

INTRODUCTION TO ALGEBRAIC TOPOLOGY TUTORIAL 8

JIE WU

Problem 1. Suppose that \tilde{X} is path-connected. Show that the function $\psi: \pi_1(X, x_0) \rightarrow p^{-1}(x_0)$ is onto.

Problem 2. Let M be a path-connected manifold with $\dim(M) \geq 2$. Show that the configuration spaces $F(M, n)$ and $B(M, n)$ are path-connected. Deduce that there is an epimorphism of groups $\psi: \pi_1(B(M, n)) \twoheadrightarrow \Sigma_n$ with $\text{Ker}(\psi) \cong \pi_1(F(M, n))$, where Σ_n is the symmetric group.

Problem 3. Prove that if $n \geq 2$ then there does not exist any continuous map $\phi: S^n \rightarrow S^1$ such that $\phi(-x) = -\phi(x)$.

Problem 4. Let ζ be a primitive (complex) k -th root of the identity 1. Let \mathbb{Z}/k act on \mathbb{C}^n by $\zeta \cdot (z_1, \dots, z_n) = (\zeta z_1, \dots, \zeta z_n)$ for $z_j \in \mathbb{C}$. Show that for $k, n \geq 2$ no \mathbb{Z}/k -map $f: \mathbb{C}^n \rightarrow \mathbb{C}$ can be norm-preserving, that is if $f(\zeta z) = \zeta f(z)$ for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, then for some $z \in \mathbb{C}^n$ $|f(z)| \neq \|z\|$.

Deduce that for $n \geq 2$ any map $h: \mathbb{C}^n \rightarrow \mathbb{C}$ has some $z \in \mathbb{C}^n$ such that $z \neq 0$ and

$$\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z) = 0.$$

Problem 5. A map-maker wishes to represent the Earth's surface in an atlas. Explain how any representation of the Earth's surface on a sheet of paper which neither

- i) excludes at least one point of the Earth's surface (e.g. North pole),
- ii) includes some point of the Earth surface more than once, nor
- iii) contains a 'cut', so that some nearby points on the Earth are noticeably separated on paper,

corresponds to an injective map from S^2 to \mathbb{R}^2 . Hence show that no such representation is possible.

Problem 6. Prove that

- a) for $n \geq 2$ any continuous map $\mathbb{R}P^n \rightarrow S^1$ is null homotopic.

- b) for $n \geq 1$ any continuous map $L^n(p) \rightarrow S^1$ is null homotopic, where $L^n(p)$ is the lens space.