

**INTRODUCTION TO ALGEBRAIC TOPOLOGY
ANSWERS TO TUTORIAL 8**

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Problem 2. Let M be a path-connected manifold with $\dim(M) \geq 2$. Show that the configuration space $F(M, n)$ is path-connected. Deduce that there is an epimorphism of groups $\phi: \psi_1(B(M, n)) \longrightarrow \Sigma_n$ with $\text{Ker}(\psi) \cong \pi_1(F(M, n))$, where Σ_n is the symmetric group.

Hint. First show that $M \setminus \{\text{any finite points}\}$ is path-connected. (You need the condition that $\dim(M) \geq 2$.)

Now you can prove the statement by induction on n . The statement is obvious for $n = 1$. Suppose that the statement holds for $< n$. Let (a_1, \dots, a_n) and (b_1, \dots, b_n) be two points in $F(M, n)$. Let a'_n be a point in a small neighbourhood of a_n such that 1) $a'_n \neq a_i$ for $1 \leq i \leq n - 1$ and 2) $a'_n \neq b_i$ for $1 \leq i \leq n - 1$.

i) The subspace

$$\{(a_1, a_2, \dots, a_{n-1})\} \times (M \setminus \{a_1, a_2, \dots, a_{n-1}\}) \subseteq F(M, n)$$

is homeomorphic to $M \setminus \{a_1, a_2, \dots, a_{n-1}\}$ and it is path-connected. Thus there is a path

$$\lambda^{(1)} = (a_1, \dots, a_{n-1}, \lambda_n^{(1)})$$

in $F(M, n)$ from (a_1, \dots, a_n) to $(a_1, \dots, a_{n-1}, a'_n)$.

ii) The subspace

$$F(M \setminus \{a'_n\}, n - 1) \times \{a'_n\} \subseteq F(M, n)$$

is homeomorphic to $F(M \setminus \{a'_n\})$ and hence it is path-connected by induction. Thus there is a path

$$\lambda^{(2)} = (\lambda_1^{(2)}, \dots, \lambda_{n-1}^{(2)}, a'_n)$$

in $F(M, n)$ from $(a_1, \dots, a_{n-1}, a'_n)$ to $(b_1, \dots, b_{n-1}, a'_n)$.

iii) The subspace

$$\{(b_1, b_2, \dots, b_{n-1})\} \times (M \setminus \{b_1, b_2, \dots, b_{n-1}\}) \subseteq F(M, n)$$

is homeomorphic to $M \setminus \{b_1, b_2, \dots, b_{n-1}\}$ and it is path-connected. Thus there is a path

$$\lambda^{(3)} = (b_1, \dots, b_{n-1}, \lambda_n^{(3)})$$

in $F(M, n)$ from $(b_1, \dots, b_{n-1}, a'_n)$ to $(b_1, \dots, b_{n-1}, b_n)$.

It follows that the product $\lambda^{(1)} * \lambda^{(2)} * \lambda^{(3)}$ is a path in $F(M, n)$ from (a_1, \dots, a_n) to (b_1, \dots, b_n) . \square

Problem 4. Let ζ be a primitive (complex) k -th root of the identity 1. Let \mathbb{Z}/k acts on \mathbb{C}^n by $\zeta \cdot (z_1, \dots, z_n) = (\zeta z_1, \dots, \zeta z_n)$ for $z_j \in \mathbb{C}$. Show that for $k, n \geq 2$ no \mathbb{Z}/k -map $f: \mathbb{C}^n \rightarrow \mathbb{C}$ can be norm-preserving, that is if $f(\zeta z) = \zeta f(z)$ for all $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, then for some $z \in \mathbb{C}^n$ $|f(z)| \neq \|z\|$.

Deduce that for $n \geq 2$ any map $h: \mathbb{C}^n \rightarrow \mathbb{C}$ has some $z \in \mathbb{C}^n$ such that $z \neq 0$ and

$$\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z) = 0.$$

Hint. i) Assume that there are such an $f: \mathbb{C}^n \rightarrow \mathbb{C}$. Then f induces a \mathbb{Z}/k -map $\bar{f}: S^{2n-1} \rightarrow S^1$. Use the arguments in proof of Borsuk-Ulam Theorem to find a contradiction.

ii) Assume that $\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z) \neq 0$ for all $z \neq 0$ in \mathbb{C}^n . Let

$$f(z) = \frac{\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z)}{\|\sum_{i=0}^{k-1} \zeta^i h(\zeta^{k-i} z)\|}$$

for $z \in S^{2n-1}$. Check that f is a \mathbb{Z}/k -map.) \square