m,\mathrm{Map}_*(S^n,X))\cong\mathrm{Map}_*(S^n\wedge S^m, X)\cong\mathrm{Map}_*(S^{n+m},X)$$

(by Theorem 2.9.9). Assertion 2 follows. □

Problem 3. Show that $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}\cong S^{2n}$.

Proof. Let

$$S^{2n}_+=\{z=(z_1,\ldots,z_n,r)\in\mathbb{C}^n\times\mathbb{R}\subseteq\mathbb{C}^{n+1}|\|z\|=1, r\geq 0.\}$$

Then $S^{2n}_+\cong D^{2n}$. Let $\phi\colon S^{2n}_+\to\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}$ be the composite

$$S^{2n}_+\subseteq\mathbb{C}^{n+1}\setminus\{0\}\xrightarrow{\text{proj.}}\mathbb{C}\mathrm{P}^n\xrightarrow{\text{proj.}}\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}.$$

Then the map ϕ factors through the quotient

$$S^{2n}_+/\partial S^{2n}_+\cong D^{2n}/S^{2n-1}\cong S^{2n}.$$

Let $\bar\phi\colon S^{2n}_+/\partial S^{2n}_+\to\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}$ be the map induced by ϕ. We show that $\bar\phi$ is a homeomorphism. Since $S^{2n}_+/\partial S^{2n}_+$ is compact and $\mathbb{C}\mathrm{P}^n/\mathbb{C}\mathrm{P}^{n-1}$ is Hausdorff, it suffices to show that $\bar\phi$ is one-to-one and onto.

To show that $\bar\phi$ is onto. Let $l\in\mathbb{C}\mathrm{P}^n\setminus\mathbb{C}\mathrm{P}^{n-1}$ and let $z=(z_1,\ldots,z_{n+1})$ be a unit vector representing l. Since $l\notin\mathbb{C}\mathrm{P}^{n-1}$, $z_{n+1}\neq 0$ and

$$z\sim\frac{\bar z_{n+1}}{|z_{n+1}|}z=(\frac{1}{|z_{n+1}|}z_1\bar z_{n+1},\ldots,\frac{1}{|z_{n+1}|}z_n\bar z_{n+1},|z_{n+1}|).$$

Thus l lies in the image of $\bar{\phi}$ or $\bar{\phi}$ is onto.

To show that $\bar{\phi}$ is one-to-one. Let $(z_1, \ldots, z_n, r) \neq (z_1', \ldots, z_n', r')$ be two points in $S^{2n}_+ \setminus \partial S^{2n}_+$. Then $r, r' > 0$. We claim that $z' = (z_1', \ldots, z_n', r')$ does not lie in the complex line determined by $z = (z_1, \ldots, z_n, r)$. Otherwise there is a complex number α such that $z' = \alpha z$. In particular, $r' = \alpha r$. Since z, z' are unit vectors, the length $|\alpha| = 1$. It follows that $\alpha = 1$ or $z = z'$ because $r, r' > 0$. This contradicts to $z \neq z'$. Thus $\bar{\phi}$ is one-to-one and hence the result. □

Problem 4. Show that and the complex projective space $\mathbb{C}P^n$ is a $2n$-manifold.

Proof. Let l be a complex line in $\mathbb{C}^{n+1}$ passing the origin and let $l^\perp$ be the n-dimensional complex vector space perpenticular to l. $l^\perp$ is isomorphic to $\mathbb{C}^n$. This gives an embedding $e_l \colon \mathbb{C}P^{n-1} \hookrightarrow CP^n$. Since l does not lie in the image of $\mathbb{C}P^{n-1}$ under the embedding e_l and $\mathbb{C}P^n / e_l(\mathbb{C}P^{n-1}) \cong \mathbb{C}P^n / \mathbb{C}P^{n-1} \cong S^{2n}$, there is an open neighborhood of l in $\mathbb{C}P^n$ that is homeomorphic to an open set in $\mathbb{R}^{2n}$. □