

# MODULAR REPRESENTATIONS AND THE HOMOTOPY OF LOW RANK $p$ -LOCAL CW-COMPLEXES

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ABSTRACT. We construct an infinite set of retractions from certain loop spaces of suspended low rank CW-complexes  $X$  localized at an odd prime  $p$ . These retracts are themselves loop spaces of arbitrary suspensions of  $X$ . This is accomplished with the help of modular representations and their applications to topology.

## 1. INTRODUCTION

Recently there has been sizeable interest in certain general homotopy decompositions of loop spaces, but the main limitation to finding any applications seems to be the lack of information concerning each of the factors in these decompositions. In the worst cases even the homology groups remain unknown. In contrast specific homotopy decompositions for specific spaces have been more successful. One of the basic examples is Serre's odd  $p$ -primary decomposition  $\Omega S^{2m} \simeq S^{2m-1} \times \Omega S^{4m-1}$  [7], which implies the  $p$ -components of homotopy groups for even dimensional spheres are determined by those of the odd dimensional spheres. Selick's odd  $p$ -primary decomposition [5] of the homotopy fiber  $\Omega^2 S^{2p+1}\{p\}$  in terms of the 1-connected cover of  $\Omega^2 S^3$  gives an upper bound of  $p$  for the  $p$ -exponent of  $S^3$ , while odd  $p$ -primary decompositions of loops on Moore spaces have proved even more useful, leading to the computation for the  $p$ -exponents on the higher dimensional spheres by Cohen, Moore, Neisendorfer [1].

Our main interest in this paper is to give more information about the factors in some of the known general homotopy decompositions of loop spaces. We will fix  $p$  to be an odd prime throughout, and we assume all spaces have been localized at  $p$ . The cell structure of a space (if any reference is made) is taken to be in the  $p$ -local sense, and as we will be mostly using mod- $p$  homology, we shall denote it by  $H_*(X)$  for any space  $X$ . Our main result is as follows.

**Theorem 1.1.** *Let  $X$  be any suspended  $p$ -local CW-complex, and  $V = \tilde{H}_*(X)$ . Let  $M$  denote the sum of the degrees of the generators of  $V$ , and define the sequence of integers  $b_i$  recursively, with  $b_0 = 0$  and*

$$b_i = (1 + \dim V)b_{i-1} + M.$$

*Suppose either  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$ , and  $1 < \dim V \leq p$ .*

- (i) *If  $\dim V < p - 1$ , then  $\Omega \Sigma^{b_i+1} X$  is a retract of  $\Omega \Sigma X$  for each  $i \geq 1$ ;*

- (ii) if  $\dim V = p$ , there exist spaces  $Y_i$  such that  $\Omega\Sigma Y_i$  is a retract of  $\Omega\Sigma X$ , and  $\tilde{H}_*(Y_i) \cong \tilde{H}_*(\Sigma^{b_i} X)$  for  $i \geq 1$ .

For the simplest case  $\dim V = 2$  we prove a stronger version of Theorem 1.1 by showing that sequence of retracts are factors in a decomposition of  $\Omega\Sigma X$ , and that the connectivity of these factors grows at a slower rate.

**Theorem 1.2.** *Fix  $p \geq 5$ . Let  $X$  be any any suspended  $p$ -local CW-complex with  $\dim V = 2$ , and either  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$ . Let  $M$  denoting the sum of the degrees of the two generators of  $V$ .*

*Suppose  $0 < k_1 < k_2 < \dots$  is any sequence satisfying the following properties:*

- (1)  $2k_i + 1$  is prime to  $p$ ;
- (2)  $2k_i + 1$  is not a multiple of  $2k_j + 1$  whenever  $i > j$ .

*Then there exists a decomposition*

$$\Omega\Sigma X \simeq \prod_j (\Omega\Sigma^{k_j M+1} X) \times (\text{Some other space}).$$

Both Theorems (1.1) and (1.2) will depend on computational work dealing with representations of symmetric groups in Sections (2), (3) and (4), as well as the use of a certain general homotopy decomposition of loop suspensions stated as Theorem 5.1 in Section (5). A consequence of Theorem (1.1) is that the stable homotopy groups of the spaces defined there are (in a precise sense) retracts of their regular homotopy groups. This property leads us to a small application towards the Moore conjecture at the end of this paper. As for another application, recall Cohen and Neisendorfer [2] extended Serre's decomposition  $\Omega S^{2m} \simeq S^{2m-1} \times \Omega S^{4m-1}$  to loops spaces of certain low rank CW-complexes:

**Theorem 1.3** (Cohen, Neisendorfer). *Let  $X$  be any suspended  $p$ -local CW-complex, and let  $V = \tilde{H}_*(X)$ . Suppose  $V_{\text{even}} = 0$ , and  $1 \leq \dim V < p - 1$ . Then there exists a functorial decomposition*

$$\Omega\Sigma X \simeq A(X) \times \Omega Q(X)$$

*such that  $A(X)$  is a finite  $H$ -space with*

$$H_*(A(X)) \cong \Lambda(V)$$

*as primitively generated algebras, and*

$$H_*(\Omega Q(X)) \cong S([L(V), L(V)]) = \bigotimes_{i=2}^{\infty} (S(L_i(V))),$$

*where  $L(V)$  is the free Lie algebra generated by  $V$ , and  $[L(V), L(V)]$  is the sub Lie algebra of  $L(V)$  generated by Lie brackets of length greater than one.  $\square$*

This decomposition was in some sense generalized by Selick, Theriault, and Wu [14]. In their construction they showed that  $\Sigma X$  can be replaced with any coassociative  $co$ - $H$ -space, in which case  $A(X)$  becomes an  $H$ -space whose mod- $p$  homology  $A^{min}(V)$  is the *minimal* functorial coalgebra retract of the tensor algebra  $T(V)$  containing  $V$ . Instead of looking into general forms of Cohen's and Neisendorfer's decomposition, we obtain a stronger version of their original result. Proved in Section (5), the main idea is that there are often many more finite  $H$ -spaces hiding inside  $\Omega Q(X)$ :

**Theorem 1.4.** *Let  $X$  be any suspended  $p$ -local CW-complex, and let  $V = \tilde{H}_*(X)$ . Suppose  $V_{even} = 0$ ,  $1 < \dim V < p - 1$ , and  $\dim V$  is even.*

*Let  $M$  denote the sum of the degrees of the generators of  $V$ , and define the sequence of integers  $b_i$  recursively, with  $b_0 = 0$  and  $b_i = (1 + \dim V)b_{i-1} + M$ . Then there exists a decomposition*

$$\Omega \Sigma X \simeq \prod_{i=0}^{\infty} A(\Sigma^{b_i} X) \times (\text{Some other space}),$$

where (as in Theorem (1.3))

$$H_*(A(\Sigma^{b_i} X)) \cong \Lambda(\Sigma^{b_i} V),$$

and  $A(\Sigma^{b_i} X)$  is a finite  $H$ -space that is a retract of  $\Omega \Sigma^{b_i+1} X$ .

## 2. ASPECTS OF THE REPRESENTATION OF SYMMETRIC GROUPS ON TENSOR PRODUCTS

In this section, as well as sections (3) and (4), we shall let  $R$  denote either the field  $\mathbb{Z}_p$ , or the ring of  $p$ -local integers  $\mathbb{Z}_{(p)}$ . The symmetric group on  $k$  letters is denoted by  $S_k$ , and  $R[S_k]$  is the group ring over  $R$  generated by  $S_k$ .

Let  $V$  be any graded  $R$ -module, and let  $V^{\otimes k}$  denote the  $k$ -fold tensor product. Consider the action of  $R[S_k]$  on  $V^{\otimes k}$  that is defined permuting factors in a graded sense. In this case the action of a single element  $\sigma \in R[S_k]$  on  $V^{\otimes k}$  induces a self-map

$$V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k},$$

which we also denote by  $\sigma$  for convenience. More generally, we let  $R[S_j]$  act on  $V^{\otimes k}$  for  $j < k$  by permuting only the first  $j$  factors in a graded sense, and leaving the remaining factors fixed.

The elements  $\hat{s}_k$  and  $\bar{s}_k$  in  $R[S_k]$  are defined as the sums

$$\hat{s}_k = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma,$$

$$\bar{s}_k = \sum_{\sigma \in S_k} \sigma.$$

One sees that the action of  $\hat{s}_k$  and  $\bar{s}_k$  on  $V^{\otimes k}$  sends a tensor in  $V^{\otimes k}$  to a certain linear combination of all permutations of that tensor. In particular, for any  $x_1 \otimes \cdots \otimes x_k \in V^{\otimes k}$  and  $\sigma \in S_k$  we have  $\hat{s}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) = \text{sgn}(\sigma) \hat{s}_k(x_1 \otimes \cdots \otimes x_k)$  if each  $x_i$  has even degree, and  $\bar{s}_k(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}) = \text{sgn}(\sigma) \bar{s}_k(x_1 \otimes \cdots \otimes x_k)$  if each  $x_i$  has odd degree.

The *Dynkin-Specht-Wever* elements  $\beta_k \in \Sigma_k$  are defined inductively starting with  $\beta_2 = 1 - (2, 1)$  and

$$\beta_k = (1 + (2, 3, \dots, k, 1))(1 \otimes \beta_{k-1}).$$

The action of  $\beta_k$  on  $V^{\otimes k}$  is given by sending a tensor  $x_1 \otimes \dots \otimes x_k \in V^{\otimes k}$  to the commutator  $[x_1, \dots, x_k] \in V^{\otimes k}$ . That is,  $\beta_2(xy) = x \otimes y - (-1)^{|x||y|}y \otimes x$ , and

$$\beta_k(x_k \otimes \dots \otimes x_1) = \beta_{k-j+1}(x_k \otimes x_{k-1} \otimes \dots \otimes x_{j+1} \otimes \beta_j(x_j \otimes \dots \otimes x_1))$$

for each  $2 \leq j \leq k - 1$ .

In the following theorem we define certain integers  $c_{n,\ell}$  and  $d_{n,\ell}$  that will be used throughout this paper.

**Theorem 2.1.** *There exist non-negative integers  $c_{n,\ell}$  and  $d_{n,\ell}$  such that for any graded free  $R$ -module  $V$  with  $\dim V = \ell > 1$ ,*

$$(\hat{s}_\ell^{\otimes n} \otimes 1) \circ \beta_{\ell n+1} \circ (\hat{s}_\ell^{\otimes n} \otimes 1)(x) = \pm c_{n,\ell} (\hat{s}_\ell^{\otimes n} \otimes 1)(x)$$

if  $V_{\text{odd}} = 0$ , and

$$(\bar{s}_\ell^{\otimes n} \otimes 1) \circ \beta_{\ell n+1} \circ (\bar{s}_\ell^{\otimes n} \otimes 1)(x) = \pm d_{n,\ell} (\bar{s}_\ell^{\otimes n} \otimes 1)(x)$$

if  $V_{\text{even}} = 0$ , independent of our choice of  $V$  and  $x \in V^{\otimes(n\ell+1)}$ .

*Proof.* Let  $V$  be a free graded  $R$ -module such that  $\ell = \dim V > 1$  and  $V_{\text{odd}} = 0$ , and let  $g$  denote the self-map  $(\hat{s}_\ell^{\otimes n} \otimes 1) : V^{\otimes(n\ell+1)} \rightarrow V^{\otimes(n\ell+1)}$ . Take a basis  $\{v_1, \dots, v_\ell\}$  of  $V$ , and let  $y = v_1 \otimes \dots \otimes v_\ell \in V^{\otimes \ell}$ , and  $z_i = y^{\otimes n} \otimes v_i \in V^{\otimes(n\ell+1)}$ . Observe that  $V_{\text{odd}} = 0$  and  $\ell = \dim V$  implies that for every  $x \in V^{\otimes(n\ell+1)}$ , we can write  $g(x)$  as a linear combination of the elements  $g(z_i)$  for each  $i$ , and in turn each  $g(z_i)$  is a linear combination of tensors  $\sigma_1(y) \otimes \dots \otimes \sigma_n(y) \otimes v_i$  for all choices of  $\sigma_1, \dots, \sigma_n \in S_n$ .

