INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWERS TO TUTORIAL 5

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Problem 1. Let S^1 be identified with $I/\partial I = [0,1]/\{0,1\}$. Show that S^1 is a co-H-group under the comultiplication μ' defined by

$$\mu'(t) = \begin{cases} (2t, *) & 0 \leq t \leq 1/2 \\ (*, 2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

and a homotopy inverse ν defined by $\nu(t) = 1 - t$.

Hint. *Homotopy identity.* We need to show that $(c, \mathrm{id}) \circ \mu' \simeq \mathrm{id} \simeq (\mathrm{id}, c) \circ \mu'$. By the definition,

$$(c, \mathrm{id}) \circ \mu'(t) \begin{cases} 0 & \text{for} \quad 0 \leq t \leq 1/2 \\ 2t-1 & \text{for} \quad 1/2 \leq t \leq 1 \end{cases}$$

and

$$(\mathrm{id}, c) \circ \mu'(t) \begin{cases} 2t & \text{for} \quad 0 \leq t \leq 1/2 \\ 1 & \text{for} \quad 1/2 \leq t \leq 1 \end{cases}$$

To see $(c, \mathrm{id}) \circ \mu' \simeq \mathrm{id}$ we construct a (pointed) homotopy $F \colon S^1 \times I \to S^1$ by

$$F(t,s) = \begin{cases} 0 & 0 \leq t \leq (1-s)/2 \\ \frac{2}{1+s}(t - \frac{1-s}{2}) & (1-s)/2 \leq t \leq 1. \end{cases}$$

To see $(\mathrm{id}, c) \circ \mu' \simeq \mathrm{id}$ we construct a homotopy

$$F(t,s) = \begin{cases} \frac{2}{1+s} t & 0 \leq t \leq \frac{1+s}{2} \\ 1 & \frac{1+s}{2} \leq t \leq 1. \end{cases}$$

Homotopy coassociative We need to show that $(\mu' \vee \mathrm{id}) \circ \mu' \simeq (\mathrm{id} \vee \mu') \circ \mu'$. Again write down these two maps:

$$(\mu' \vee \mathrm{id}) \circ \mu'(t) = \begin{cases} (4t, *, *) & 0 \leq t \leq 1/4 \\ (*, 4t-1, *) & 1/4 \leq t \leq 1/2 \\ (*, *, 2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

$$(\mathrm{id} \vee \mu') \circ \mu'(t) = \begin{cases} (2t, *, *) & 0 \leq t \leq 1/2 \\ (*, 4t-2, *) & 1/2 \leq t \leq 3/4 \\ (*, *, 4t-3) & 3/4 \leq t \leq 1. \end{cases}$$

Construct a (pointed) homotopy as follows:

$$F(t,s)=\begin{cases} (\frac{4}{1+s}t,*,*) & 0\leq t\leq \frac{1+s}{4}\\ (*,4t-(1+s),*) & \frac{1+s}{4}\leq t\leq \frac{2+s}{4}\\ (*,*,\frac{4}{2-s}(t-\frac{s+2}{4})) & \frac{2+s}{4}\leq t\leq 1.\end{cases}$$

Homotopy Inverse. We need to show that $(\mathrm{id},\nu)\circ\mu'$ and $(\nu,\mathrm{id})\circ\mu'$ are null homotopic. Note

$$(\mathrm{id},\nu)\circ\mu'(t)=\begin{cases} 2t & 0\leq t\leq 1/2\\ 2-2t & 1/2\leq t\leq 1\end{cases}$$

$$(\nu,\mathrm{id})\circ\mu'(t)=\begin{cases} 1-2t & 0\leq t\leq 1/2\\ 2t-1 & 1/2\leq t\leq 1.\end{cases}$$

We construct the following (pointed) homotopies $F,G\colon S^1\times I\to S^1$.

$$F(t,s)=\begin{cases} 2st & 0\leq t\leq 1/2\\ 2s-2st & 1/2\leq t\leq 1\end{cases}$$

$$G(t,s)=\begin{cases} 1-2st & 0\leq t\leq 1/2\\ 2st+1-2s & 1/2\leq t\leq 1.\end{cases}$$

Then $F_0=G_0=c$, $F_1=(\mathrm{id},\nu)\circ\mu'$ and $G_1=(\nu,\mathrm{id})\circ\mu'$.

□

Problem 2. Let μ_1 and μ_2 be two multiplications on Y such that Y is an H-space under μ_1 and μ_2. Show that

$$\Omega\mu_1\simeq\Omega\mu_2\colon\Omega(Y\times Y)\to\Omega Y.$$

Hint. By Theorem 3.3.13, we have

$$\mu_{1*}=\mu_{2*}\colon[\Sigma X,Y\times Y]=[X,\Omega(Y\times Y)]\to[\Sigma X,Y]=[X,\Omega Y]$$

for any X. Choose $X=\Omega(Y\times Y)$. Then

$$[\Omega\mu_1]=\mu_{1*}([\mathrm{id}_{\Omega(Y\times Y)}])=\mu_{2*}([\mathrm{id}_{\Omega(Y\times Y)}])=[\Omega\mu_2].$$

□

Problem 3. Let λ and μ be paths in X from x to y. Suppose that X is simply connected. Then $\lambda\simeq\mu$.

