Supplement to 1.9 - 1.10

Proposition 1.9.5. *If* $\varliminf_{n\to\infty} a_n = \varlimsup_{n\to\infty} a_n = A$ *(finite), then* $\{a_n\}$ *converges and* $\lim_{n\to\infty} a_n = A$.

Proof. Given $\epsilon > 0$, since $\varlimsup_{n\to\infty} a_n = A$, it follows from Proposition 1.9.4(i) that there exists $N_1 \in \mathbb{Z}^+$ such that

$$a_n < A + \epsilon \quad \forall n > N_1. \tag{*}$$

Similarly, since $\varliminf_{n\to\infty} a_n = A$, it follows from Proposition 1.9.4(ii) that there exists $N_2 \in \mathbb{Z}^+$ such that

$$a_n > A - \epsilon \quad \forall n > N_2. \tag{**}$$

Let $N = \max\{N_1, N_2\}$. Combining (*) and (**), it follows that

$$\begin{aligned} & A - \epsilon < a_n < A + \epsilon \quad \forall n > N \\ \Rightarrow \quad & -\epsilon < a_n - A < \epsilon \quad \forall n > N \\ \Rightarrow \quad & |a_n - A| < \epsilon \quad \forall n > N. \end{aligned}$$

Hence, $\lim_{n\to\infty} a_n = A$.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Proposition 1.10.1. *Every Cauchy sequence is bounded.*

Proof. Given that $\{a_n\}$ is a Cauchy sequence. Set $\epsilon = 1$. Then there exists $N \in \mathbb{Z}^+$ such that

$$
\begin{aligned}
& |a_n - a_m| < 1 \quad \forall m, n > N \\
\Rightarrow \quad & |a_n - a_{N+1}| < 1 \quad \forall n > N \quad \text{(here } m = N+1\text{)} \\
\Rightarrow \quad & a_{N+1} - 1 < a_n < a_{N+1|} + 1 \quad \forall n > N.
\end{aligned}
$$

Now let

$$
\begin{aligned}
M &= \max\{a_1, a_2, \cdots, a_N, a_{N+1} + 1\}, \\
m &= \min\{a_1, a_2, \cdots, a_N, a_{N+1} - 1\}.
\end{aligned}
$$

Then one can check that

$$m \leq a_n \leq M \quad \forall n.$$

(Check!) Hence $\{a_n\}$ is bounded.

Theorem 1.10.2 [Cauchy's criterion].
A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Proof. "$\Rightarrow$"part, i.e., "every convergent sequence is Cauchy".
Given that $\{a_n\}$ is convergent, say $\lim_{n\to\infty} a_n = A$.
Then for any given $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$|a_n - A| < \frac{\epsilon}{2} \quad \forall n > N.$$

Now for any $m, n > N$,

$$\begin{aligned} & |a_n - a_m| \\ = {} & |(a_n - A) - (a_m - A)| \\ \le {} & |a_n - A| + |a_m - A| \\ < {} & \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{(since both } m, n > N) \\ = {} & \epsilon. \end{aligned}$$

Therefore, $\{a_n\}$ is a Cauchy sequence.

"$\Leftarrow$"part, i.e., "every Cauchy sequence is convergent".
Given that $\{a_n\}$ is Cauchy.
By Proposition 1.10.1, $\{a_n\}$ is bounded.
As in 1.9, we let

$$\begin{aligned} b_n &= \sup_{k\ge n} a_k = \sup\{a_n, a_{n+1}, \cdots\}, \\ c_n &= \inf_{k\ge n} a_k = \inf\{a_n, a_{n+1}, \cdots\}. \end{aligned}$$

Then $\{b_n\}$ and $\{c_n\}$ are also bounded (check!). Hence both $\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} b_n$ and $\underline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} c_n$ exist. Write

$$\overline{\lim_{n\to\infty}} a_n = \lim_{n\to\infty} b_n = B \quad \text{and} \quad \underline{\lim_{n\to\infty}} a_n = \lim c_n = C.$$

Given any $\epsilon > 0$, since $\{a_n\}$ is a Cauchy sequence, there exists $N \in \mathbb{Z}^+$ such that

$$\begin{aligned} & |a_n - a_m| < \frac{\epsilon}{3} \quad \forall m, n > N, \\ \Rightarrow \qquad & a_n - \frac{\epsilon}{3} < a_m < a_n + \frac{\epsilon}{3} \quad \forall m, n > N. \end{aligned} \tag{1}$$

In particular, letting $m = n, n+1, n+2, \cdots,$, etc., one sees from the second inequality in (1) that

$$a_n + \frac{\epsilon}{3} > a_n, \ a_n + \frac{\epsilon}{3} > a_{n+1}, \cdots, \text{etc..}$$

Hence, $a_n + \frac{\epsilon}{3}$ is an upper bound of $\{a_n, a_{n+1}, \cdots\}$. Hence,

$$\begin{aligned} & a_n + \frac{\epsilon}{3} \geq \sup\{a_n, a_{n+1}, \cdots\} = b_n \\ \Rightarrow \qquad & a_n \geq b_n - \frac{\epsilon}{3}. \end{aligned} \tag{2}$$

Similarly, one can use the first inequality in (1) that

$$a_n \leq c_n + \frac{\epsilon}{3}. \tag{3}$$

Combining (2) and (3), it follows that for any given $\epsilon > 0$, there exists N such that

$$\begin{aligned} & b_n - \frac{\epsilon}{3} \leq a_n \leq c_n + \frac{\epsilon}{3} \quad \forall n > N \\ \Rightarrow \qquad & b_n - \frac{\epsilon}{3} \leq c_n + \frac{\epsilon}{3} \quad \forall n > N \\ \Rightarrow \qquad & b_n - c_n \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon \quad \forall n > N. \end{aligned}$$

Hence for any given $\epsilon > 0$, there exists N such that

$$\begin{aligned} & (b_n - c_n) - 0 < \epsilon \quad \forall n > N \\ \Rightarrow \qquad & -\epsilon < 0 \leq (b_n - c_n) - 0 < \epsilon \quad \forall n > N \\ \Rightarrow \qquad & |(b_n - c_n) - 0| < \epsilon \quad \forall n > N. \end{aligned}$$

Hence $\lim_{n\to\infty} (b_n - c_n) = 0 \Rightarrow B = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = C$. In other words, we have $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$.

Then by Proposition 1.9.5, $\{a_n\}$ converges, and $\lim_{n\to\infty} a_n = B = C$.

This finishes the proof of Theorem 1.10.2.