Let  $\gamma \in R[S_{n\ell+1}]$  be any element. Since the factor  $v_1$  occurs in  $y$ ,

$$(1) \quad g \circ \gamma(y^{\otimes n} \otimes v_1) = \pm c_\gamma g(y^{\otimes n} \otimes v_1)$$

for some integer  $c_\gamma \geq 0$ . If we let  $y_j \in V^{\otimes \ell}$  be the permutation of the tensor  $y$  such that the first and  $j^{\text{th}}$  factors are interchanged, it is clear that  $g \circ \gamma(y_j^{\otimes n} \otimes v_j) = \pm c_\gamma g(y_j^{\otimes n} \otimes v_j)$  for each  $j$ , as this is the same as replacing  $v_1$  with  $v_j$  and  $v_j$  with  $v_1$  in equation (1), and both  $v_1$  and  $v_j$  have either even degree or odd degree. Then since  $\hat{s}_n(y) = \pm \hat{s}_n(y_j)$  and  $\hat{s}_n(\sigma(y)) = \text{sgn}(\sigma) \hat{s}_n(y)$  for any  $\sigma \in S_n$ ,  $g \circ \gamma(\sigma_1(y) \otimes \dots \otimes \sigma_n(y) \otimes v_j) = \pm c_\gamma g \circ \gamma(\sigma_1(y) \otimes \dots \otimes \sigma_n(y) \otimes v_j)$  for each  $j$  and  $\sigma_1, \dots, \sigma_n \in S_n$ . Since  $g(x)$  takes the form of a linear combination as stated above, thus

$$g \circ \gamma \circ g(x) = \pm c_\gamma g(x)$$

for any  $x \in V^{\otimes(n\ell+1)}$ .

Next, let  $W$  be any free grade  $R$ -module such that  $\dim W = \dim V$  and  $W_{\text{odd}} = 0$ . If  $\{\omega_1, \dots, \omega_\ell\}$  is a basis of  $W$ , there is an isomorphism  $V \xrightarrow{\theta} W$  of ungraded  $R$ -modules defined by sending  $v_i$  to  $\omega_i$ . Since both  $W_{\text{odd}} = 0$  and  $V_{\text{odd}} = 0$ , the isomorphism  $V^{\otimes(n\ell+1)} \xrightarrow{\theta^{\otimes(n\ell+1)}} W^{\otimes(n\ell+1)}$  of ungraded  $R$ -modules is equivariant with respect to the (graded) action of  $R[S_{n\ell+1}]$ . Thus  $c_\gamma$  is independent of  $V$ . We finish by setting  $\gamma = \beta_{n\ell+1}$ , and  $c_{n,\ell} = c_\gamma$ . The proof for the  $V_{\text{even}} = 0$  case is identical.  $\square$

The values for several instances of the integers  $c_{n,\ell}$  and  $d_{n,\ell}$  are given in the following theorem. The next two sections are geared towards a proof of part (i) and (ii), while part (iii) follows with the aid of a computer. None-the-less,  $n$  and  $\ell$  need to be kept small in part (iii) due to the computational complexity of calculating  $c_{n,\ell}$  and  $d_{n,\ell}$ .

**Theorem 2.2.**

- (i)  $c_{1,\ell} = d_{1,\ell} = (\ell + 1)((\ell - 1)!)$  for  $\ell > 1$ ;
- (ii)  $c_{n,2} = d_{n,2} = 3^n$ ;
- (iii)  $c_{2,3} = 64, d_{2,3} = 32; c_{3,3} = 512, d_{3,3} = 64; c_{2,4} = 420, d_{2,4} = 900$ .

**Remark 2.3.** *Observing the pattern in Theorem (2.2), one would guess that in general at least one of  $c_{n,\ell}$  or  $d_{n,\ell}$  is equal to  $(\ell + 1)^n((\ell - 1)!)^n$ .*

### 3. CALCULATING $c_{1,\ell}$ AND $d_{1,\ell}$ FOR $\ell > 2$

In order to show  $c_{1,\ell} = (\ell + 1)((\ell - 1)!)$ , it suffices to show  $(\hat{s}_\ell \otimes 1) \circ \beta_{\ell+1} \circ (\hat{s}_\ell \otimes 1)(x) = \pm(\ell + 1)((\ell - 1)!)(\hat{s}_\ell \otimes 1)(x)$  for any particular choice of graded free  $R$ -module  $V$  satisfying  $\ell = \dim V > 1$  and  $V_{\text{odd}} = 0$ , and any particular choice of  $x \in V$  satisfying  $(\hat{s} \otimes 1)(x) \neq 0$ . We will work in a more general context of by showing that this equation holds for any graded  $R$ -module  $V$ , so long as  $x$  is of the form  $x_{k-1} \otimes \dots \otimes x_1 \otimes x_i$  for some  $1 \leq i \leq k - 1$ , and where each  $x_i$  is a homogeneous element in  $V$  of even degree. This is done in Proposition (3.5), together with an analogous calculation for  $d_{1,\ell}$ .

The following elements in  $R[S_k]$  will be of use:

$$\hat{t}_{j,k} = \sum_{\sigma \in T_{j,k}} \text{sgn}(\sigma)\sigma,$$

and

$$\bar{t}_{j,k} = \sum_{\sigma \in T_{j,k}} \sigma,$$

where  $T_{j,k} \subseteq S_k$  is the subset consisting of the  $k$  elements in  $S_k$  that switch the position of the  $j^{\text{th}}$  letter in the identity element  $(12 \dots k)$ . Given this, these elements satisfy the equations

$$\hat{s}_k = \hat{t}_{1,k} \circ (1 \otimes \hat{s}_{k-1}) = \hat{t}_{k,k} \circ (\hat{s}_{k-1} \otimes 1),$$

and

$$\bar{s}_k = \bar{t}_{1,k} \circ (1 \otimes \bar{s}_{k-1}) = \bar{t}_{k,k} \circ (\bar{s}_{k-1} \otimes 1).$$

We begin with a few lemmas.

**Lemma 3.1.** *Let  $k > 2$ , and  $V$  be any graded  $R$ -module. Then for every  $y \in V^{\otimes k}$*

$$\hat{s}_k \circ \beta_k(y) = 0,$$

and

$$\bar{s}_k \circ \beta_k(y) = 0.$$

*Proof.* It is sufficient to show the lemma holds for homogeneous elements in  $V^{\otimes k}$ . We proceed by induction. By inspection we have  $\hat{s}_3 \otimes \beta_3(x) = 0$  for any  $x \in V^{\otimes 3}$ . Let us assume  $\hat{s}_{k-1} \circ \beta_{k-1}(x) = 0$  for every  $x \in V^{\otimes(k-1)}$ . Take any homogeneous element  $x_k \otimes \cdots \otimes x_1 \in V^{\otimes k}$  and let  $c$  denote  $|x_{k-1} \otimes \cdots \otimes x_1|$ . Then

$$\begin{aligned} & \hat{s}_k \circ \beta_k(x_k \otimes \cdots \otimes x_1) \\ &= \hat{s}_k(x_k \otimes \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1)) - (-1)^{c|x_k|} \hat{s}_k(\beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k) \\ &= t_{1,k}(1 \otimes \hat{s}_{k-1})(x_k \otimes \beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1)) \\ &\quad - (-1)^{c|x_k|} t_{k,k}(\hat{s}_{k-1} \otimes 1)(\beta_{k-1}(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k) \\ &= t_{1,k}(x_k \otimes (\hat{s}_{k-1} \circ \beta_{k-1})(x_{k-1} \otimes \cdots \otimes x_1)) \\ &\quad - (-1)^{c|x_k|} t_{k,k}((\hat{s}_{k-1} \circ \beta_{k-1})(x_{k-1} \otimes \cdots \otimes x_1) \otimes x_k) \\ &= t_{1,k}(0) - (-1)^{c|x_k|} t_{k,k}(0) \\ &= 0. \end{aligned}$$

The proof for  $\bar{s}_k$  is identical. □

**Lemma 3.2.** *Let  $k > 2$ , and  $V$  be any graded  $R$ -module. Then for every  $y \in V^{\otimes k}$  and each  $3 \leq j \leq k$ ,*

$$\hat{s}_k \circ (1^{\otimes(k-j)} \otimes \beta_j)(y) = 0,$$

and

$$\bar{s}_k \circ (1^{\otimes(k-j)} \otimes \beta_j)(y) = 0.$$

*Proof.* It is sufficient to prove that the statement holds for homogeneous elements in  $V^{\otimes k}$ . For each  $m, j$  such that  $1 < j \leq m$  and  $m < k$ , and every  $x \in V^{\otimes m}$ , assume

$$\hat{s}_m \circ (1^{\otimes(m-j-1)} \otimes \beta_{j+1})(x) = 0$$

holds. We must show it also holds for  $m = k$ , and each  $1 < j < k$ . For our second inductive assumption let us assume

$$\hat{s}_k \circ (1^{\otimes(k-j-1)} \otimes \beta_{j+1})(y) = 0$$

holds for every  $y \in V^{\otimes k}$  and some  $j$ . The base case  $j = k$  follows by Lemma 3.1. Note that by our first inductive assumption we have

$$\begin{aligned}
 & \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1) \otimes x_{j+1}) \\
 &= t_{k,k}(\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1)) \otimes x_{j+1}) \\
 &= t_{k,k}(0x_{j+1}) \\
 &= 0.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 0 &= \hat{s}_k \circ (1^{k-j-1} \otimes \beta_{j+1})(x_k \otimes \cdots \otimes x_1) \\
 &= \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_{j+1}(x_{j+1} \otimes \cdots \otimes x_1)) \\
 &= \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1)) \\
 &\quad - (-1)^{c|x_{j+1}|} \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes \beta_j(x_j \otimes \cdots \otimes x_1) \otimes x_{j+1}) \\
 &= \hat{s}_k(x_k \otimes \cdots \otimes x_{j+2} \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1)),
 \end{aligned}$$

which completes the induction step. The proof for  $\bar{s}_k$  is identical.  $\square$

**Lemma 3.3.** *Let  $k > 2$ , and  $V$  be any graded  $R$ -module. Take any homogeneous element  $y = x_k \otimes \cdots \otimes x_1 \in V^{\otimes k}$ . If  $|x_i|$  is even for each  $i$ , then*

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(y) = (\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes ((123) - (132) - 2(231)))(y),$$

and if  $|x_i|$  is odd for each  $i$ ,

$$(\bar{s}_{k-1} \otimes 1) \circ \beta_k(y) = (\bar{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes ((123) + (132) - 2(231)))(y).$$

*Proof.* Let  $c$  denote  $|x_{k-1} \otimes \cdots \otimes x_1|$ . Assume  $|x_i|$  is even for each  $i$ . Using Lemma 3.2

$$\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1)) = 0$$

for each  $3 < j \leq k$ , so

$$\begin{aligned}
 & (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1)) \\
 &= (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes x_j \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1)) \\
 &\quad - (-1)^{c|x_k|} (\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1))) \otimes x_j \\
 &= (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes x_j \otimes \beta_{j-1}(x_{j-1} \otimes \cdots \otimes x_1)).
 \end{aligned}$$

By induction

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1) = (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_{j+1} \otimes \beta_j(x_j \otimes \cdots \otimes x_1))$$

for each  $3 \leq j \leq k$ . In particular when  $j = 3$ , the fact that

$$\beta_3(x_3 \otimes x_2 \otimes x_1) = ((123) - (132) - (231) + (321))(x_3 \otimes x_2 \otimes x_1)$$

implies

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1) = (\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes ((123) - (132) - (231) + (321)))(x_k \otimes \cdots \otimes x_1)$$

Since (321) is an odd permutation of (231) leaving the last letter fixed

$$(\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes (321))(x_k \otimes \cdots \otimes x_1) = -(\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes (231))(x_k \otimes \cdots \otimes x_1).$$

Hence

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1) = (\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes ((123) - (132) - 2(231)))(x_k \otimes \cdots \otimes x_1).$$

The proof of the second case, when the degrees of each  $x_i$  are odd, is similar. □

As an immediate consequence of Lemma 3.3, we obtain the following lemma.