Hint. $[\lambda*\mu^{-1}]=[\epsilon_x]$ because $\pi_1(X,x)=0$. It follows that

$$[\mu]=[\epsilon_x][\mu]=[\lambda*\mu^{-1}][\mu]=[\lambda][\mu^{-1}][\mu]=[\lambda].$$

□

Problem 4. Let X and Y be locally compact Hausdorff pointed spaces. Show that $\Sigma(X \vee Y)$ is a weak retract of $\Sigma(X \times Y)$.

Hint. Let $j\colon X \vee Y \to X \times Y$ be the inclusion. Show that

$$\Sigma j^*\colon [\Sigma(X \times Y), Z] \to [\Sigma(X \vee Y), Z] = [\Sigma X, Z] \times [\Sigma Y, Z]$$

is onto for any Z.

(The result will follow by taking $Z = \Sigma(X \vee Y)$.)

Let $i_1\colon X \to X \vee Y$ and $i_2\colon Y \to X \vee Y$ be the inclusions. Then

$$\Sigma i_1^* \circ \Sigma j^* = (\Sigma j \circ \Sigma i_1)^*\colon [\Sigma(X \times Y), Z] \to [\Sigma X, Z]$$

is onto because for any $f\colon \Sigma X \to Z$ we have

$$[f] = [f \circ \mathrm{id}_{\Sigma X}] = [f \circ \Sigma p_X \circ \Sigma j \circ \Sigma i_1] = (\Sigma j \circ \Sigma i_1)^*([f \circ \Sigma p_X],$$

where $p_X\colon X \times Y \to X$ is the coordinate projection. Similarly

$$\Sigma i_2^* \circ \Sigma j^*\colon [\Sigma(X \times Y), Z] \to [\Sigma Y, Z]$$

is onto. Now let $([f], [g]) \in [\Sigma(X \vee Y), Z] = [\Sigma X, Z] \times [\Sigma Y, Z]$ be any element. The arguments above show that there exist elements $[f'], [g'] \in [\Sigma(X \times Y), Z]$ such that

$$\Sigma j^*([f']) = ([f], 1) \quad \Sigma j^*([g']) = (1, [g]).$$

Thus

$$\Sigma j^*([f'][g']) = ([f], 1)(1, [g]) = ([f], [g]).$$

□

Note:

(1) By expecting the proof, we have a stronger statement:
Let $j\colon X \vee Y \to X \times Y$ be the inclusion and let Z be any H-space. Then $j^*\colon [X \times Y, Z] \to [X \vee Y, Z]$ is onto (an epimorphism).

(2) A similar result for loops is
$\Omega(X \times Y)$ is defomable into $\Omega(X \vee Y)$.

Problem 5. Suppose that X is a locally compact Hausdorff space. Show that ΣX is a retract of $\Sigma\Omega\Sigma X$.

Hint. Let $e\colon \Sigma\Omega\Sigma X \to \Sigma X$ be the evaluation map and let $E\colon X \to \Omega\Sigma X$ be the inclusion, that is $E(x)(t) = t \wedge x$ for $x \in X$. The map E is continuous because $E = \alpha(\mathrm{id}_{\Sigma X})$, where $\alpha\colon \mathrm{Map}_*(\Sigma X, \Sigma X) \to \mathrm{Map}_*(X, \Omega X)$ is the association map. Then $e \circ \Sigma E = \mathrm{id}_{\Sigma X}$. □

Problem 6. Let Y be a locally compact Hausdorff space. Suppose that X and Y are co-H-spaces. Show that $X \wedge Y$ is a homotopy associative and homotopy commutative co-H-space.

Hint. By Theorem 3.1.10, $[X \wedge Y, Z] \cong [X, \mathrm{Map}_*(Y, Z)]$ for any pointed space Z. By Theorem 3.3.7, $\mathrm{Map}_*(Y, Z)$ is an H-space. According to Theorem 3.3.13, $[X, \mathrm{Map}_*(Y, Z)]$ is associative and commutative under the multiplication induced by either the comultiplication on X or the comultiplication on Y. Thus

> $[X \wedge Y, Z]$ *is associative and commutative under the multiplication induced by either the comultiplication on* X *or the comultiplication on* Y *for any* Z.

By choosing particular space $Z = (X \wedge Y) \vee (X \wedge Y)$, show that

$$\mu'_X \wedge \mathrm{id}_Y \simeq \mathrm{id}_X \wedge \mu'_Y \colon X \wedge Y \to (X \wedge Y) \vee (X \wedge Y).$$

Let $\mu'_{X\wedge Y} = \mu'_X \wedge \mathrm{id}_Y \simeq \mathrm{id}_X \wedge \mu'_Y$. Let f_i be the composite

$$(X \wedge Y) \vee (X \wedge Y) \xrightarrow{p_i} X \wedge Y \xhookrightarrow{j_i} (X \wedge Y) \vee (X \wedge Y)$$

for $i = 1, 2$, where p_i is the i-th coordinate projection and j_i is the i-coordinate inclusion. By checking the definition of the multiplication on homotopy classes,

$$[\mu'_{X\wedge Y}] = [j_1 \circ p_1][j_2 \circ p_2]$$

and

$$[T \circ \mu'_Z] = [j_2 \circ p_2][j_1 \circ p_1]$$

in the commutative monoid $[Z, Z \vee Z]$ and hence $[\mu'_{X\wedge Y}] = [T \circ \mu'_{\mu'_{X\wedge Y}}]$ or $\mu'_{X\wedge Y}$ is homotopy commutative.

Similarly show that $\mu'_{X\wedge Y}$ is coassociative by considering the case

$$Z = (X \wedge Y) \vee (X \wedge Y) \vee (X \wedge Y).$$

□