Supplement to 3.9

Example 3.9.2.

1. Consider the function

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

Then we will show that $f(x)\neq$ its Taylor series at $x_0=0$.

Proof. First we show that for any $n\in\mathbb{Z}^+$,

$$\lim_{x\to 0}\frac{1}{x^n e^{\frac{1}{x^2}}}=0. \tag{*}$$

To see this, we substitute $y=\frac{1}{x^2}$ to get

$$\begin{aligned}\lim_{x\to 0}\frac{1}{x^n e^{\frac{1}{x^2}}} &= \lim_{x\to 0}\frac{1}{x^{2n}e^{\frac{1}{x^2}}}\cdot x^n\\ &= \left(\lim_{y\to+\infty}\frac{y^n}{e^y}\right)\cdot\lim_{x\to 0}x^n\\ &= 0\cdot 0 \quad \text{(by L'Hopital's rule)}\\ &= 0.\end{aligned}$$

Next we compute $f'(0)$.

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$$
\begin{aligned}
f'(0) &= \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \\
&= \lim_{x \to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} \\
&= \lim_{x \to 0} \frac{1}{x e^{\frac{1}{x^2}}} \\
&= 0 \quad \text{(by (*))}.
\end{aligned}
$$

For $x \neq 0$,

$$
f'(x) = \frac{d}{dx}\big(e^{-\frac{1}{x^2}}\big) = 2x^{-3} e^{-\frac{1}{x^2}}.
$$

Thus,

$$
f'(x) = \begin{cases} 2x^{-3} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}
$$

Next we compute $f''(x)$.

$$
\begin{aligned}
f''(0) &= \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \\
&= \lim_{x \to 0} \frac{2x^{-3} e^{-\frac{1}{x^2}} - 0}{x} \\
&= 2 \lim_{x \to 0} \frac{1}{x^4 e^{\frac{1}{x^2}}} \\
&= 0 \quad \text{(by (*))}.
\end{aligned}
$$

Again, for $x \neq 0$,

$$
f''(x) = \frac{d}{dx}\big(2x^{-3} e^{-\frac{1}{x^2}}\big) = (-6e^{-4} + 4x^{-6}) e^{-\frac{1}{x^2}}.
$$

Similar calculations will lead to

$$f(0) = f'(0) = f''(0) = f^{(3)}(0) = f^{(4)}(0) = \cdots = 0.$$

Thus we have

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k &= \sum_{k=0}^{\infty} \frac{0}{k!} x^k \\ &= 0 + 0x + 0x^2 + \cdots \\ &= 0.\end{aligned}$$

Clearly, at any $x \neq 0$, $f(x) = e^{-\frac{1}{x^2}} \neq 0$.

Therefore, $f(x) \neq \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$.