1. Denote the set of rational numbers by $\mathbb{Q}$. Consider the set
$$S = \{x \in \mathbb{Q} | 0 \leq x < 1\}.$$
Find $\sup S$ and $\inf S$. Justify your answers.
2. Let A and B be two non-empty bounded set of real numbers such that $A \subseteq B$. Show that $\inf A \geq \inf B$.
3. Let A and B be two non-empty bounded set of real numbers
   i) Show that $\sup A \cup B = \max\{\sup A, \sup B\}$.
   ii) Is it true that $\sup A \cap B = \min\{\sup A, \sup B\}$? Justify your answer.
4. Consider the sequence $\{a_n\}$ defined recursively by
$$a_1 = 2, \quad a_n = \sqrt{6 + a_{n-1}}, \quad , n = 2, 3, 4, \cdots .$$
   i) Show that $a_n \leq 3$ for all n.
   ii) Show that $\{a_n\}$ is monotone increasing.
   iii) Using parts i) and ii), show that $\{a_n\}$ converges, and find its limit.
5. Consider the sequence $\{x_n\}$ defined recursively by
$$x_1 = \tfrac{3}{4}, \quad x_{n+1} = 2x_n - x_n^2, \quad n = 1, 2, 3, \cdots .$$
Show that $\{x_n\}$ converges, and find its limit. (Hint: Show that $x_n \leq 1$ for all n and $\{x_n\}$ is monotone increasing.)
6. Find the lim sup and lim inf of the sequences:
   (a). $\{4 + \cos \frac{n\pi}{2}\}$.
   (b). $\{\frac{1+(-1)^n}{n}\}$.
7. For each of the following series, calculate the n-th partial sum S_n, and determine whether the series is convergent or divergent
   i) $\sum_{n=1}^{\infty} \ln \frac{n+2}{n+3}$.
   ii) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.
8. Determine the convergence or divergence of each of the following series
   (a). $\sum_{n=1}^{\infty} \frac{n^2-1}{2n^2+n}$.
   (b). $\sum_{n=1}^{\infty} \sin \frac{n\pi}{2}$.
   (c). $\sum_{n=1}^{\infty} \frac{n^2+\sqrt{n}}{n^3-n+4}$.
   (d). $\sum_{n=1}^{\infty} \frac{3+\sin n}{n^2}$.
   (e). $\sum_{n=1}^{\infty} \frac{2^n+3}{3^{n+1}-n}$.
   (f). $\sum_{n=1}^{\infty} \frac{2}{n^{1+\frac{1}{n}}}$.
   (g). $\sum_{n=1}^{\infty} \frac{4+(-1)^n}{2n}$.
   (h). $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln n)}$.
   (i). $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$.
   (j). $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.01}}$.
   (k). $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.
   (l). $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$.

Some suggested answers:

1. $\sup S = 1$ and $\inf S = 0$.
4. $\lim_{n\to\infty} a_n = 3$.
5. $\lim_{n\to\infty} x_n = 1$.
6. a) $\limsup = 5$ and $\liminf = 3$.
6. b) $\limsup = \liminf = \lim = 0$.
7. i) $S_n = \ln 3 - \ln(n+3)$ and $\sum_{n=1}^{\infty} \ln \frac{n+2}{n+3}$ is divergent.
7. ii) $S_n = \frac{1}{2}[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}]$ and $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim S_n = \frac{3}{4}$.
8. a) divergent by the divergence test.
8. b) divergent by the divergence test.
8. c) divergent by the limit comparison test (comparing with $\sum_{n=1}^{\infty} \frac{1}{n}$).
8. d) convergent by the comparison test (comparing with $\sum_{n=1}^{\infty} \frac{4}{n^2}$).
8. e) convergent by the limit comparison test (comparing with $\sum_{n=1}^{\infty} (\frac{2}{3})^n$).
8. f) divergent by the limit comparison test (comparing with $\sum_{n=1}^{\infty} \frac{1}{n}$).
8. g) divergent by the comparison test (comparing with $\sum_{n=1}^{\infty} \frac{3}{2n}$).
8. h) divergent by the integral test. (Hint: $\int \frac{1}{x(1+\ln x)} dx = \ln|1 + \ln x|$.)
8. i) convergent by the integral test. (Hint: $\int \frac{1}{x(1+\ln^2 x)} = \arctan(\ln x)$.)
8. j) convergent by the integral test. (Hint: $\int \frac{1}{x(\ln x)^{1.01}} dx = \frac{-100}{(\ln x)^{0.01}}$.)
8. k) divergent by the integral test. (Hint: $\int \frac{1}{x \ln x} dx = \ln \ln x$.)
8. l) divergent by the integral test. (Hint: $\int \frac{x}{x^2+1} dx = \frac{1}{2}\ln(x^2+1)$.)