A Problem on Modules over the General Linear Groups
(Jie Wu)

Let the ground ring $\mathbf{k}$ be a field of characteristic $p > 0$ and let $\mathrm{GL(m,\mathbf{k})}$ be the general linear group, that is, $\mathrm{GL(m,\mathbf{k})}$ consists of $m \times m$ matrices over $\mathbf{k}$ with non-zero determinant. The multiplication on $\mathrm{GL(m,\mathbf{k})}$ is the matrix multiplication.

Let V be an m-dimensional vector space over $\mathbf{k}$. Then $\mathrm{GL(m,\mathbf{k})}$ acts on V in the canonical way, $\alpha_{m\times 1} \mapsto A_{m\times m} \cdot \alpha_{m\times 1}$. Let $V^{\otimes n}$ be the n-fold self tensor product of V over $\mathbf{k}$. Let $\mathrm{GL(m,\mathbf{k})}$ act on $V^{\otimes n}$ diagonally, that is,

$$A \cdot x_1 \otimes \cdots \otimes x_n = (A \cdot x_1) \otimes \cdots \otimes (A \cdot x_n)$$

for vectors $x_i \in V$ and $m \times m$ matrices $A \in \mathrm{GL(m,\mathbf{k})}$.

Let $a \in V^{\otimes k}$ and let $b \in V^{\otimes l}$. The commutative $[a, b]$ is defined by $[a, b] = a \otimes b - b \otimes a \in V^{\otimes k+l}$. Let $L_n(V)$ be the vector subspace of $V^{\otimes n}$ spanned by the iterated commutators

$$[[x_1, x_2], x_3], \ldots, x_n]$$

for $x_i \in V$. Then clearly $L_n(V)$ is a $\mathrm{GL(m,\mathbf{k})}$-submodule of $V^{\otimes n}$, that is, $L_n(V)$ is an invariant subspace of $V^{\otimes n}$ under $\mathrm{GL(m,\mathbf{k})}$-action.

A subspace M of $L_n(V)$ is called *T_n-projective* if (1) M is an $\mathrm{GL(m,\mathbf{k})}$-submodule of $L_n(V)$ and (2) there is a morphism of $\mathrm{GL(m,\mathbf{k})}$-modules $r \colon V^{\otimes n} \to M$ such that the composite

$$M \subseteq L_n(V) \subseteq V^{\otimes n} \xrightarrow{r} M$$

is the identity map. In other words, M is a $\mathrm{GL(m,\mathbf{k})}$-summand of $V^{\otimes n}$. The subspace M of $L_n(V)$ is called *maximal T_n-projective* if (1) M is a T_n-projective subspace of $L_n(V)$ and (2) $\dim M \geq \dim N$ for any T_n-projective subspace N of $L_n(V)$.

Problem. Let $\mathbf{k}$ be an algebraic closed field of characteristic $p > 0$. Determine the maximal T_n-projective subspace of $L_n(V)$.

Note. 1. The answer of this problem helps to determine the homology of the *functorially atomic retracts* of $\Omega\Sigma X$ for the spaces X with cells less than or equal to $m = \dim V$.
2. The problem remains open even when $\dim V = 2$.