Supplement to 1.7 - 1.8

Theorem 1.7.2 [Monotone Convergence Theorem].
(i) If $\{a_n\}$ is monotone increasing and bounded above, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = \sup_n a_n$.
(ii) If $\{a_n\}$ is monotone decreasing and bounded below, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = \inf_n a_n$.

Proof. (i) Suppose $\{a_n\}$ is monotone increasing and bounded above. Then by the Completeness Axiom of $\mathbb{R}$, $\sup_n a_n$ exists (finite). Now, given $\epsilon > 0$, since $\sup_n a_n - \epsilon < \sup_n a_n$, it follows that $\sup_n a_n - \epsilon$ is not an upper bound of $\{a_n\}$. In other words, there exists an $N \in \mathbb{Z}^+$ such that $a_N > \sup_n a_n - \epsilon$. Then for all $n > N$, we have

$$\sup_n a_n - \epsilon < a_N \leq a_n \quad (\text{since } n > N).$$

In other words, we have $\sup_n a_n - \epsilon < a_n$ for all $n > N$.
Hence we have $-\epsilon < a_n - \sup_n a_n \leq 0 < \epsilon$ for all $n > N$.
Equivalently, we have $\left|a_n - \sup_n a_n\right| < \epsilon$ for all $n > N$.
Hence $\lim_{n\to\infty} a_n = \sup_n a_n$ (exists).
The proof of (ii) is similar.

Corollary 1.7.3. *If $\{a_n\}$ is monotone increasing (decreasing), then either*
(i) $\{a_n\}$ *is convergent or*
(ii) $\lim\limits_{n\to\infty} a_n = +\infty(-\infty)$.

Proof. Suppose $\{a_n\}$ is monotone increasing, then either $\{a_n\}$ is bounded above or not bounded above.
Case (a): If $\{a_n\}$ is bounded above, then by the Monotone Convergence Theorem, $\{a_n\}$ converges.
Case (b): If $\{a_n\}$ is not bounded above, then $\{a_n\}$ has no upper bounds. Thus for any given $k > 0$, k is not an upper bound of $\{a_n\}$. In other words, there exists $N \in \mathbb{Z}^+$ such that

$$a_N > k.$$

Since $\{a_n\}$ is monotone increasing, it follows that for all $n > N$,

$$a_n \geq a_N > k.$$

Therefore, $\lim\limits_{n\to\infty} a_n = +\infty$.
The proof for the case when $\{a_n\}$ is monotone decreasing is similar.

Theorem 1.8.1. *Suppose $\lim\limits_{n\to\infty} a_n = A$. Then every subsequence of $\{a_n\}$ also converges to A, i.e.,*

$$\lim_{k\to\infty} a_{n_k} = A.$$

Proof. For any given $\epsilon > 0$, since $\lim\limits_{n\to\infty} a_n = A$, there exists $N \in \mathbb{Z}^+$ such that

$$|a_n - A| < \epsilon \quad \forall n > N. \tag{*}$$

Then for all $k > N$, we have

$$n_k \geq k > N.$$

Hence

$$|a_{n_k} - A| < \epsilon \quad \forall k > N \quad \text{(by (*))}.$$

Therefore, $\lim\limits_{k\to\infty} a_{n_k} = A$.