Supplement to 2.2

Theorem 2.2.4 [Comparison Test].

Consider 2 positive series $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$. *Suppose that eventually* $0 \le a_k \le b_k$.

(i) If $\sum_{k=1}^{\infty} b_k$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(ii) If $\sum_{k=1}^{\infty} a_k$ *diverges, then* $\sum_{k=1}^{\infty} b_k$ *diverges.*

Proof. Let $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$. Then $A_n \le B_n$ for all n.

If $\sum_{k=1}^{\infty} b_k$ converges, then $\{B_n\}$ converges.

Hence $\{B_n\}$ is bounded above, say, $B_n \le M$ for all n.

Then $A_n \le B_n \le M$ for all n.

Therefore, $\{A_n\}$ is also bounded above (by M).

Since $a_k \ge 0$ for all k, $\{A_n\}$ is monotone increasing.

Thus by the Monotone Convergence Theorem, the sequence of partial sums $\{A_n\}$ converges.

In other words, $\sum_{k=1}^{\infty} a_k$ converges.

(ii) follows immediately from (i).

Corollary 2.2.5 [Limit Comparison Test].
If $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *are (eventually) positive, and*

$$\lim_{k\to\infty} \frac{b_k}{a_k} = L(\neq 0, \neq \infty),$$

then either both series converge or both series diverge.

Proof. Let $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$.

$$\begin{aligned} & \lim_{k\to\infty} \frac{b_k}{a_k} = L \ (\neq 0, \neq \infty) \\ \Rightarrow \quad & \left\{\frac{b_k}{a_k}\right\} \text{ is bounded above, say, by } M \\ \Rightarrow \quad & 0 \leq b_k \leq M a_k \quad \forall k \end{aligned} \tag{*}$$

Similarly,

$$\begin{aligned} & \lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{L} \ (\neq 0, \neq \infty) \\ \Rightarrow \quad & \left\{\frac{a_k}{b_k}\right\} \text{ is bounded above, say, by } M' \\ \Rightarrow \quad & 0 \leq a_k \leq M' b_k \quad \forall k \end{aligned} \tag{**}$$

If $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} M a_k = M \sum_{k=1}^{\infty} a_k$ is also convergent. By (*) and the comparison test, it follows that $\sum_{k=1}^{\infty} b_k$ is also convergent.

If $\sum_{k=1}^{\infty} b_k$ is convergent, then $\sum_{k=1}^{\infty} M' b_k = M' \sum_{k=1}^{\infty} b_k$ is also convergent. By (**) and the comparison test, it follows that $\sum_{k=1}^{\infty} a_k$ is also convergent.

Hence $\sum_{k=1}^{\infty} a_k$ is convergent iff $\sum_{k=1}^{\infty} b_k$ is convergent.

Hence $\sum_{k=1}^{\infty} a_k$ is divergent iff $\sum_{k=1}^{\infty} b_k$ is divergent.