**Lemma 3.4.** *Take any homogeneous element  $x_k \otimes \cdots \otimes x_1 \in V^{\otimes k}$ . If  $|x_i|$  is even for each  $i$ ,*

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1) = \begin{cases} (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_i \text{ for some } i > 3 \\ 0, & \text{if } x_1 = x_2 \\ 3(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_3 \end{cases}$$

Similarly, if  $|x_i|$  is odd for each  $i$ ,

$$(\bar{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1) = \begin{cases} (\bar{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_i \text{ for some } i > 3 \\ 0, & \text{if } x_1 = x_2 \\ 3(\bar{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1), & \text{if } x_1 = x_3 \end{cases}$$

*Proof.* Suppose each  $x_i$  is of even degree, and  $x_1 = x_3$ . By Lemma 3.3

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1) = (\hat{s}_{k-1} \otimes 1) \circ (1^{\otimes(k-3)} \otimes ((123) - (132) - 2(231)))(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1)$$

which in turn is equal to

$$(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1) - (\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_1 \otimes x_2) - 2(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_2 \otimes x_1 \otimes x_1).$$

Notice that

$$(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_1 \otimes x_2) = 0,$$

and since  $\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_2 \otimes x_1) = -\hat{s}_{k-1}(x_k \otimes \cdots \otimes x_1 \otimes x_2)$

$$(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_2 \otimes x_1 \otimes x_1) = -(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1).$$

Therefore we have

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1) = 3(\hat{s}_{k-1} \otimes 1)(x_k \otimes \cdots \otimes x_1 \otimes x_2 \otimes x_1).$$

The proof for the rest of the cases is similar. □

Part (i) of Theorem (2.2) follows from the following proposition.

**Proposition 3.5.** *Take any homogeneous element  $y = x_{k-1} \otimes \cdots \otimes x_1 \in V^{\otimes k-1}$ , for  $V$  a graded  $R$ -module. If  $|x_i|$  is even for each  $i$ , then for each  $1 \leq j \leq k-1$*

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\hat{s}_{k-1} \otimes 1)(y \otimes x_j) = k((k-2)!)(\hat{s}_{k-1} \otimes 1)(y \otimes x_j),$$

and if  $|x_i|$  is odd for each  $i$ ,

$$(\bar{s}_{k-1} \otimes 1) \circ \beta_k(\bar{s}_{k-1} \otimes 1)(y \otimes x_j) = k((k-2)!)(\bar{s}_{k-1} \otimes 1)(y \otimes x_j).$$

*Proof.* For each  $1 \leq j, n \leq k-1$ , let  $S_{k-1}^{j,n} \subseteq S_{k-1}$  be the subset of all permutations  $\sigma \in S_{k-1}$  satisfying  $\sigma(j) = n$ . Notice that

$$|S_{k-1}^{j,n}| = (k-2)!,$$

and for  $m \neq n$

$$|S_{k-1}^{j,n} \cap S_{k-1}^{j,m}| = 0,$$

and

$$|S_{k-1} - (S_{k-1}^{j,n} \cup S_{k-1}^{j,m})| = (k-1)! - 2(k-2)!.$$

Let  $A_j$  denote the set  $S_{k-1} - (S_{k-1}^{j,1} \cup S_{k-1}^{j,2})$ . For each  $1 \leq j \leq k-1$  we can write  $\hat{s}_k \in S_k$  as

$$\begin{aligned} \hat{s}_k &= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma \\ &= \sum_{\sigma \in A_j} \text{sgn}(\sigma) \sigma + \sum_{\sigma \in S_{k-1}^{j,1}} \text{sgn}(\sigma) \sigma + \sum_{\sigma \in S_{k-1}^{j,2}} \text{sgn}(\sigma) \sigma. \end{aligned}$$

Notice that for every  $\sigma \in S_{k-1}$  we have

$$\hat{s}_{k-1} \sigma = \text{sgn}(\sigma) \hat{s}_{k-1}.$$

So for each  $\sigma \in S_{k-1}^{j,2}$ , Lemma 3.2 implies

$$\begin{aligned} (\hat{s}_{k-1} \otimes 1) \circ \beta_k(\sigma(y) \otimes x_j) &= 3(\hat{s}_{k-1} \otimes 1)(\sigma(y) \otimes x_j) \\ &= 3\hat{s}_{k-1}(\sigma(y)) \otimes x_j \\ &= 3\text{sgn}(\sigma)(\hat{s}_{k-1}(y) \otimes x_j). \end{aligned}$$

Likewise Lemma 3.2 implies that for every  $\sigma \in A_j$ ,

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\sigma(y) \otimes x_j) = \text{sgn}(\sigma)(\hat{s}_{k-1}(y) \otimes x_j)$$

and for every  $\sigma \in S_{k-1}^{j,1}$  we have

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\sigma(y) \otimes x_j) = 0.$$

Hence

$$\begin{aligned} (\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_{k-1}^{j,2}} \text{sgn}(\sigma) \sigma(y) \right) \otimes x_j &= \sum_{\sigma \in S_{k-1}^{j,2}} (\hat{s}_{k-1} \otimes 1) \circ \beta_k(\text{sgn}(\sigma) \sigma(y) \otimes x_j) \\ &= \sum_{\sigma \in S_{k-1}^{j,2}} (3 \text{sgn}(\sigma)^2 \hat{s}_{k-1}(y) \otimes x_j) \\ &= 3(k-2)! (\hat{s}_{k-1}(y) \otimes x_j). \end{aligned}$$

Similarly

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in A_j} \text{sgn}(\sigma) \sigma(y) \right) \otimes x_j = ((k-1)! - 2(k-2)!) (\hat{s}_{k-1}(y) \otimes x_j),$$

and

$$(\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_{k-1}^{j,1}} \text{sgn}(\sigma) \sigma(y) \right) \otimes x_j = 0.$$

Putting these facts together,

$$\begin{aligned} &(\hat{s}_{k-1} \otimes 1) \circ \beta_k(\hat{s}_{k-1} \otimes 1)(y \otimes x_j) \\ &= (\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sigma(y) \right) \otimes x_j \\ &= (\hat{s}_{k-1} \otimes 1) \circ \beta_k \left( \sum_{\sigma \in A_j} \text{sgn}(\sigma) \sigma(y) + \sum_{\sigma \in S_{k-1}^{j,1}} \text{sgn}(\sigma) \sigma(y) + \sum_{\sigma \in S_{k-1}^{j,2}} \text{sgn}(\sigma) \sigma(y) \right) \otimes x_j \\ &= ((k-1)! - 2(k-2)!) (\hat{s}_{k-1}(y) \otimes x_j) + 0 + 3(k-2)! (\hat{s}_{k-1}(y) \otimes x_j) \\ &= k(k-2)! (\hat{s}_{k-1} \otimes 1)(y \otimes x_j). \end{aligned}$$

The proof for second part of the proposition is similar.  $\square$

#### 4. CALCULATING $c_{n,2}$ AND $d_{n,2}$ FOR $n \geq 1$

Like in the previous section we will work in the more general context of graded  $R$ -modules, and calculate  $c_{n,2}$  (and  $d_{n,2}$ ) by proving the equality  $(\hat{s}_2^{\otimes n} \otimes 1) \circ \beta_{2n+1} \circ (\hat{s}_2^{\otimes n} \otimes 1)(x) = \pm(3^n)(\hat{s}_2^{\otimes n} \otimes 1)(x)$  holds for certain homogeneous tensors  $x$  whose factors are of even degree (or odd degree for the calculation of  $d_{n,2}$ ). We begin working our way towards a proof of this starting with a few technical lemmas.

**Lemma 4.1.** *Let  $V$  be any graded  $R$ -module, and  $x_1, x_2 \in V$  any homogeneous elements. Let  $\sigma_1, \dots, \sigma_k \in S_2$  be any  $k > 1$  choices of the two elements in  $S_2 = \{(12), (21)\}$ , and take*

$$y = x_{\sigma_1(1)} \otimes (x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \in V^{\otimes 2k},$$

and

$$z = (x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \otimes x_{\sigma_1(1)} \in V^{\otimes 2k}.$$

(i) *Suppose  $|x_1|$  and  $|x_2|$  are both odd, and let  $n \geq 0$  be the number of times  $\sigma_i = \sigma_1$  for  $i > 1$ . Then*

$$\bar{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y) = (-1)^{k-1} (-2)^n (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

Furthermore,

$$\bar{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = -(\bar{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y)).$$

(ii) *Suppose  $|x_1|$  and  $|x_2|$  are both even. If  $\sigma_{2i} = \sigma_1$  for some  $i$ , then*

$$\hat{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y) = 0.$$

Otherwise, if  $m \geq 0$  is the number of times  $\sigma_{2i+1} = \sigma_1$  for  $i > 0$ ,

$$\hat{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y) = (-2)^m (3^{\lfloor \frac{k}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

Furthermore,

$$\hat{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k (\hat{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y)).$$

*Proof of part (i).* With our choice of  $y \in V^{\otimes 2k}$  defined in the statement of the lemma, it will be convenient to let  $y' \in V^{\otimes 2k-3}$  denote  $(x_{\sigma_{k-1}(1)} \otimes \cdots \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)})$  - that is, the tensor of the last  $2k-3$  factors of  $y$ . Also, let  $n'$  be the number of choices of  $i$  such that  $1 < i < k$  and  $\sigma_i = \sigma_1$ , and  $n$  be the number of choices of  $i$  such that  $1 < i \leq k$  and  $\sigma_i = \sigma_1$ .

Part (i) holds for  $k = 2$  by inspection. Assume it holds for some  $k-1 \geq 2$ . In particular, our inductive assumptions are

$$(2) \quad (\bar{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = -(\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')),$$

and

$$(3) \quad (\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = (-1)^{k-2} (-2)^{n'} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1}.$$

Noting  $x_1$  and  $x_2$  are of odd degree, one has the following equality:

$$\begin{aligned}
(4) \quad (1 \otimes \beta_{2k-1})(y) &= x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes \beta_{2k-2}(x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \\
&\quad - x_{\sigma_1(1)} \otimes \beta_{2k-2}(x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \otimes x_{\sigma_k(1)} \\
&= x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \\
&\quad + x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \\
&\quad - x_{\sigma_1(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \\
&\quad - x_{\sigma_1(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}.
\end{aligned}$$

Let us assume  $\sigma_1 \neq \sigma_k$ . Then  $x_{\sigma_k(2)} = x_{\sigma_1(1)}$ ,  $x_{\sigma_k(1)} = x_{\sigma_1(2)}$ . We replace  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(1)}$ , and  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(2)}$  in equation (4). Since  $\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(1)}) = 0$ , and using equations (4) and (2),

$$\begin{aligned}
&(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) \\
&= \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes (\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\
&\quad - \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes (\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\
&\quad - (\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}). \\
&= -(\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}).
\end{aligned}$$

Then by equation (3),

$$(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = -(-1)^{k-2}(-2)^{n'}(\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k},$$

and since our assumption is that  $\sigma_1 \neq \sigma_k$ , then  $n' = n$ , and so we are done.

