

1. Prove the following limits by using $\epsilon - N$ definition

i) $\lim_{n \rightarrow \infty} \frac{2n^2}{3n^2+2} = \frac{2}{3}$.

ii) $\lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = 1$.

2. For each of the following statements, either prove that the statement is true or give a counter example to show that the statement is false:

i) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n + b_n\}$ diverges.

[Hint: You may use Theorem 1.4.3.]

ii) If $\{a_n\}$ converges and $\{b_n\}$ diverges, then $\{a_n b_n\}$ diverges.

[From Question 3 onwards, you may assume the limits of the standard sequences.]

3. Evaluate the following limits:

i) $\lim_{n \rightarrow \infty} \frac{n^2+5n^3-1}{3n^3+6n+4}$;

ii) $\lim_{n \rightarrow \infty} \frac{3^n+n^8}{2n^2+7^n}$;

iii) $\lim_{n \rightarrow \infty} \sqrt{\frac{n^4+4n^3+1}{n^3+2n^2}}$.

4. Use the Squeeze theorem to find the following limits:

i) $\lim_{n \rightarrow \infty} \frac{1+|\sin n|}{2n}$;

ii) $\lim_{n \rightarrow \infty} \left(\frac{2n-5}{3n+1}\right)^n$.

5. Do the following sequences tend to $+\infty$ or $-\infty$? Justify your answer.

i) $\left\{\frac{e^n}{n^{100}}\right\}$;

ii) $\left\{\frac{n}{\ln \frac{1}{n+2}}\right\}$.

6. Evaluate the following limits (you may assume the limits of the standard sequences and use the Squeeze theorem, etc.)

(a). $\lim_{n \rightarrow \infty} \sqrt[3]{\frac{2n^4+n+1}{16n^4+n^2+2}}$;

(b). $\lim_{n \rightarrow \infty} \left(3 + \ln\left(\cos \frac{1}{\sqrt{n}}\right) + \frac{n^2}{1.1^n}\right)$;

(c). $\lim_{n \rightarrow \infty} \frac{n^4+8^n}{9^n+n+8^n}$;

(d). $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n}{3}}$;

(e). $\lim_{n \rightarrow \infty} (\sqrt{3n-2} - \sqrt{3n-3})$;

(f). $\lim_{n \rightarrow \infty} \left(\frac{3+(-1)^n}{5}\right)^n$;

- (g). $\lim_{n \rightarrow \infty} \frac{7^n + \ln n - n!}{n! + n^2}$;
- (h). $\lim_{n \rightarrow \infty} \frac{n^{100} 100^n}{n!}$;
- (i). $\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}$;
- (j). $\lim_{n \rightarrow \infty} (3^n + 4^n)^{\frac{1}{n}}$;
- (k). $\lim_{n \rightarrow \infty} n \sin \frac{3}{n}$; (Hint: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$.)
- (l). $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{3n}\right)^{2n}$;
- (m). $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n^3}\right)^{n^3+2}$.
7. Let S and T be two bounded sets of real numbers. Show that $S \cup T$ is also a bounded set.
8. i) Show that a sequence $\{a_n\}$ is bounded if and only if $\{|a_n|\}$ is bounded.
 ii) Using i) or otherwise, show that if $\lim_{n \rightarrow \infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

Some suggested answers:

2. i) True ii) False counter-example: $a_n = 1/n$, $b_n = n$.
3. i) $\frac{5}{3}$ ii) 0 iii) 0.
4. i) 0 ii) 0
5. i) $\lim_{n \rightarrow \infty} \frac{e^n}{n^{100}} = +\infty$ ii) $\lim_{n \rightarrow \infty} \frac{n}{\ln \frac{1}{n+2}} = -\infty$.
6. (a) 1/2 (b) 3 (c) 0 (d) 1 (e) 0 (f) 0 (g) -1 (h) 0 (i) 0 (j) 4
 (k) 3 (l) $e^{-\frac{2}{3}}$ (m) e .