

1. Evaluate $\sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} \frac{x^n(1-x^2)}{\sqrt{1+x}} dx$ in simplest form. Justify your answer.
2. Let $\sum_{k=0}^{\infty} a_k$ be an absolutely convergent series.
 - i) Show that $\sum_{k=0}^{\infty} a_k \sin kx$ converges uniformly on $(-\infty, +\infty)$.
 - ii) Hence evaluate $\int_0^{2\pi} \sum_{k=1}^{\infty} a_k \sin kx dx$. Justify your answer.
3. Consider the function $f(x) = \sum_{n=1}^{\infty} \frac{\cos^n x}{n^3}$. Is f differential on $(-\infty, +\infty)$? Justify your answer.
4. Find the radius of convergence of each of the following power series:
 - i) $\sum_{n=1}^{\infty} \left(1 + \frac{3}{n}\right)^{n^2} x^n$.
 - ii) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n!} (x-1)^n$.
 - iii) $\frac{x}{5} + \left(\frac{x}{6}\right)^2 + \left(\frac{x}{5}\right)^3 + \left(\frac{x}{6}\right)^4 + \left(\frac{x}{5}\right)^5 + \left(\frac{x}{6}\right)^6 + \dots$.
5. Find the interval of convergence of each of the following power series:
 - i) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n x^n}{n^{3/2}}$.
 - ii) $\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n+1}$.
6. By computing derivatives, find the Taylor series of e^{2x} at $x = 3$.
7. i) By integrating from $t = 0$ to $t = x$ the power series $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$, $|t| < 1$, show that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$
 for all $|x| < 1$.
 - ii) Use part i) to find the Taylor series of $\ln(1+2x^2)$ at $x_0 = 0$.
8. Use series of estimate the integrals' values with an error of magnitude less than 10^{-3} .

i) $\int_0^{0.2} \sin x^2 dx.$

ii) $\int_0^{0.1} \frac{1}{\sqrt{(1+x^4)}} dx.$

9. Use series to evaluate the limits

i) $\frac{\arctan y - \sin y}{y^3 \cos y}.$

ii) $\lim_{x \rightarrow \infty} x^2(e^{-1/x^2} - 1).$

10. According to the Alternating Series Estimation Theorem, how many terms of the Maclaurin series for $\arctan 1$ would you have to add to be sure of finding $\pi/4$ with an error of magnitude less than 10^{-3} ? Give reasons for your answer.

Some suggested answers:

1. $\sqrt{\frac{3}{2}} - \frac{2}{3}.$

2(ii). 0.

3. Yes.

4(i) $\frac{1}{e^3}.$

4(ii) $\infty.$

4(iii) 5.

5(i). $[-\frac{1}{2}, \frac{1}{2}].$

5(ii). $[\frac{5}{3}, \frac{7}{3}].$

6 $\sum_{n=0}^{\infty} \frac{2^n e^6 (x-3)^n}{n!}.$

7(ii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n x^{2n}}{n}.$

8(i). 0.00267.

8(ii). 0.1.

9(i). $-\frac{1}{6}.$

9(ii). -1.

10. 500 terms.