Next, let us assume  $\sigma_1 = \sigma_k$ . Then  $x_{\sigma_k(1)} = x_{\sigma_1(1)}$ , and  $x_{\sigma_k(2)} = x_{\sigma_1(2)}$ . Thus we replace  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(1)}$  and  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(2)}$  in equation (4), and using the fact that  $\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(1)}) = 0$ , we obtain

$$\begin{aligned}
(5) \quad (\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) &= -\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes \bar{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) \\
&\quad - \bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes \bar{s}_2(x_{\sigma_1(2)} \otimes x_{\sigma_1(1)}) \\
&= \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes \bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\
&\quad - \bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes (-\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))
\end{aligned}$$

So by equation (3),

$$\begin{aligned}
&(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 2(-1)^{k-2}(-2)^{n'}(\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k} \\
&= (-1)^{k-1}(-2)^{n'+1}(\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.
\end{aligned}$$

Since  $\sigma_1 = \sigma_k$ , then  $n' + 1 = n$ , and we are done.

To complete the induction we have to show

$$\bar{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = -(\bar{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y)),$$

where  $z$  is as defined in the statement of the lemma. Having found  $(\bar{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y)$  above, this is the same as proving the equality

$$\bar{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k (-2)^n (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

Thus we follow a similar argument for  $z$  as we did for  $y$  above. First, since  $z = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes y' \otimes x_{\sigma_1(1)}$ ,

$$(6) \quad \begin{aligned} (\beta_{2k-1} \otimes 1)(z) &= x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_1(1)} \\ &\quad + x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_1(1)} \\ &\quad - x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \otimes x_{\sigma_1(1)} \\ &\quad - \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_1(1)}. \end{aligned}$$

As before, let us first assume  $\sigma_1 \neq \sigma_k$ . Then

$$\begin{aligned} (\bar{s}_2^{\otimes k}) \circ (\beta_{2k-1} \otimes 1)(y) &= (-\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \otimes \bar{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) \\ &\quad - (\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes (-\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})) \\ &\quad - (\bar{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) \otimes (-\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})). \end{aligned}$$

So by equations (2) and (3),

$$(\bar{s}_2^{\otimes k}) \circ (\beta_{2k-1} \otimes 1)(y) = (-1)^k (-2)^{n'} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k},$$

and since we are assuming  $\sigma_1 \neq \sigma_k$ , then  $n' = n$ , so we are done.

Next, assume  $\sigma_1 = \sigma_k$ . By equation (6)

$$\begin{aligned} (\bar{s}_2^{\otimes k}) \circ (\beta_{2k-1} \otimes 1)(y) &= \bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes \bar{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) \\ &\quad + \bar{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes (-\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})), \end{aligned}$$

and so using equation (3) we obtain

$$(\bar{s}_2^{\otimes k}) \circ (\beta_{2k-1} \otimes 1)(y) = (-1)^k (-2)^{n'+1} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

Since  $\sigma_1 = \sigma_k$ , then  $n' + 1 = n$ . This completes the induction for part (i).  $\square$

*Proof of part (ii).* We follow along a similar line as the proof of part (i), with  $y \in V^{\otimes 2k}$  and  $y' \in V^{\otimes 2k-3}$  defined as before (but this time with  $x_1$  and  $x_2$  having both even degree). Let  $m'$  be the number of choices of  $i$  such that  $1 < 2i + 1 < k$  and  $\sigma_{2i+1} = \sigma_1$ , and  $m$  be the number of choices of  $i$  such that  $1 < 2i + 1 \leq k$  and  $\sigma_{2i+1} = \sigma_1$ .

Part (ii) holds for  $k = 2$  by inspection. Let us assume part (ii) holds for some  $k - 1 \geq 2$ . In particular, our inductive assumptions are as follows: first,

$$(7) \quad (\hat{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = (-1)^{k-1}(\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')),$$

and whenever  $\sigma_1 = \sigma_{2i}$  for some  $i$  such that  $2i < k$ , we have

$$(8) \quad (\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = 0.$$

Otherwise,

$$(9) \quad (\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = (-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1}.$$

Next, observe the following equality:

$$(10) \quad \begin{aligned} (1 \otimes \beta_{2k-1})(y) &= x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes \beta_{2k-2}(x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \\ &\quad - x_{\sigma_1(1)} \otimes \beta_{2k-2}(x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)}) \otimes x_{\sigma_k(1)} \\ &= x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \\ &\quad - x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \\ &\quad - x_{\sigma_1(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \\ &\quad + x_{\sigma_1(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}. \end{aligned}$$

Let us assume  $\sigma_1 \neq \sigma_k$ . Then  $x_{\sigma_k(2)} = x_{\sigma_1(1)}$ ,  $x_{\sigma_k(1)} = x_{\sigma_1(2)}$ , and so we replace  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(1)}$ , and  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(2)}$  in equation (10). Since  $\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(1)}) = 0$ ,

$$\hat{s}_2^{\otimes k}(x_{\sigma_1(1)} \otimes x_{\sigma_1(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)}) = 0.$$

Therefore by equations (10) and (7) we have

$$(11) \quad \begin{aligned} &(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) \\ &= \hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes (\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\ &\quad + \hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes (\hat{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) \\ &\quad + \hat{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y') \otimes (-\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))). \\ &= (1 - (-1)^{k-1})\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes (\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\ &\quad + (\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes \hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}). \end{aligned}$$

Now suppose  $\sigma_{2i} = \sigma_1$  for some  $2i < k$ . Then  $(\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = 0$  by our inductive assumption, and so

$$(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 0$$

by equation (11). On the other hand, suppose  $\sigma_{2i} \neq \sigma_1$  for all  $i$ . Recall  $m'$  is the number of choices of  $i$  such that  $1 < 2i + 1 < k$  and  $\sigma_{2i+1} = \sigma_1$ , and  $m$  is the number of choices of  $i$  such that  $1 < 2i + 1 \leq k$  and  $\sigma_{2i+1} = \sigma_1$ . By equation (9)

$$\begin{aligned} & (\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) \\ &= (1 - (-1)^{k-1})(-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k} \\ &\quad + (-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k} \\ &= (2 + (-1)^k)(-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}. \end{aligned}$$

Since  $\sigma_1 \neq \sigma_k$ ,  $m' = m$ . So when  $k$  is odd,  $\lfloor \frac{k-1}{2} \rfloor = \lfloor \frac{k}{2} \rfloor$ , and  $(2 + (-1)^k)(-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor}) = (-2)^m(3^{\lfloor \frac{k}{2} \rfloor})$ . Likewise, when  $k$  is even,  $(2 + (-1)^k)(-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor}) = 3(-2)^m(3^{\lfloor \frac{k-1}{2} \rfloor}) = (-2)^m(3^{\lfloor \frac{k}{2} \rfloor})$ .

So in any case

$$(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = (-2)^m(3^{\lfloor \frac{k}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

This finishes the the case  $\sigma_1 \neq \sigma_k$ .

Let us assume  $\sigma_1 = \sigma_k$ . Then we have  $x_{\sigma_k(1)} = x_{\sigma_1(1)}$ , and  $x_{\sigma_k(2)} = x_{\sigma_1(2)}$ . So replacing  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(1)}$  and  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(2)}$  in equation (10), using equation (7), and the fact that  $\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(1)}) = 0$ , one obtains

$$\begin{aligned} & (\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = -(-1)^{k-1}\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}) \otimes \hat{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\ &\quad + \hat{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \otimes (-\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)})). \end{aligned}$$

Suppose  $\sigma_1 = \sigma_{2i}$  for some choice of  $i$  such that  $2i < k$ . Then  $\hat{s}_2^{\otimes k-1}(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = 0$  by equation (8), and so

$$(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 0$$

using equation (5).

Suppose  $\sigma_1 \neq \sigma_{2i}$  for all choices of  $i$  such that  $2i < k$ . Then by equations (9) and (5),

$$(12) \quad (\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = ((-1)^k - 1)(-2)^{m'}(3^{\lfloor \frac{k-1}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

Since we are assuming  $\sigma_1 = \sigma_k$ , by the statement of our lemma one would expect that  $(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = 0$  whenever  $k$  is even. By equation (12) this is indeed the case. On the other hand when  $k$  is odd, since  $\sigma_1 = \sigma_k$  we have  $m = m' + 1$ , and therefore by equation (12)

$$(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y) = (-2)^m(3^{\lfloor \frac{k}{2} \rfloor})(\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

This finishes the  $\sigma_1 = \sigma_k$  case.

To complete the induction we have to show

$$\hat{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k(\hat{s}_2^{\otimes k} \circ (1 \otimes \beta_{2k-1})(y))$$

(where  $z$  is defined in the statement of the lemma), as an equality of this form in equation (7) is assumed in our induction step. Having found  $(\hat{s}_2^{\otimes k}) \circ (1 \otimes \beta_{2k-1})(y)$  above, this is the same as proving the equality

$$\hat{s}_2^{\otimes k} \circ (\beta_{2k-1} \otimes 1)(z) = (-1)^k (-2)^m (3^{\lfloor \frac{k}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k}.$$

Here we have  $z = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes y' \otimes x_{\sigma_1(1)}$ , whereas before  $y = x_{\sigma_1(1)} \otimes x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes y'$ . None-the-less, an argument similar to the one above shows that this equality is correct.  $\square$

The following is a consequence of Lemma (4.1).

**Lemma 4.2.** *Let  $V$  be any graded  $R$ -module, and  $x_1, x_2 \in V$  any homogeneous elements. Let  $\sigma_1, \dots, \sigma_k \in S_2$  be any  $k > 1$  choices of the two elements in  $S_2 = \{(12), (21)\}$ , and let*

$$y = (x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \in V^{\otimes 2k-1}.$$

(i) *Suppose  $|x_1|$  and  $|x_2|$  are both odd, and let  $n \geq 0$  be the number of times  $\sigma_i = \sigma_1$  for  $i > 1$ . Then*

$$(\bar{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1}(y) = (-1)^{k-1} (-2)^n (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.$$

(ii) *Suppose  $|x_1|$  and  $|x_2|$  are both even. If  $\sigma_{2i} = \sigma_1$  for some  $i$ , then*

$$(\hat{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1}(y) = 0.$$

*Otherwise, if  $m \geq 0$  is the number of times  $\sigma_{2i+1} = \sigma_1$  for  $i > 0$ , then*

$$(\hat{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1}(y) = (-1)^{k-1} (-2)^m (3^{\lfloor \frac{k}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.$$

*Proof of part (i).* Part (i) holds for  $k = 2$  by inspection. For our inductive assumption, assume part (i) holds for some  $k - 1 > 2$ . Let  $y$  be a choice of tensor as defined in the statement of the lemma, and for convenience let  $y' = x_{\sigma_{k-1}(1)} \otimes x_{\sigma_{k-1}(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)} \otimes x_{\sigma_1(2)} \in V^{\otimes 2k-3}$ . Let  $n$  be the number of times  $\sigma_i = \sigma_1$  for  $1 < i \leq k$ , and  $n'$  be the number of times  $\sigma_i = \sigma_1$  for  $1 < i < k$ . Note the following equality:

$$\begin{aligned} (13) \quad (\beta_{2k-1})(y) &= x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \\ &\quad + x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \\ &\quad - x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \\ &\quad - \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}. \end{aligned}$$

Let us assume  $\sigma_k \neq \sigma_1$ , so we can replace  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(1)}$  and  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(2)}$  in equation (13). In this case, observe that the factor  $x_{\sigma_1(2)}$  occurs more often than the factor  $x_{\sigma_1(1)}$  in the tensor  $x_{\sigma_k(1)} \otimes y'$ , thus

$$\bar{s}_2^{\otimes k-1}(x_{\sigma_k(1)} \otimes \beta_{2k-3}(y')) = 0.$$

Also, if  $\sigma_1 \neq \sigma_{2i}$  for all  $1 < 2i < k$ , then by our inductive assumption

$$(\bar{s}_2^{\otimes k-2} \otimes 1) \circ \beta_{2k-3}(y') = (-1)^{k-2} (-2)^{n'} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-2} \otimes x_{\sigma_1(2)},$$

and by Lemma (4.1),

$$(\bar{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = -(\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')).$$

Therefore, by equation (13)

$$\begin{aligned} & (\bar{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) \\ &= (-1)^{k-2} (-2)^{n'} \bar{s}_2(x_{\sigma_1(2)} \otimes x_{\sigma_1(1)}) \otimes (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-2} \otimes x_{\sigma_1(2)} \\ &= -(-1)^{k-2} (-2)^{n'} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}. \end{aligned}$$

Since  $\sigma_k \neq \sigma_1$ ,  $n = n'$ , and we are done.

