WHAT ARE HOMOTOPY GROUPS?

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Some people say that homotopy theory studies continuous transformations. This is true in certain sense that I am going to explain below. In one of the next topics, I am going to explain that this is not exactly true from other point of views.

First of all let's consider the definition of homotopy. Let f and g be continuous functions from a space X to a space Y. The intuitive view of deformation that the graph of f can be moved continuously to that of g. This means that we have a collection of maps $f_t\colon X \to Y$ labeled by a parameter t with $0 \le t \le 1$ such that $f_0 = f$ and $f_1 = g$. This collection of maps $\{f_t\}$ must be *continuously* changed when t changes from 0 to 1. Now I give a precise definition of homotopy: A *homotopy* from f to g means a continuous function F from the product space $X \times [0, 1]$ to Y such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all x in X. If such an F exists, the map f is homotopic to g denoted by $f \simeq g$. Given a parameter $0 \le t \le 1$, we have a continuous map $f_t(x) = F(x, t)$ for $x \in X$ from X to Y. This collection of maps continuously moves because the function F is continuous. Given two maps f and g, sometimes f is homotopic to g and sometimes it is not. So a fundamental problem in homotopy theory could be described as how to determine one map homotopic to another.

In mathematics, it often happens that a particular point has some special properties. For instance, in a group we have a particular element so-called the identity. When we are going to set the Cartesian system, we choose a particular point in our space as the origin. This idea is applied to topological spaces. A pointed space X means a (topological) space X with a given point x_0 so-called the base-point. Let X and Y be pointed spaces. A pointed map $f\colon X \to Y$ means a continuous function which sends the base-point to the base-point, that is, $f(x_0) = y_0$. Let f and $g\colon X \to Y$ be two pointed map. In this case, we already have $f(x_0) = g(x_0) = y_0$ by definition. A *pointed homotopy* from f to g means a continuous function F from the product space $X \times [0, 1]$ to Y such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all x in X, and $F(x_0, t) = f(x_0) = g(x_0) = y_0$ for all t. In other words, a pointed homotopy means a homotopy which is constant at the base-point x_0 or the base-point is punctured under the deformation from the graph of f to that of g. A pointed space X is called *homotopy equivalent* to another pointed space Y if there exist pointed maps $f\colon X \to Y$ and $g\colon Y \to X$ such that the composite $g \circ f\colon X \to X$ is (pointed) homotopic to the identity map of X and $f \circ g\colon Y \to Y$ is (pointed) homotopic to the identity map of Y. Intuitively a space X is homotopy equivalent to Y means that X can be continuously deformed to Y. For instance our Euclidean space $\mathbb{R}^n$ is homotopy equivalent a single point. So the homotopy equivalence relation gives no difference between a point and $\mathbb{R}^n$. On the other hand, we found that many spaces are not homotopy equivalent to each other. For instance, an m-dimensional sphere is NOT homotopy equivalent to an n-dimensional sphere unless $m = n$. One can intuitively that the sphere S^2 has a 3-dimensional hole and S^3 has a 4-dimensional hole. A *continuous deformation* will never be able to move a 3-dimensional hole to a 4-dimensional hole.

Of course S^n is NOT homotopy equivalent to a point because roughly speaking the hole inside the sphere could be never removed under continuous deformation. Also the Klein bottle is NOT homotopy equivalent to say the sphere S^2. One can intuitively think that the Klein bottle has a very unusual *hole*. Although we have a lot of results about how to classify spaces under homotopy equivalence, it is REALLY a hard problem to how a space is or is not homotopy equivalent to another in general. This makes homotopy theory to be one of the hardest area in mathematics. If you like to attack very hard mathematics problem, it is a good idea to study homotopy theory. There are a lot of problems in this area what are described as out of human ability by some people. The problems about homotopy groups are just some of these questions. On the other hand, you need not to be scared away from the hardness of homotopy theory because you could always get excellent results if you are trying to think the problems more and more. More hard problems in an area also give you more change to show your mathematical ability.

Now I am going to explain the definition of homotopy groups. Given two spaces X and Y, let $\mathrm{M}ap_*(X,Y)$ be the set of ALL pointed maps from X to Y. This set is too huge to handle and so we should try to a relatively much smaller set. A canonical way is to put an equivalence relation on the set $\mathrm{M}ap_*(X,Y)$ and then consider the quotient set. This relation is simply given by the homotopy equivalence relation, namely a pointed map f is equivalent to g if f is pointed homotopic to g. By modulo this relation, we obtain the quotient set, denoted by $[X,Y]$, called the set of homotopy classes from X to Y. The homotopy class represented by f is denoted by $[f]$. So if $f \simeq g$, then $[f]=[g]$. The set $[X,Y]$ is really much smaller than $\mathrm{M}ap_*(X,Y)$. It often happens that the $[X,Y]$ has only finitely many elements while $\mathrm{M}ap_*(X,Y)$ has uncountably many. Now given a pointed space X the n^{th} homotopy group $\pi_n(X)$ is defined to be the set of homotopy classes $[S^n,X]$. In other words, by definition, to get $\pi_n(X)$, first we should considered ALL pointed maps from the sphere S^n to X and then modulo the homotopy equivalence relation. Now I am going to explain the group structure on $\pi_n(X)$.

