1. Show that the function
Solution 1. Since $f(z) = e^{z^2}$, $f(z)$ is entire and $f'(z) = 2ze^{z^2}$.
Solution 2. $u(x, y) = e^{x^2-y^2}\cos(2xy)$ and $v(x, y) = e^{x^2-y^2}\sin(2xy)$. Since

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy)$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy),$$

$f(z) = u + iv$ is entire and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy)i + 2xe^{x^2-y^2}\sin(2xy)i.$$

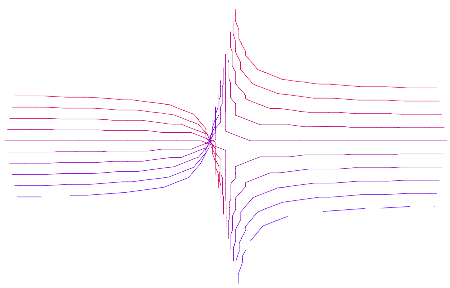
2.

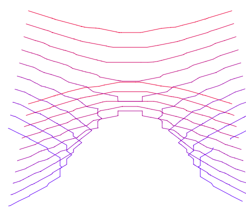
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

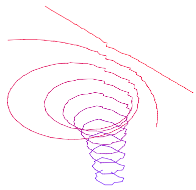
3. $ax^2 + bxy - ay^2$.
4. a) Yes. b) No. For example $u = v = xy$. c) Yes.
5. a) $v = \cos x \sinh y + C$, b) $v = \tan^{-1}(y/x) + C = \operatorname{Arg} z + C$, c) $v = -\operatorname{Re}(e^{z^2}) + C$, d) $u = -\operatorname{Im}(1/z) = \operatorname{Re}(i/z)$. Thus $v = \operatorname{Im}(i/z) + C = \operatorname{Re}(1/z) + C = x/(x^2 + y^2) + C$.

6. a)





u=x^2−y^2 , v=2xy



b) $u = \frac{x^2-1+y^2}{(x+1)^2+y^2}$,

v=$\frac{2y}{(x+1)^2+y^2}$

7. $\phi(x, y) = xy - 1$.

8. a)$\phi(x, y) = \text{Re}(z^2 + 5z + 1)$. b) $\phi(x, y) = 2\,\text{Re}(z^2/(z + 2i))$.