INTRODUCTION TO ALGEBRAIC TOPOLOGY ANSWERS TO TUTORIAL 4

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Problem 3. Let X and Y be topological spaces. Then X and Y are path-connected if and only if $X \times Y$ is path-connected.

Hint. $\pi_0(X \times Y) = \pi_0(X) \times \pi_0(Y)$. □

Problem 4. The *comb space* Y is defined by

$$Y = \{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq 1, x = 0, 1/n \text{ or } y = 0, 0 \leq x \leq 1\}.$$

Show that the identity map of Y is homotopic to the constant map to $(0, 1) \in Y$.

Hint. See the lecture notes. □

Problem 5. Show that A is a weak deformation retract of X if and only if A is a weak retract of X and X is deformable into A.

Hint. Since A is a weak retract of X and X is deformable to A, the inclusion $i \colon A \to X$ has a left inverse r and a right homotopy inverse r' from X to A, that is, $r \circ i \simeq \mathrm{id}_A$ and $i \circ r' \simeq \mathrm{id}_X$. It follows that

$$r = r \circ \mathrm{id}_X \simeq r \circ i \circ r' \simeq \mathrm{id}_A \circ r' = r'.$$

Thus $r \simeq r'$ is a both-sided homotopy inverse of i, that is i is a homotopy equivalence. □

Problem 6. Show that S^n is a strong deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$.

Hint. Consider $F(x, t) = (1 - t)x + tx/\|x\|$ for $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and $t \in I$. □

Problem 7. Suppose that X is deformable into a retract A. Show that A is a deformation retraction of X.

Hint. Let $r \colon X \to A$ be a retraction and let $i \colon A \to X$ be the inclusion. Then r is a left homotopy inverse of i. Since X is deformable into A, i has a right homotopy inverse and so r is a right homotopy inverse. Thus $i \circ r \simeq \mathrm{id}_X$, that is A is a deformation retract of X. □

Problem 8. Show that $S^n \subseteq D^{n+1}$ is a cofibration.

Hint. Discussed in class. □