Next we assume  $\sigma_k = \sigma_1$ . Replacing  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(1)}$  and  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(2)}$  in equation (13), the factor  $x_{\sigma_1(2)}$  occurs more often than the factor  $x_{\sigma_1(1)}$  in the tensors  $x_{\sigma_k(2)} \otimes y'$  and  $y' \otimes x_{\sigma_k(2)}$ , implying  $\bar{s}_2^{\otimes k-1}(x_{\sigma_k(2)} \otimes \beta_{2k-3}(y')) = 0$ , and  $\bar{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_k(2)}) = 0$ . Also, by Lemma (4.1)

$$(\bar{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) = (-1)^{k-2} (-2)^{n'} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1}.$$

Following along a similar line as the previous case,

$$(\bar{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) = 2(-1)^{k-2} (-2)^{n'} (\bar{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.$$

Because we are assuming  $\sigma_k \neq \sigma_1$ , then  $n = n' + 1$ , and so  $2(-1)^{k-2} (-2)^{n'} = (-1)^{k-1} (-2)^n$ . This completes the induction.  $\square$

*Proof of part (ii).* The structure of the proof is similar to that of part (i). Here, part (ii) holds for  $k = 2$  by inspection, and our inductive assumption is that it holds for some  $k - 1 > 2$ . Let  $y$  and  $y'$  be as in the proof of part (i), and let  $m$  be the number of times  $\sigma_{2i+1} = \sigma_1$  for  $0 < 2i + 1 \leq k$ , and  $m'$  be the number of times  $\sigma_{2i+1} = \sigma_1$  for  $0 < 2i + 1 < k$ . Note the following equality:

$$\begin{aligned} (14) \quad & (\beta_{2k-1})(y) = x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \\ & \quad - x_{\sigma_k(1)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \\ & \quad - x_{\sigma_k(2)} \otimes \beta_{2k-3}(y') \otimes x_{\sigma_k(1)} \\ & \quad + \beta_{2k-3}(y') \otimes x_{\sigma_k(2)} \otimes x_{\sigma_k(1)}. \end{aligned}$$

Let us assume  $\sigma_k \neq \sigma_1$ , so we replace  $x_{\sigma_k(2)}$  with  $x_{\sigma_1(1)}$  and  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(2)}$  in equation (14). In this case, observe that the factor  $x_{\sigma_1(2)}$  occurs more often than the factor  $x_{\sigma_1(1)}$  in the tensor  $x_{\sigma_k(1)} \otimes y'$ , thus

$$\hat{s}_2^{\otimes k-1}(x_{\sigma_k(1)} \otimes \beta_{2k-3}(y')) = 0.$$

Also, if  $\sigma_1 \neq \sigma_{2i}$  for all  $1 < 2i < k$ , then by our inductive assumption

$$(15) \quad (\hat{s}_2^{\otimes k-2} \otimes 1) \circ \beta_{2k-3}(y') = (-1)^{k-2} (-2)^{m'} (3^{\lfloor \frac{k-1}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-2} \otimes x_{\sigma_1(2)},$$

and by Lemma (4.1),

$$(16) \quad (\hat{s}_2^{\otimes k-1})(\beta_{2k-3}(y') \otimes x_{\sigma_1(1)}) = (-1)^{k-1} (\hat{s}_2^{\otimes k-1})(x_{\sigma_1(1)} \otimes \beta_{2k-3}(y')) \\ = (-1)^{k-1} (-2)^{m'} (3^{\lfloor \frac{k-1}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1}.$$

Therefore

$$(\hat{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) = (2(-1)^{k-1} - 1) (-2)^{m'} (3^{\lfloor \frac{k-1}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.$$

When  $k$  is even,  $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor + 1$  and  $(2(-1)^{k-1} - 1) = -3 = (-1)^{k-1} 3$ ; when  $k$  is odd,  $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor$  and  $(2(-1)^{k-1} - 1) = 1 = (-1)^{k-1}$ . Since  $\sigma_k \neq \sigma_1$ ,  $m = m'$ , and we are done. If  $\sigma_1 = \sigma_{2i}$  for some  $1 < 2i < k$ , both equations (15) and (16) are zero by our inductive assumption and Lemma (4.1), and so  $(\hat{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) = 0$ .

Finally, let us assume  $\sigma_k = \sigma_1$ . Replacing  $x_{\sigma_k(1)}$  with  $x_{\sigma_1(1)}$  and  $x_{\sigma_2(1)}$  with  $x_{\sigma_1(2)}$  in equation (14), notice that the factor  $x_{\sigma_1(2)}$  occurs more often than the factor  $x_{\sigma_1(1)}$  in the tensors  $x_{\sigma_k(2)} \otimes y'$  and  $y' \otimes x_{\sigma_k(2)}$ , so  $\hat{s}_2^{\otimes k-1}(x_{\sigma_k(2)} \otimes \beta_{2k-3}(y')) = 0$ , and  $\hat{s}_2^{\otimes k-1}(\beta_{2k-3}(y') \otimes x_{\sigma_k(2)}) = 0$ . The rest being similar as before, we have  $(\hat{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) = 0$  when  $\sigma_1 = \sigma_{2i}$  for some  $1 < 2i < k$ , and

$$(\hat{s}_2^{\otimes k-1} \otimes 1) \circ (\beta_{2k-1})(y) = ((-1)^{k-2} - 1) (-2)^{m'} (3^{\lfloor \frac{k-1}{2} \rfloor}) (\hat{s}_2(x_{\sigma_1(1)} \otimes x_{\sigma_1(2)}))^{\otimes k-1} \otimes x_{\sigma_1(2)}.$$

whenever  $\sigma_1 \neq \sigma_{2i}$  for all  $1 < 2i < k$ . When  $k$  is even, by the statement of the lemma one would expect this equation to be zero, as we are assuming  $\sigma_k \neq \sigma_1$ . This is in fact the case since  $((-1)^{k-2} - 1) = 0$  whenever  $k$  is even. On the other hand, when  $k$  is odd we have  $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{k-1}{2} \rfloor$  and  $((-1)^{k-2} - 1) = -2 = (-1)^{k-1} (-2)$ . Also, because  $\sigma_k = \sigma_1$ , then  $m = m' + 1$ . This completes the induction.  $\square$

Part (ii) of Theorem (2.2) follows from the following proposition.

**Proposition 4.3.** *Let  $V$  be any graded  $R$ -module, and  $x_1, x_2 \in V$  any homogeneous elements. Let  $\sigma_1, \dots, \sigma_k \in S_2$  be any  $k > 1$  choices of the two elements in  $S_2 = \{(12), (21)\}$ , and let*

$$y = (x_{\sigma_k(1)} \otimes x_{\sigma_k(2)} \otimes \cdots \otimes x_{\sigma_2(1)} \otimes x_{\sigma_2(2)}) \otimes x_{\sigma_1(2)} \in V^{\otimes 2k-1}.$$

(i) *If  $|x_1|$  and  $|x_2|$  are both odd, then*

$$(\bar{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} \circ (\bar{s}_2^{\otimes k-1} \otimes 1)(y) = \pm 3^{k-1} (\bar{s}_2^{\otimes k-1} \otimes 1)(y).$$

(ii) *If  $|x_1|$  and  $|x_2|$  are both even, then*

$$(\hat{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} \circ (\hat{s}_2^{\otimes k-1} \otimes 1)(y) = \pm 3^{k-1} (\hat{s}_2^{\otimes k-1} \otimes 1)(y).$$

*Proof.* Without loss of generality assume  $\sigma_1 = (21)$  (so  $x_{\sigma_1(1)} = x_2$  and  $x_{\sigma_1(2)} = x_1$ ), and take the tensor

$$x = (x_1 \otimes x_2)^{\otimes k-1} \otimes x_1 \in V^{\otimes 2k-1}.$$

Notice that  $(\bar{s}_2^{\otimes k-1} \otimes 1)(y) = (\text{sgn}(\sigma_2) \cdots \text{sgn}(\sigma_k))(\bar{s}_2^{\otimes k-1} \otimes 1)(x)$  when  $x_1$  and  $x_2$  have odd degree, and  $(\hat{s}_2^{\otimes k-1} \otimes 1)(y) = (\text{sgn}(\sigma_2) \cdots \text{sgn}(\sigma_k))(\hat{s}_2^{\otimes k-1} \otimes 1)(x)$  when  $x_1$  and  $x_2$  have even degree. Since  $(\text{sgn}(\sigma_2) \cdots \text{sgn}(\sigma_k)) = \pm 1$ , it is sufficient we prove that the proposition holds for  $x$  in place of  $y$ . In this case, for any collection  $\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2$  we shall write  $((x_{\bar{\sigma}_{k-1}(1)} \otimes x_{\bar{\sigma}_{k-1}(2)}) \otimes \cdots \otimes (x_{\bar{\sigma}_1(1)} \otimes x_{\bar{\sigma}_1(2)})) \otimes x_1$  as  $(\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x)$  for short.

Assume  $x_1$  and  $x_2$  have odd degree. Then

$$(17) \quad \begin{aligned} & (\bar{s}_2^{\otimes k-1} \otimes 1)(x) \\ &= \sum_{\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2} (\text{sgn}(\bar{\sigma}_1) \cdots \text{sgn}(\bar{\sigma}_{k-1})) (\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x) \\ &= \sum_{n=0}^{k-1} \left( \sum_{\substack{\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2, \\ \bar{\sigma}_i = (21) \text{ for } n \text{ choices of } i}} (-1)^n (\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x) \right). \end{aligned}$$

Also, by part (i) of Lemma (4.2),

$$\begin{aligned} & (\bar{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} ((\bar{\sigma}_{k-1} \otimes \cdots \otimes \bar{\sigma}_1 \otimes 1)(x)) \\ &= (-1)^{k-1} (-2)^n (\bar{s}_2(x_1 \otimes x_2))^{\otimes k-1} \otimes x_1 \\ &= (-1)^{k-1} (-2)^n (\bar{s}_2^{\otimes k-1} \otimes 1)(x) \end{aligned}$$

whenever  $\bar{\sigma}_i = (21)$  for exactly  $n$  choices of  $i$ . Since each  $\bar{\sigma}_i$  can either be (12) or (21), there are  $\binom{k-1}{n}$  choices of  $\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2$  with the property that  $\bar{\sigma}_i = (21)$  for exactly  $n$  choices of  $i$ . Therefore by equation (17)

$$\begin{aligned} & (\bar{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} \circ (\bar{s}_2^{\otimes k-1} \otimes 1)(x) \\ &= (-1)^{k-1} \sum_{n=0}^{k-1} \left( \sum_{\substack{\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2, \\ \bar{\sigma}_i = (21) \text{ for } n \text{ choices of } i}} (2^n) (\bar{s}_2^{\otimes k-1} \otimes 1)(x) \right) \\ &= (-1)^{k-1} \left( \sum_{n=0}^{k-1} \binom{k-1}{n} (2^n) \right) (\bar{s}_2^{\otimes k-1} \otimes 1)(x) \\ &= (-1)^{k-1} 3^{k-1} (\bar{s}_2^{\otimes k-1} \otimes 1)(x), \end{aligned}$$

where the last equality follows by the binomial formula.

