The Exponent Problem in Homotopy Theory
(Jie Wu)

The fundamental problem in homotopy theory is to determine the homotopy groups $\pi_*(X)$ of a given space X. The determination of the homotopy groups is a very challenging problem. So far the general homotopy groups of spheres (except the circle) are far unknown.

The definition of homotopy groups is as follows: Let X be a (topological) space. Choose a point x_0 in X as a base point. Consider the n-dimensional sphere S^n, that is, S^n is the space of unit vectors in the $(n+1)$-dimensional Euclidian space. For instance, $S^0 = \{-1, 1\} \subseteq \mathbb{R}$, S^1 is the circle and S^2 is the usual sphere. Let $N = (0, \ldots, 0, 1)$ be the North pole of S^n. Let $\mathrm{Map}_*(S^n, X)$ be the set of continuous maps $f\colon S^n \to X$ with $f(N) = x_0$. Define an equivalence relation $\simeq$ in $\mathrm{Map}_*(S^n, X)$ as follows: $f \simeq g$ if and only if there exists a continuous map $F\colon S^n \times [0,1] \to X$ such that

1) $F(x, 0) = f(x)$ for $x \in S^n$;
2) $F(x, 1) = g(x)$ for $x \in S^n$ and
3) $F(N, t) = x_0$ for $0 \leq t \leq 1$.

Given a parameter $0 \leq t \leq 1$, we have a continuous map $F_t(x) = F(x, t)$ from the sphere S^n to X. Condition 1 above says that $F_0 = f$, condition 2 above says that $F_1 = g$ and condition 3 above says that F_t maps the north pole N to the base point x_0 for each parameter t. Thus, roughly speaking, $f \simeq g$ if and only if there is a *continuous* deformation from f to g. The *homotopy group* $\pi_n(X)$ is defined by $\pi_n(X) = \mathrm{Map}_*(S^n, X)/\simeq$ the quotient set of $\mathrm{Map}_*(S^n, X)$ by the equivalence relation $\simeq$.

$\pi_0(X)$ is the set of path connected components of X. This is NOT a group. $\pi_0(X)$ consists of one element if and only if X is path-connected, that is, for any two points in X there is a continuous path from one to another.

The fundamental group is $\pi_1(X)$. The multiplication on $\pi_1(X)$ is defined as follows: We may consider S^1 as the quotient of the interval $I = [0, 1]$ by making the identification of 0 with 1. A continuous map $f\colon S^1 \to X$ with $f(N) = x_0$ is then described as a (continuous) loop in X starting and ending at the base point x_0. Given two continuous maps $f, g\colon S^1 \to X$ with $f(N) = g(N) = x_0$, we define the product $f * g$ by $f * g(t) = f(2t)$ for $0 \leq t \leq 1/2$ and $g(2t-1)$ for $1/2 \leq t \leq 1$. In other words, $f * g$ is the loop which first goes along the loop induced by f and then the loop induced by g. The multiplication in $\pi_1(X)$ is induced by the product $f * g$. In general $\pi_1(X)$ is a non-commutative group.

The multiplication on $\pi_n(X)$ with $n \geq 2$ is defined as follows: Let $S^n \vee S^n$ be the quotient of S^n by pinching the equator $\{(x_1, \ldots, x_n, 0) | x_1^2 + \cdots + x_n^2 = 1\}$ of

S^n to a point and pinching one line of longitude to the point. The space $S^n \vee S^n$ can be regarded as two spheres joining at the north pole. Let $f, g\colon S^n \to X$ with $f(N) = g(N) = x_0$. We obtain a map $\phi\colon S^n \vee S^n \to X$ where ϕ restricted to the top sphere of $S^n \vee S^n$ is f and ϕ restricted to the bottom sphere is g. The product $f+g\colon S^n \to X$ is then defined to be the composition of the quotient $S^n \to S^n \vee S^n$ with the map ϕ. Roughly speaking $f+g$ is obtained by taking values from f on the upper hemisphere and taking values from g on the lower hemisphere. The multiplication on $\pi_n(X)$ is induced by $f+g$. For $n \geq 2$, $\pi_n(X)$ is always a commutative group.

Examples. Let $n \geq 1$. Then $\pi_j(S^n) = 0$ for $j < n$, $\pi_n(S^n) = \mathbb{Z}$ the group of integers, $\pi_j(S^1) = 0$ for $j > 1$, $\pi_3(S^2) = \mathbb{Z}$, $\pi_{n+1}(S^n) = \mathbb{Z}_2 = \{0, 1\}$ for $n \geq 3$ and $\pi_{n+2}(S^n) = \mathbb{Z}_2$ for $n \geq 2$. Observe that the set $\mathrm{Map}_*(S^{n+2}, S^n)$ is large set (at least uncountable) but by taking homotopy equivalence relation $\pi_{n+2}(S^n)$ has only two elements for $n \geq 2$.

A space X is called simply connected if X is path connected and the fundamental group $\pi_1(X)$ is trivial. For instance S^n is simply connected for $n \geq 2$.

Since it is difficult to determine the homotopy groups in general, people study the properties of homotopy groups. One of these properties is about the exponent of homotopy groups. Let G be an abelian group and let p be a prime integer. The p-torsion component $\mathrm{Tor}_p(G)$ of G is the subgroup of G consisting of elements $x \in G$ with the property that $p^r x = 0$ for some r. Recall that any finitely generated abelian group G admits a decomposition that $G = F \oplus \mathrm{Tor}_2(G) \oplus \mathrm{Tor}_3(G) \oplus \mathrm{Tor}_5(G) \oplus \ldots$, where F is a direct sum of $\mathbb{Z}$. By tensoring with rational numbers $\mathbb{Q}$, $G \otimes \mathbb{Q} = F \otimes \mathbb{Q}$ is a vector space over $\mathbb{Q}$.

Let X be a simply connected space and let $b_n(X)$ be the dimension of $\pi_n(X) \otimes \mathbb{Q}$. Observe that the number $b_n(X)$ (possibly $+\infty$) depends on n. A space X is a called *rationally elliptic* if each $b_n(X) < +\infty$ and there exists a polynomial function $f(x)$ such that $b_n(X) \leq f(n)$ for each n.

The following conjecture is the famous conjecture in homotopy theory.

Moore Conjecture. Let X be a simply connected finite CW-complex. Suppose that X is rationally elliptic. Then for each prime integer p there exists a positive integer $r > 0$ such that $p^r x = 0$ for any $x \in \mathrm{Tor}_p(\pi_n(X)) = 0$ and any n. In other words the p-torsion components of the homotopy groups $\pi_n(X)$ have a bounded exponent.

Let $X = S^m$ be a sphere with $m \geq 2$. Then $b_n(S^m) = 0$ for $0 < n < m$, $b_n(S^m) = 1$. If m is odd, then $b_n(S^m) = 0$ for $n > m$. If m is even, then $b_{2m-1}(S^m) = 1$ and $b_n(S^m) = 0$ for $n > m$ and $n \neq 2m-1$. Thus S^m satisfies the conditions in the Moore conjecture. It was known by Cohen-Moore-Neisendorfer that the Moore conjecture holds for $X = S^m$. For general spaces X the Moore conjecture remains largely open.