INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWERS TO TUTORIAL 6

JIE WU

2. Show that $\pi_n(S^1) = 0$ for $n > 1$.

Hint. Let $f\colon S^n = I^n/\partial I^n \to S^1$ be a pointed map and let $q\colon I^n \to S^n$ be the quotient map. Since I^n is starlike from 0, there is a unique lifting $\tilde{f}\colon I^n \to \mathbb{R}$ such that $e \circ \tilde{f} = f \circ q$ and $\tilde{f}(0) = 0$. It follows that

$$\tilde{f}(\partial I^n) \subseteq e^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}.$$

Since ∂I^n is connected when $n > 1$, $\tilde{f}(\partial I^n)$ is a connected subspace of $\mathbb{Z}$ and so $\tilde{f}|_{\partial I^n}$ is a constant map. Because $\tilde{f}(0) = 0$, $\tilde{f}(x) = 0$ for any $x \in \partial I^n$ and so $\tilde{f}$ induces a map $\bar{f}\colon I^n/\partial I^n \to \mathbb{R}$ such that $\tilde{f} = \bar{f} \circ q$. It follows that $f = e \circ \bar{f}\colon S^n \to S^1$. Since $\mathbb{R}$ is contractible, $\bar{f}$ is null homotopic and so $f = e \circ \bar{f}$ is null homotopic. □

3. Let λ and μ be paths in X from x to y. Suppose that X is simply connected. Then $\lambda \simeq \mu$.

Hint. $[\lambda * \mu^{-1}] = [\epsilon_x]$ because $\pi_1(X, x) = 0$. It follows that

$$[\mu] = [\epsilon_x][\mu] = [\lambda * \mu^{-1}][\mu] = [\lambda][\mu^{-1}][\mu] = [\lambda].$$

□