1. For each of the following sequence of functions, determine whether it converges pointwise to a function, and find the limiting if it exists. Justify your answers.

i) $\left\{\left(1+\frac{x}{n}\right)^{nx}\right\}$, $x \in (-\infty, +\infty)$.

ii) $\{x^{n+1}\}$, $x \in [-1, 1]$.

iii) $\left\{\frac{x^{2n}}{1+x^{2n}}\right\}$, $x \in [0, 1]$.

2. Determine whether the following sequences of functions converge uniformly on the indicated intervals. Justify your answers.

i) $F_n(x) = \frac{n^2}{n^2+x^2}$, $x \in [0, 1]$.

ii) $F_n(x) = x^n(1-x)$, $x \in [0, 1]$.

iii) $f_n(x) = \frac{n \ln x}{x^n}$, $x \in [1, \infty)$.

iv) $f_n(x) = \frac{n \ln x}{x^n}$, $x \in [4, \infty)$.

3. Let $\{F_n\}$ be a sequence of functions on an interval I. It is given that $\{F_n\}$ converges uniformly on some function F on I. Suppose also that for each $n \in \mathbb{Z}^+$, there exists a real number $M_n > 0$ such that

$$|F_n(x)| \le M_n \quad \text{for all } x \in I.$$

Show that there exists a real number M such that $|F(x)| \le M$ for all $x \in I$.
(Hint: Recall the definition of uniform convergence.)

4. Evaluate the limits. Justify your answers.

i) $\lim_{n\to\infty} \int_0^1 \frac{n+e^x}{n+x^2} dx$.

ii) $\lim_{n\to\infty} \int_1^2 \left(\frac{x^2+1}{8}\right)^n \sin nx dx$.

5. Prove that each of the following series of functions converges uniformly on the indicated interval.

i) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+x^2}$, $x \in (-\infty, +\infty)$.

ii) $\sum_{n=1}^{\infty} \frac{1}{1+n^3x^2}$, $x \in [2, \infty)$.

iii) $\sum_{n=1}^{\infty} \frac{xe^{-nx}}{n^2}$, $x \in (0, \infty)$.

6. Consider the function

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^k x^k}{1 + x^{2k}}, \quad x \in (0, \frac{2}{3}).$$

Show that F is continuous on the interval $(0, \frac{2}{3})$.

(Hint: You may need the Weierstrass M-test.)