When $n=0$, $\pi_0(X)$ is just a set in general. By definition, S^0 means the boundary of the 1-dimensional disc $[-1,1]$ and so S^0 has only two points -1 and 1. We need to fixed one of them, say -1, as the base-point. Let x_0 be the base-point of X and let $f\colon S^0 \to X$ be a pointed map. Because f is pointed, we have $f(-1)=x_0$ and then $f(1)$ could be any point in X. This gives the identification of the set $\mathrm{M}ap_*(S^0,X)$ with X via $f \mapsto f(1)$. Let f and g be two pointed maps from S^0 to X. Suppose that f pointed homotopic to g. Then we have a homotopy $F\colon S^0 \times I \to X$ from f to g. By definition, $F(-1,t)=x_0$ for $0 \le t \le 1$ and $F(1,t)$ is a continuous function from $[0,1]$ to X such that $F(1,0)=f(1)$ and $F(1,1)=g(1)$. By changing the parameter t, we obtain a (continuous) path from the point $f(1)$ and $g(1)$. Conversely, if there is a (continuous) path from $f(1)$ to $g(1)$, one gets a pointed homotopy from f to g. By using the identification of $\mathrm{M}ap_*(S^0,X)$ with X, one gets the conclusion that the set $\pi_0(X)$ is one-to-one correspondent to the set of path-connected components of X, where two points in X lies in the same path-connected component if there is a (continuous) path joining them. In particular, if X is path-connected, that is, any two points can joined by a (continuous) path, then the $\pi_0(X)$ has only one element.

$\pi_1(X)$ is called the fundamental group of X. This one really a group but not commutative in general. To get the group structure on $\pi_1(X)$, we need to define a canonical multiplication. Recall that $\pi_1(X) = [S^1, X]$ is the quotient of $Map_*(S^1, X)$. Any element in $\pi_1(X)$ can be represented by a pointed map $f\colon S^1 \to X$. Recall that the unit circle S^1 is given by the complex number of length 1, that is, elements in S^1 are given by $e^{i2\pi t}$ for $0 \le t \le 1$. Choose $1 = e^{i2\pi}$ as the base-point of S^1. Since f is pointed map, $f(1) = x_0$. When the parameter t changes from 0 to 1, $f(e^{i2\pi t})$ forms a (continuous) path in X starting from the base-point x_0 and ending with the base-point x_0, namely $f(e^{i2\pi t})$ forms a loop in X. Now given two pointed maps $f, g\colon S^1 \to X$, that is, two loops in X, we can make a new loop, denoted by $f * g$, which goes through the loop of f first and then goes through that of g. More precise $(f * g)(e^{i2\pi t}) = f(e^{i2\pi 2t})$ for $0 \le t \le 1/2$ and $(f * g)(e^{i2\pi t}) = g(e^{i2\pi(2t-1)})$ for $1/2 \le t \le 1$. The multiplication on $\pi_1(X)$ is exactly induced by this loop product, namely $[f]\cdot[g] = [f * g]$. Certainly one has to prove that this really gives a group structure, name this multiplication is associative with the identity and inverse, but the proof can be found in any text book of algebraic topology. By drawing pictures, you might be convinced that this multiplication is NOT commutative because $[f][g]$ means taking the loop of f first and the loop of g while $[g][f]$ means taking the loop of g first and then the loop of f. A typical example is that X is given by joining two circles at one point. In this example, $\pi_1(X)$ is a free group of rank 2, namely $\pi_1(X)$ is given by the form product of the words like x_1x_2, $x_1^{-1}x_2x_1$, and etc.

For $n \ge 2$, $\pi_n(X)$ is a commutative group. Choose the North pole as the base-point of S^n. Then we pinch the $(n-1)$-dimensional unit sphere containing the North pole and the South pole to one point. Then we obtain the quotient space Y of S^n. From intuition, one can see that Y is the join of two S^n at one point. (It is a good exercise to prove it.) So the space Y can be described as follows: Take two copies of S^n and then join them together at the North pole. Now given two pointed maps $f, g\colon S^n \to X$ we define a function $f \vee g\colon Y \to X$ such that $f \vee g$ restricted to one copy of the two spheres in Y is f and restricted to another is g. This function is well-defined and continuous. On the other hand, since Y is a quotient space of S^n, we have a continuous map (this is called *pinching map*) from S^n to Y. The map $f \vee g$ composing with the pinching map gives a new map from $S^n \to X$. Now the multiplication on $\pi_n(X)$ is induced by this product. In the text books of algebraic topology, there are theorems which state that $\pi_n(X)$ is a commutative group under this multiplication.

Roughly speaking a space X could be handled in some sense if we know the homotopy groups of X together with various application in many areas of mathematics and theoretical physics. However the determination of the homotopy groups is a very challenging problem, particularly higher homotopy groups. For the case X is a sphere, all homotopy groups of the circle S^1 are known, but the general homotopy groups of S^n with $n \ge 2$ are far unknown. The determination of the homotopy groups of spheres is one of the central problems in homotopy theory. You may find out some information on $\pi_k(S^n)$ in this web or many references. The spheres are very nice-looking spaces. Unfortunately people still understand so few about the spheres. You may also know that there is famous conjecture so-called Poincaré conjecture on S^3. That one is the one of the central problems in low dimensional topology.

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