On the other hand, assume  $x_1$  and  $x_2$  have even degree. Then

$$(\hat{s}_2^{\otimes k-1} \otimes 1)(x) = \sum_{m=0}^{k-1} \left( \sum_{\substack{\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2, \\ \bar{\sigma}_i = (21) \text{ for } m \text{ choices of } i}} (-1)^m (\bar{\sigma}_{k-1} \otimes \dots \otimes \bar{\sigma}_1 \otimes 1)(x) \right).$$

By part (ii) of Lemma (4.2), if  $\bar{\sigma}_{2i} = (21)$  for some  $i$ ,  $(\hat{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1}((\bar{\sigma}_{k-1} \otimes \dots \otimes \bar{\sigma}_1 \otimes 1)(x)) = 0$ .

Otherwise if  $\bar{\sigma}_{2i+1} = (21)$  for exactly  $m$  choices of  $i$ , then we have

$$\begin{aligned} & (\hat{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1}((\bar{\sigma}_{k-1} \otimes \dots \otimes \bar{\sigma}_1 \otimes 1)(x)) \\ &= (-1)^{k-1} (-2)^m (3^{\lfloor \frac{k}{2} \rfloor}) (\hat{s}_2^{\otimes k-1} \otimes 1)(x). \end{aligned}$$

So because there are  $\binom{\lfloor \frac{k-1}{m} \rfloor}{m}$  choices of  $\bar{\sigma}_1, \dots, \bar{\sigma}_{k-1} \in S_2$  with the property that  $\bar{\sigma}_{2i+1} = (21)$  for exactly  $m$  choices of  $i$ , and  $\bar{\sigma}_{2i} \neq (21)$  for each  $i$ , then

$$\begin{aligned} & (\hat{s}_2^{\otimes k-1} \otimes 1) \circ \beta_{2k-1} \circ (\hat{s}_2^{\otimes k-1} \otimes 1)(x) \\ &= (-1)^{k-1} (3^{\lfloor \frac{k}{2} \rfloor}) \left( \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{\lfloor \frac{k-1}{2} \rfloor}{m} (2^m) \right) (\hat{s}_2^{\otimes k-1} \otimes 1)(x) \\ &= (-1)^{k-1} (3^{\lfloor \frac{k}{2} \rfloor}) (3^{\lfloor \frac{k-1}{2} \rfloor}) (\hat{s}_2^{\otimes k-1} \otimes 1)(x) \\ &= (-1)^{k-1} (3^{k-1}) (\hat{s}_2^{\otimes k-1} \otimes 1)(x), \end{aligned}$$

where again the last equalities follow using the binomial formula.  $\square$

## 5. GEOMETRICALLY REALIZING THEOREM (2.1)

Considerable work has been done on the applications of representation theory to algebraic topology, in particular the geometric realization of certain spaces and maps using the representation theory of the symmetric group (see references [11, 13] for example). We begin by going over a few of the basic concepts using  $\mathbb{Z}_p$  as our ground ring.

Let  $X$  to be a  $p$ -local suspended  $CW$ -complex, and let us denote the reduced  $\mathbb{Z}_p$ -homology of  $X$  with  $V$ . The  $k$ -fold self-smash of  $X$  is written as  $X^{(k)}$ , and its reduced mod- $p$  homology given by  $\tilde{H}_*(X^{(k)}) \cong V^{\otimes k}$ . Recall from section (2) the action of  $\mathbb{Z}_p[S_k]$  on  $V^{\otimes k}$  induces a self-map  $V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}$  for each  $\sigma \in \mathbb{Z}_p[S_k]$ . Since  $X^{(k)}$  is a suspension whenever  $X$  is, by permuting factors of the smash product and taking  $co$ - $H$ -space sums, for each  $\sigma \in \mathbb{Z}_p[S_k]$  one can construct a self-map  $X^{(k)} \xrightarrow{f_\sigma} X^{(k)}$  with property that  $(f_\sigma)_*$  induces  $V^{\otimes k} \xrightarrow{\sigma} V^{\otimes k}$  on mod- $p$  homology. In this case we will denote the mapping telescope

$$X^{(k)} \xrightarrow{f_\sigma} X^{(k)} \xrightarrow{f_\sigma} \dots$$

by  $T(f_\sigma)$ . If  $\sigma \in \mathbb{Z}_p[S_k]$  happens to be an idempotent, we have an isomorphism

$$\tilde{H}_*(T(f_\sigma)) \simeq Im(\sigma : V^{\otimes k} \longrightarrow V^{\otimes k}),$$

and one obtains a splitting  $X^{(k)} \simeq T(f_\sigma) \vee T(f_{1-\sigma})$ . The inclusion  $X^{(k)} \xrightarrow{\iota} T(f_\sigma)$  is a retraction that induces this splitting, and its left homotopy inverse  $T(f_\sigma) \xrightarrow{\kappa} X^{(k)}$  can be chosen so that the composition  $X^{(k)} \xrightarrow{\iota} T(f_\sigma) \xrightarrow{\kappa} X^{(k)}$  is homotopic to  $f_\sigma$ .

It is a well known fact that  $\beta_k \beta_k = k \beta_k \in \mathbb{Z}_p[S_k]$  [8, 10], implying  $\frac{1}{k} \beta_k \in \mathbb{Z}_p[S_k]$  is an idempotent whenever  $k$  is prime to  $p$ . By taking mapping telescopes one has  $T(f_{\beta_k})$  a retract of  $X^{(k)}$  when  $k$  is prime to  $p$ . We shall denote  $T(f_{\beta_k})$  by  $L_k(X)$ . The mod- $p$  homology of  $L_k(X)$  is the image of  $V^{\otimes k} \xrightarrow{\beta_k} V^{\otimes k}$ , so it is the submodule of length  $k$  Lie brackets in  $V^{\otimes k}$ , and is denoted by  $L_k(V)$ . These spaces  $L_k(X)$  turn out to have some significance, as is apparent in the following homotopy decomposition (see [12]).

**Theorem 5.1.** *Let  $X$  be a  $p$ -local suspension, and let  $1 < k_1 < k_2 < \dots$  be any sequence satisfying the following properties:*

- (1)  $k_i$  is prime to  $p$ ;
- (2)  $k_i$  is not a multiple of  $k_j$  whenever  $i > j$ .

*Then there exists a homotopy decomposition*

$$\Omega \Sigma X \simeq \prod_j \Omega \Sigma L_{k_j}(X) \times (\text{Some other space}).$$

□

The factors in this decomposition are of interest if one is to study the homotopy theory of  $\Omega \Sigma X$ . Ideally these factors can themselves be further broken down into more familiar spaces. Since  $L_k(X)$  is a retract of  $X^{(k)}$ , the most manageable way of doing this is to obtain splittings of  $L_k(X)$  by searching for splittings of  $X^{(k)}$  using the tools of representation theory. A comprehensive look at the finest possible 2-primary splittings of  $X^{(k)}$  for  $X$  a 2-cell complex can be found in [13].

In the following theorem we give criteria for constructing a certain retraction of  $L_{n\ell+1}(X)$ . This can be seen as a geometric realization of the compositions  $(\hat{s}_\ell^{\otimes n} \otimes 1) \circ \beta_{\ell n+1} \circ (\hat{s}_\ell^{\otimes n} \otimes 1)$  and  $(\bar{s}_\ell^{\otimes n} \otimes 1) \circ \beta_{\ell n+1} \circ (\bar{s}_\ell^{\otimes n} \otimes 1)$  from the Theorem (2.1).

**Proposition 5.2.** *Fix  $n > 0$  and  $\ell > 1$  such that  $n\ell + 1$  is prime to  $p$ , and take the integers  $c_{n,\ell}$  and  $d_{n,\ell}$  in Theorem (2.1).*

*Suppose  $X$  is any suspended  $p$ -local CW-complex with  $\dim V = \ell > 1$  (where  $V$  denotes  $\tilde{H}_*(X)$ ). Let  $M$  denote the sum of the degrees of the generators of  $V$ . If  $c_{n,\ell}$  is prime to  $p$  and  $V_{\text{odd}} = 0$ , or  $d_{n,\ell}$  is prime to  $p$  and  $V_{\text{even}} = 0$ , then*

- (i) *there exist a space  $Y$  that is a retract of  $L_{n\ell+1}(X)$ , and  $\tilde{H}_*(Y) \cong \tilde{H}_*(\Sigma^{nM} X)$ ;*
- (ii) *if  $\ell \leq p - 1$ , then  $\Sigma^{nM} X$  is a retract of  $L_{n\ell+1}(X)$ .*

*Proof of part (i).* Recall the elements  $\bar{s}_\ell, \hat{s}_\ell \in \mathbb{Z}_p[S_\ell]$  defined in Section (2). If  $V_{\text{even}} = 0$ , let  $s_\ell = \bar{s}_\ell$  and assume  $c = c_{n,\ell}$  is prime to  $p$ . Otherwise if  $V_{\text{odd}} = 0$ , let  $s_\ell = \hat{s}_\ell$  and assume  $c = d_{n,\ell}$  is prime

to  $p$ . We have self-maps  $X^{(\ell)} \xrightarrow{f_{s_\ell}} X^{(\ell)}$  and  $X^{(n\ell+1)} \xrightarrow{f_{\beta_{n\ell+1}}} X^{(n\ell+1)}$  inducing  $V^{\otimes \ell} \xrightarrow{s_\ell} V^{\otimes \ell}$  and  $V^{\otimes(n\ell+1)} \xrightarrow{\beta_{n\ell+1}} V^{\otimes(n\ell+1)}$  on mod- $p$  homology. Consider the composition

$$g : X^{(n\ell+1)} \xrightarrow{f_{s_\ell}^{(n)} \wedge \mathbb{1}} X^{(n\ell+1)} \xrightarrow{f_{\beta_{n\ell+1}}} X^{(n\ell+1)},$$

where  $\mathbb{1}$  is the identity map on  $X$ , and  $f_{s_\ell}^{(n)}$  is the  $n$ -fold self-smash of  $f_{s_\ell}$ . On mod- $p$  homology  $g$  induces

$$g_* : V^{\otimes(n\ell+1)} \xrightarrow{s_\ell^{\otimes n} \otimes \mathbb{1}} V^{\otimes(n\ell+1)} \xrightarrow{\beta_{n\ell+1}} V^{\otimes(n\ell+1)}.$$

Let  $T(g)$  be the telescope of  $g$ . By Theorem (2.1),

$$(18) \quad (s_\ell^{\otimes n} \otimes \mathbb{1}) \circ \beta_{n\ell+1} \circ (s_\ell^{\otimes n} \otimes \mathbb{1}) = c(s_\ell^{\otimes n} \otimes \mathbb{1}).$$

Thus  $g_* \circ g_* = c(g_*)$ . Since  $c$  is prime to  $p$ , this implies  $\tilde{H}_*(T(g)) \cong \text{Im}(g_*)$ . Notice  $\text{Im}(g_*) \subseteq \text{Im}(s_\ell^{\otimes n} \otimes \mathbb{1})$ , and  $\text{Im}((s_\ell^{\otimes n} \otimes \mathbb{1}) \circ g_*) \subseteq \text{Im}(g_* \circ g_*) = \text{Im}(g_*)$ . By equation (18),  $\text{Im}((s_\ell^{\otimes n} \otimes \mathbb{1}) \circ g_*) = \text{Im}(c(s_\ell^{\otimes n} \otimes \mathbb{1})) = \text{Im}(s_\ell^{\otimes n} \otimes \mathbb{1})$ . Therefore  $\text{Im}(g_*) = \text{Im}(s_\ell^{\otimes n} \otimes \mathbb{1})$ . Also,  $\text{Im}(s_\ell^{\otimes n})$  is a submodule of  $V^{\otimes n\ell}$  with dimension 1, whose single generator has degree  $nM$ . So  $\text{Im}(s_\ell^{\otimes n} \otimes \mathbb{1}) \cong \Sigma^{nM}V$  as graded  $\mathbb{Z}_p$ -modules, where  $\Sigma^{nM}V$  is the  $nM$ -fold suspension of the graded  $\mathbb{Z}_p$ -module  $V$ . Hence

$$\tilde{H}_*(T(g)) \cong \text{Im}(g_*) \cong \Sigma^{nM}V \cong \tilde{H}_*(\Sigma^{nM}X).$$

