**Supplement to 1.4. Proofs of Standard Limits.**

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any fixed $p > 0$.

*Proof.*

$$\lim_{n\to\infty} \frac{1}{n^p} = \Big(\lim_{n\to\infty} \frac{1}{n}\Big)^p = 0^p = 0.$$

2. $\lim_{n\to\infty} c^n = 0$ for any fixed $c$ where $|c| < 1$.

*Proof.* Case 1: When $c = 0$, the statement is obvious.
Case 2: When $c > 0$, we have

$$\ln\big(\lim_{n\to\infty} c^n\big) = \lim_{n\to\infty} \ln c^n = \lim_{n\to\infty} n \ln c = -\infty.$$

Thus, $\lim_{n\to\infty} c^n = 0$.
Case 3: When $c < 0$, we have $-|c|^n \le c^n \le |c|^n$ for all $n$.
By Case 2, we have $\lim_{n\to\infty}(-|c|^n) = 0 = \lim_{n\to\infty} |c|^n$. Hence by Squeeze theorem, we also have $\lim_{n\to\infty} c^n = 0$.

3. $\lim_{n\to\infty} c^{\frac{1}{n}} = 1$ for any fixed $c > 0$.

*Proof.* $\lim_{n\to\infty} c^{\frac{1}{n}} = c^{\lim_{n\to\infty} \frac{1}{n}} = c^0 = 1$.

4. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

*Proof.*

$$\ln\big(\lim_{n\to\infty} \sqrt[n]{n}\big) = \lim_{n\to\infty} \ln \sqrt[n]{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0 \text{ (by L'Hopital's rule)}.$$

Thus, $\lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1$.

5. $\lim_{n\to\infty} \frac{n^p}{c^n} = 0$ for any fixed $p$ and $c > 1$.

*Proof.*

$$\begin{aligned}\ln\Big(\lim_{n\to\infty} \frac{n^p}{c^n}\Big) &= \lim_{n\to\infty}(p \ln n - n \ln c)\\ &= \lim_{n\to\infty} \frac{p\ln n - n\ln c}{n} \lim_{n\to\infty} n\\ &= (-\ln c)\cdot(+\infty) = -\infty.\end{aligned}$$

Thus, $\lim_{n\to\infty} \frac{n^p}{c^n} = 0$.

6. $\displaystyle\lim_{n\to\infty} \frac{c^n}{n!} = 0$ for any fixed $c$.

*Proof.* Observe that $a_n = \dfrac{c^n}{n!} = \dfrac{c \cdot c \cdot \cdots \cdot c}{n(n-1) \cdot \cdots \cdot 1}$. Now fix an integer $M > c$. Then for any $n > M$,

$$a_n = \frac{c \cdot c \cdot \cdots \cdot c}{n(n-1) \cdot \cdots \cdot (M+1)} a_M < a_M.$$

Thus we have, for $n > M$, $0 \leq a_n < \dfrac{c}{n} a_M$.
Since $\displaystyle\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{c}{n} a_M$, by the Squeeze theorem, $\lim_{n\to\infty} a_n = 0$.

7. $\displaystyle\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$ for any fixed $x$.

*Proof.*

$$\begin{aligned} \ln\left(\lim_{n\to\infty} (1 + \frac{x}{n})^n\right) &= \lim_{n\to\infty} \ln\big((1 + \frac{x}{n})^n\big) \\ &= \lim_{n\to\infty} \frac{\ln(1 + \frac{x}{n})}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{(1 + \frac{x}{n}) \cdot \frac{-x}{n^2}}{-\frac{1}{n^2}} \\ &= x. \end{aligned}$$

Thus, $\displaystyle\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$.