Let us also consider the composition

$$h = (c - g) : X^{(n\ell+1)} \xrightarrow{\psi} X^{(n\ell+1)} \vee X^{(n\ell+1)} \xrightarrow{c \vee -g} X^{(n\ell+1)} \vee X^{(n\ell+1)} \xrightarrow{\nabla} X^{(n\ell+1)},$$

where  $\psi$  is the pinch map,  $c$  is the degree  $c$  map on  $X^{(n\ell+1)}$ ,  $-g$  is the composition of  $g$  and the degree  $-1$  map on  $X^{(n\ell+1)}$ , and  $\nabla$  is the fold map. On mod- $p$  homology we have  $h_* = c - g_*$ , and  $c$  is prime to  $p$ . Since  $g_* \circ g_* = c(g_*)$ , this implies  $\text{Im}(h_*) \cong (V^{\otimes(n\ell+1)} - \text{Im}(g_*))$  and  $\text{Im}(g_*) \cap \text{Im}(h_*) = \emptyset$ . Therefore  $V^{\otimes(n\ell+1)}$  splits as a sum of  $\mathbb{Z}_p$ -submodules  $\text{Im}(g_*) \oplus \text{Im}(h_*)$ . Notice that

$$h_* \circ h_* = (c - g_*) \circ (c - g_*) = c^2 - 2c(g_*) + (g_* \circ g_*) = c^2 - 2c(g_*) + c(g_*) = c(h_*),$$

so taking the telescope  $T(h)$ , we have  $\tilde{H}_*(T(h)) \cong \text{Im}(h_*)$ . Thus we have the following splitting of graded  $\mathbb{Z}_p$ -modules,

$$\tilde{H}_*(X^{(n\ell+1)}) = V^{\otimes(n\ell+1)} = \text{Im}(g_*) \oplus \text{Im}(h_*) \cong \tilde{H}_*(T(g)) \oplus \tilde{H}_*(T(h)).$$

As the inclusions  $X^{(n\ell+1)} \xrightarrow{\iota_g} T(g)$  and  $X^{(n\ell+1)} \xrightarrow{\iota_h} T(h)$  induce projections of  $\text{Im}(g_*)$  and  $\text{Im}(h_*)$  isomorphically onto  $\tilde{H}_*(T(g))$  and  $\tilde{H}_*(T(h))$  in mod- $p$  homology, the map

$$f : X^{(n\ell+1)} \xrightarrow{\psi} X^{(n\ell+1)} \vee X^{(n\ell+1)} \xrightarrow{\iota_g \vee \iota_h} T(g) \vee T(h)$$

induces an isomorphism on mod- $p$  homology, so it is a homotopy equivalence.

Let  $f^{-1} : T(g) \vee T(h) \longrightarrow X^{(n\ell+1)}$  denote the inverse homotopy equivalence of  $f$ . Since  $f \circ f^{-1} : X^{(n\ell+1)} \longrightarrow X^{(n\ell+1)}$  is homotopic to the identity, and  $X^{(n\ell+1)} \xrightarrow{\iota_g} T(g)$  maps  $Im(g_*)$  isomorphically onto  $\tilde{H}_*(T(g))$  on mod- $p$  homology, the composition

$$\kappa_g : T(g) \longrightarrow T(g) \vee T(h) \xrightarrow{f^{-1}} X^{(n\ell+1)}$$

maps  $\tilde{H}_*(T(g))$  isomorphically onto  $Im(g_*)$  in mod- $p$  homology. Also, since  $n\ell + 1$  is prime to  $p$  and  $\beta_{n\ell+1} \circ \beta_{n\ell+1} = (n\ell + 1)\beta_{n\ell+1}$ ,  $\frac{1}{n\ell+1}\beta_{n\ell+1} \in \mathbb{Z}_p[S_{n\ell+1}]$  is an idempotent. So the inclusion  $X^{(n\ell+1)} \xrightarrow{\iota} T(f_{\beta_{n\ell+1}}) = L_{n\ell+1}(X)$  is a retraction, and we can take some left homotopy inverse  $\kappa$  such that  $X^{(n\ell+1)} \xrightarrow{\iota} L_{n\ell+1}(X) \xrightarrow{\kappa} X^{(n\ell+1)}$  is homotopic to  $X^{(n\ell+1)} \xrightarrow{f_{\beta_{n\ell+1}}} X^{(n\ell+1)}$ . Now consider the composition

$$\alpha : T(g) \xrightarrow{\kappa_g} X^{(n\ell+1)} \xrightarrow{\iota} L_{n\ell+1}(X) \xrightarrow{\kappa} X^{(n\ell+1)} \xrightarrow{\iota_g} T(g).$$

Recall  $g_* = \beta_{n\ell+1} \circ (s_\ell^{\otimes n} \otimes 1)$  by definition. Then on mod- $p$  homology  $\kappa_* \circ \iota_*$  sends  $Im(g_*)$  (surjectively) onto  $\beta_{n\ell+1}(Im(g_*)) = Im(\beta_{n\ell+1} \circ \beta_{n\ell+1} \circ (s_\ell^{\otimes n} \otimes 1)) = Im((n\ell+1)\beta_{n\ell+1} \circ (s_\ell^{\otimes n} \otimes 1)) = Im(g_*)$ . Since  $(\iota_g)_*$  projects  $Im(g_*)$  isomorphically onto  $\tilde{H}_*(T(g))$ , and  $\kappa_g$  maps  $\tilde{H}_*(T(g))$  isomorphically onto  $Im(g_*)$ ,  $\alpha_*$  is an isomorphism on mod- $p$  homology. Therefore  $\alpha$  is a homotopy equivalence and  $T(g)$  is a retract of  $L_{n\ell+1}(X)$ .  $\square$

*Proof of part (ii).* We continue where the proof of part (i) left off to avoid redefining things, but this time the added assumption is that  $\ell \leq p - 1$ . Recall  $s_\ell \in \mathbb{Z}_p[S_\ell]$  is either  $\bar{s}_\ell$ , or  $\hat{s}_\ell$ , depending on whether  $V_{even} = 0$  or  $V_{odd} = 0$ . In either case it is well known (and not difficult to see) that  $s_\ell s_\ell = \ell! s_\ell$ . So one can take the idempotent  $\frac{1}{\ell!} s_\ell$  when  $\ell \leq p - 1$ . In this case the inclusion  $X^{(\ell)} \xrightarrow{\bar{\iota}} T(f_{s_\ell})$  is a retraction, and since  $\tilde{H}_*(T(f_{s_\ell})) \cong Im(s_\ell)$  is a 1-dimensional submodule of  $V^{\otimes \ell}$  whose generator has degree  $M$ ,  $T(f_{s_\ell})$  is homotopy equivalent to the  $M$ -sphere  $S^M$ .

Let  $\gamma : X^{(n\ell)} \xrightarrow{\bar{\iota}^{(n)}} S^{nM}$  be the  $n$ -fold smash of  $\bar{\iota}$ . On mod- $p$  homology  $\gamma$  induces an isomorphism onto  $Im(s_\ell^{\otimes n})$ , and so the smash of  $\gamma$  and the identity on  $X$ ,

$$\gamma \wedge \mathbb{1} : X^{(n\ell)} \longrightarrow \Sigma^{nM} X,$$

induces an isomorphism onto  $Im(s_\ell^{\otimes n} \otimes 1)$ . Since the section map  $T(g) \xrightarrow{\kappa_g} X^{(n\ell+1)}$  defined in the proof of part (i) induces an isomorphism onto  $Im(s_\ell^{\otimes n} \otimes 1)$  on mod- $p$  homology, the composition  $T(g) \xrightarrow{\kappa_g} X^{(n\ell)} \xrightarrow{\gamma \wedge \mathbb{1}} \Sigma^{nM} X$  is an isomorphism on mod- $p$  homology, so it is a homotopy equivalence. Thus part (ii) follows from part (i).  $\square$

If the criteria in Proposition (5.2) are satisfied, then Proposition (5.2) together with Theorem (5.1) imply  $\Omega\Sigma Y$  is a retract of  $\Omega\Sigma X$ . Likewise, if  $\ell \leq p - 1$ , then  $\Omega\Sigma^{nM+1} X$  is a retract of  $\Omega\Sigma X$ . Given that both  $d_{n,\ell}$  and  $c_{n,\ell}$  are prime to  $p$ , one can iterate Proposition (5.2) to produce an infinite sequence of such retractions, as is stated more precisely in the following proposition.

**Proposition 5.3.** *Fix  $n > 0$  and  $\ell > 1$  such that  $n\ell + 1$  is prime to  $p$ , and suppose both  $c_{n,\ell}$  and  $d_{n,\ell}$  are prime to  $p$ .*

*Let  $X$  be any suspended  $p$ -local CW-complex with  $\dim V = \ell > 1$  (where  $V$  denotes  $\tilde{H}_*(X)$ ), and either  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$ . Let  $M$  denote the sum of the degrees of the generators of  $V$ , and define the sequence of integers  $b_{i,n}$  recursively with  $b_{0,n} = 0$ , and*

$$b_{i,n} = (n\ell + 1)b_{i-1,n} + nM.$$

*Then*

- (i) *there exist spaces  $Y_i$  such that  $\Omega\Sigma Y_i$  is a retract of  $\Omega\Sigma X$ , and  $\tilde{H}_*(Y_i) \cong \tilde{H}_*(\Sigma^{b_{i,n}} X)$  for each  $i \geq 1$ ;*
- (ii) *if  $\ell \leq p - 1$ , then  $\Omega\Sigma^{b_{i,n}+1} X$  is a retract of  $\Omega\Sigma X$  for each  $i \geq 1$ .*

*Proof.* We will prove part (ii) since part (i) is similar. This is done by induction, with the base case being  $\Omega\Sigma^{nM+1} X$  is a retract of  $\Omega\Sigma X$ . These base case holds true since  $\Sigma^{nM} X$  is a retract of  $L_{n\ell+1}(X)$  by Proposition (5.2), and by Theorem (5.1)  $\Omega\Sigma L_{n\ell+1}(X)$  is a retract of  $\Omega\Sigma X$  when  $n\ell + 1$  is prime to  $p$ .

For our inductive assumption, let us assume  $\Omega\Sigma^{b_{i,n}+1} X$  is a retract of  $\Omega\Sigma X$  for some  $i \geq 1$ , and let  $M'$  be the sum of the degrees of the generators of  $\tilde{H}_*(\Sigma^{b_{i,n}} X)$ . Notice that  $\dim \Sigma^{b_{i,n}} V = \dim V = \ell$ , and since  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$ , either  $(\Sigma^{b_{i,n}} V)_{\text{odd}} = 0$  or  $(\Sigma^{b_{i,n}} V)_{\text{even}} = 0$ . Since we have an isomorphism  $\tilde{H}_*(\Sigma^{b_{i,n}} X) \cong \Sigma^{b_{i,n}} V$  of graded  $\mathbb{Z}_p$ -modules, and since we are assuming that both  $c_{n,\ell}$  and  $d_{n,\ell}$  are prime to  $p$ , by Proposition (5.2)  $\Sigma^{nM'}(\Sigma^{b_{i,n}} X)$  is a retract of  $L_{n\ell+1}(\Sigma^{b_{i,n}} X)$ , Also, because  $n\ell + 1$  is prime to  $p$ , by Theorem (5.1)  $\Omega\Sigma L_{n\ell+1}(\Sigma^{b_{i,n}} X)$  is a retract of  $\Omega\Sigma(\Sigma^{b_{i,n}} X)$ , so  $\Omega\Sigma^{nM'+1}(\Sigma^{b_{i,n}} X)$  is also a retract of  $\Omega\Sigma(\Sigma^{b_{i,n}} X)$ . Then using our inductive assumption,  $\Omega\Sigma^{nM'+1}(\Sigma^{b_{i,n}} X)$  is a retract of  $\Omega\Sigma X$ .

To check that  $M'$  has the correct value, let  $\{v_1, \dots, v_\ell\}$  be a basis for  $V$  and  $M$  be the sum of the degrees of the generators in this basis. In this case

$$M' = \sum_{1 \leq i \leq \ell} (b_{i,n} + |v_i|) = \ell b_{i,n} + M.$$

Thus  $\Sigma^{nM'}(\Sigma^{b_{i,n}} X) = \Sigma^{b_{i+1,n}} X$ . This completes the induction. □

Proposition (5.3) becomes useful when we know  $c_{n,\ell}$  and  $d_{n,\ell}$  are prime to  $p$ . In Theorem (2.2) we found that  $c_{1,\ell} = d_{1,\ell} = (\ell + 1)((\ell - 1)!)$  for  $\ell > 1$ , so Theorem (1.1) follows as an easy consequence. For the case  $\ell = 2$  we found  $c_{n,2} = d_{n,2} = 3^n$  for  $n \geq 1$ , which wuse use to prove Theorem (1.2).

*Proof of Theorem (1.2).* Fix  $p \geq 5$ , and let  $X$  be any suspended  $p$ -local CW-complex with  $\dim V = 2$ , and either  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$ . Let  $M$  denoting the sum of the degrees of the two generators of  $V$ .

Let  $0 < k_1 < k_2 < \dots$  be any sequence satisfying  $2k_i + 1$  is prime to  $p$  and  $2k_i + 1$  is not a multiple of  $2k_j + 1$  whenever  $i > j$ . By Theorem (5.1) there is a decomposition

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{2k_j+1}(X) \times (\text{Some other space}).$$

Since  $2k_j + 1$  and  $c_{k_j,2} = d_{k_j,2} = 3^{k_j}$  are prime to  $p$  for all  $j$ , by Proposition (5.2)  $\Sigma^{k_j M} X$  is a retract of  $L_{2k_j+1}(X)$ , so in turn  $\Omega\Sigma^{k_j M+1} X$  is a retract of  $\Omega\Sigma L_{2k_j+1}(X)$ . This proves the theorem.  $\square$

**Remark 5.4.** *There is an analogous decomposition of the 2-cell complex  $X$  in Theorem (1.2) in terms of spheres, on the condition that  $V_{\text{odd}} = 0$ . In this case it is easy to show  $L_2(X) \simeq S^M$  and  $L_3(X) \simeq \Sigma^M X$ . Repeating this argument starting with  $\Sigma^M X$  in place of  $X$ , one can show  $\Omega\Sigma X \simeq \prod_{i=1}^{\infty} \Omega(S^{b_{i,1}+1}) \times (\text{Some other space})$  by using Theorem (5.1) (where  $b_{i,1}$  are the integers defined in Proposition (5.3)).*

*Proof of Theorem (1.4).* Let  $X$  be any suspended  $p$ -local CW-complex, and let  $V$  denote  $\tilde{H}_*(X)$ , and  $M$  be the sum of the degrees of the generators in  $V$ . Assume  $V_{\text{even}} = 0$ ,  $\ell = \dim V$  is even, and  $1 < \ell < p-1$ . Let  $L(V)$  is the free Lie algebra generated by  $V$ , and  $[L(V), L(V)]$  the sub Lie algebra of  $L(V)$  generated by Lie brackets of length greater than one. By the Poincaré-Birkhoff-Witt theorem, there is an isomorphism of coalgebras

$$T(V) \cong \Lambda(V) \otimes S([L(V), L(V)]).$$

This isomorphism is geometrically realized by Cohen's and Neisendorfer's decomposition (Theorem (1.3))

$$\Omega\Sigma X \simeq A(X) \times \Omega Q(X),$$

with  $H_*(A(X)) \cong \Lambda(V)$  and  $H_*(\Omega Q(X)) \cong S([L(V), L(V)])$ . By Theorem (5.1)  $\Omega\Sigma L_{\ell+1}(X)$  is a retract of  $\Omega\Sigma X$ , and the proof of this in [12] indicates the section map  $\Omega\Sigma L_{\ell+1}(X) \rightarrow \Omega\Sigma X$  for this retraction induces the natural inclusion

$$\tilde{H}_*(\Omega\Sigma L_{\ell+1}(X)) \simeq T(L_{\ell+1}(V)) \cong \bigotimes_{i=1}^{\infty} S(L_i(L_{\ell+1}(V))) \subseteq \Lambda(V) \otimes S([L(V), L(V)])$$

into the right-hand factor (where  $L_j(V)$  denotes the  $\mathbb{Z}_p$ -submodule of length  $i$  Lie brackets in  $L(V)$ , and where the isomorphism follows by the Poincaré-Birkhoff-Witt theorem). Therefore  $\Omega\Sigma L_{\ell+1}(X)$  is also a retract of  $\Omega Q(X)$ . In turn,  $\Omega\Sigma^{M+1} X$  is a retract of  $\Omega\Sigma L_{\ell+1}(X)$  using Proposition (5.3), so we obtain a decomposition

$$\Omega\Sigma X \simeq A(X) \times \Omega\Sigma^{M+1} X \times (\text{Some other space}).$$

Since  $\ell = \dim V$  is even and  $V_{\text{even}} = 0$ ,  $\tilde{H}_*(\Sigma^M X) \cong \Sigma^M V$  has only odd degree generators, so we can reapply Cohen's and Neisendorfer's decomposition to  $\Omega\Sigma^{M+1} X$ . Iterating this argument, starting by taking  $\Sigma^M X$  in place of  $X$ , and using an induction similar to the proof of Proposition (5.3),

we obtain the decomposition

$$\Omega\Sigma X \simeq \prod_{i=0}^{\infty} A(\Sigma^{b_{i,1}}X) \times (\text{Some other space}),$$

where  $b_{i,1}$  are the integers defined in Proposition (5.3). □

## 6. AN APPLICATION TO THE MOORE CONJECTURE

The  $p$ -exponent  $\exp_p(X)$  of a space  $X$  is defined as the smallest power  $p^t$  such that  $p^t$  annihilates the  $p$ -primary torsion of  $\pi_i(X)$  for all  $i > 0$ . It is known that the spheres, finite  $H$ -spaces, and certain Moore spaces all have finite exponents at odd primes [1, 9, 4]. A simply connected wedge of spheres is a simple example of a finite  $CW$ -complex without a finite  $p$ -exponent, or more generally, any wedge  $S^m \vee \Sigma X$  where  $\Sigma X$  is rationally non-trivial and  $m > 1$  [9]. The Moore conjecture addresses the question concerning which spaces have finite  $p$ -exponent and which spaces have no  $p$ -exponent. As the statement of the conjecture stands, a simply connected finite  $CW$ -complex  $X$  has  $\exp_p(X)$  finite at a prime  $p$  if and only if  $\pi_*(X) \otimes \mathbb{Q}$  is a finite dimensional vector space. Since  $X$  is rationally a wedge of spheres when it is a suspension, this can be stated equivalently as follows:

**Conjecture 6.1** (Moore conjecture for suspensions). *Let  $X$  be a simply connected finite  $CW$ -complex that is a suspension.*

- (a) *If  $\dim(H_*(X; \mathbb{Z}) \otimes \mathbb{Q}) \leq 1$ , then  $\exp_p(X)$  is finite;*
- (b) *if  $\dim(H_*(X; \mathbb{Z}) \otimes \mathbb{Q}) > 1$ , then  $\exp_p(X)$  is infinite.*

We should note that part (a) is known to hold at *almost all* primes  $p$  based on work of McGibbon and Wilkerson [3]. Working in the other direction, Selick [6] showed that part (b) holds when  $H_*(X; \mathbb{Z})$  is torsion-free. There is also a stable analogue of the Moore conjecture due Stanley [9], which is much easier to prove. That is, any simply connected finite  $CW$ -complex  $X$  has a finite  $p$ -exponent on stable homotopy  $\pi_*^s(X)$  if and only if  $X$  is rationally trivial. Using the following proposition together with Stanley's theorem, we can recover Selick's result when limiting ourselves to the spaces in Theorem (1.1):

**Proposition 6.2.** *Take the integers  $b_i$  and a suspended  $p$ -local  $CW$ -complex  $X$  as in Theorem (1.1), letting  $V = \tilde{H}_*(X)$ ,  $1 < \dim V < p - 1$ , and either  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$ . Assume  $X$  is  $(m - 1)$ -connected for some  $m \geq 1$ . Then for each  $j$  the stable homotopy group  $\pi_j^s(\Sigma X)$  is a retract of  $\pi_{j+b_i}(\Sigma X)$  for all  $i$  large enough such that  $j \leq b_i + 2m$ .*

*Proof.* By Theorem (1.1)  $\Omega\Sigma^{b_i+1}X$  is a retract of  $\Omega\Sigma X$ , so  $\pi_{j+b_i}(\Sigma^{b_i+1}X)$  is a retract of  $\pi_{j+b_i}(\Sigma X)$  for each  $j$ . By the Freudenthal suspension theorem  $\pi_{j+b_i}(\Sigma^{b_i+1}X) \cong \pi_{j+b_i}^s(\Sigma^{b_i+1}X)$  for  $j \leq b_i + 2m$ , and  $\pi_{j+b_i}^s(\Sigma^{b_i+1}X) \cong \pi_j^s(\Sigma X)$ . Thus  $\pi_j^s(\Sigma X)$  is a retract of  $\pi_{j+b_i}(\Sigma X)$  when  $j \leq b_i + 2m$ . □

Thus we see that the stable homotopy groups  $\pi_*^s(\Sigma X)$  of the space  $\Sigma X$  in Proposition (6.2) are contained in  $\pi_*(\Sigma X)$ . Since  $\Sigma X$  is rationally nontrivial,  $\exp_p(\pi_*^s(\Sigma X))$  is infinite by Stanley's theorem. So  $\exp_p(\Sigma X)$  must also be infinite.

Selick's result still leaves part (a) of the Moore conjecture open for those suspensions that have torsion in their integral homology. The restriction  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$  in Theorem (1.1) excludes such torsion spaces. In hopes of making further progress on the Moore conjecture, one might ask whether the restriction  $V_{\text{odd}} = 0$  or  $V_{\text{even}} = 0$  can be dropped in Theorem (1.1), and then Proposition (6.2) together with Stanley's result can be extended to torsion spaces. Unfortunately this seems unlikely. For example, we can let  $X$  be a wedge  $S^m \vee P^n(p^r)$  where the Moore space  $P^n(p^r)$  is the cofibre of the degree  $p^r$  map  $S^{n-1} \xrightarrow{p^r} S^{n-1}$ . Then  $\Sigma X$  has torsion in its integral homology but is rationally nontrivial, and the mod- $p$  homology  $V = H_*(X)$  has an even degree generator and an odd degree generator, so  $V_{\text{odd}} \neq 0$  and  $V_{\text{even}} \neq 0$ . If an analogue of Theorem (1.1) held true for  $X$ , the stable homotopy groups  $\pi_*^s(\Sigma X)$  would be contained in  $\pi_*(\Sigma X)$ , and  $\exp_p(\Sigma X)$  would be infinite using Stanley's theorem. But using the Hilton-Milnor theorem and certain properties of Moore spaces, one can show that  $\exp_p(\Sigma X)$  is in fact finite (as is predicted by the Moore conjecture, since  $\dim(H_*(\Sigma X; \mathbb{Z}) \otimes \mathbb{Q}) = 1$ ).

Though Theorem (1.1) might not work in general situations, it still seems likely that a lot more can be said about the factors in Theorem 5.1 for more general spaces. It may be the case that a broader range of decompositions exists in terms of spaces that are in some sense familiar, aside from the retractions we constructed for the narrow range of spaces leading to Theorem (1.1).

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