Lecture Notes on Algebraic Topology

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Contents

CHAPTER 1

Introduction

1. Sets

Let X and Y be sets. The notation $Y \subseteq X$ means that Y is a subset of X and $Y \subset X$ means that Y is a proper subset of X, that is $Y \subseteq X$ and $Y \neq X$. Let $X \setminus Y$ denote the set

$$X \setminus Y = \{x | x \in X \quad \text{and} \quad x \notin Y\}.$$

The empty set is denoted by $\emptyset$.

Let X and Y be sets. The Cartesian product is defined by

$$X \times Y = \{(x, y) | x \in X, y \in Y\}.$$

Note: If X and Y are finite sets of m and n elements, respectively, then $X \times Y$ is a finite set of mn elements.

Let $X_1, \cdots, X_n$ be sets. The Cartesian product is defined by

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \cdots, x_n) | x_i \in X_i, 1 \leq i \leq n\}.$$

The infinite Cartesian product is defined similarly. For example, let $\{X_\alpha | \alpha \in I\}$ be a family of sets. Then

$$\prod_{\alpha \in I} X_\alpha = \{(x_\alpha) | x_\alpha \in X_\alpha, \alpha \in I\}.$$

The α-coordinate projection

$$\pi_\alpha \colon \prod_{\alpha' \in I} X_{\alpha'} \to X_\alpha$$

is defined by

$$\pi_\alpha((x_{\alpha'})) = x_\alpha.$$

THEOREM 1.1. *Let $\{X_\alpha | \alpha \in I\}$ be a family of set. Then the Cartesian product $\prod_{\alpha \in I} X_\alpha$ satisfies the following universal lifting property:*

> *Let X be any set and let $f_\alpha \colon X \to X_\alpha$ be any function for each $\alpha \in I$. Then there is a unique function*
>
> $$f \colon X \to \prod_{\alpha \in I} X_\alpha$$
>
> *such that*
>
> $$f_\alpha = \pi_\alpha \circ f$$
>
> *for each α.*

PROOF. Let $f\colon X \to \prod_{\alpha\in I} X_\alpha$ be a function defined by

$$f(x) = (f_\alpha(x))$$

for each $x \in X$. Then f is a function with the property that $f_\alpha(x) = \pi_\alpha \circ f(x)$ for any x and so $f_\alpha = \pi_\alpha \circ f$. This shows the existence of the universal lifting property. Let $g\colon X \to \prod_{\alpha\in I} X_\alpha$ be any function with the property that $f_\alpha = \pi_\alpha \circ g$ for each α. Then the α-th coordinate of $g(x)$ is $f_\alpha(x)$ for each $x \in X$. Thus $g = f$ defined above. This shows the uniqueness of the universal lifting property. □

Let $f\colon X \to Y$ be a function. Then the image of f is defined by

$$\mathrm{Im}(f) = f(X) = \{y \in Y | y = f(x) \quad \text{for} \quad \text{some} \quad x \in X\}.$$

The identity function on X is denoted by id_X, id or 1. Thus $\mathrm{id}(x) = x$.

EXERCISE 1.1. *Let $f\colon X \to Y$ be a function. Let $\{X_\alpha | \alpha \in I\}$ be a family of subsets of X. Then*

1) *show that*
$$f(\bigcup_{\alpha\in I} X_\alpha) = \bigcup_{\alpha\in I} f(X_\alpha);$$
2) *show that*
$$f(\bigcap_{\alpha\in I} X_\alpha) \subseteq \bigcap_{\alpha\in I} f(X_\alpha);$$
3) *show by example that*
$$f(\bigcap_{\alpha\in I} X_\alpha) \neq \bigcap_{\alpha\in I} f(X_\alpha)$$
in general.

Let $f\colon X \to Y$ be a function. Let A be a subset of X. Then the restriction $f|_A\colon A \to Y$ is the function defined by

$$f|_A(a) = f(a)$$

for $a \in A$. Let B be a subset of Y. The pre-image $f^{-1}(B)$ is defined by

$$f^{-1}(B) = \{x \in X | f(x) \in B\}.$$

Note that $f^{-1}(B)$ could be an empty set.

EXERCISE 1.2. *Let $f\colon X \to Y$ be a function. Let $\{B_\beta | \beta \in J\}$ be a family of subsets of Y. Then*

1) *show that*
$$f^{-1}(\bigcup_{\beta\in J} B_\beta) = \bigcup_{\beta\in J} f^{-1}(B_\beta);$$
2) *show that*
$$f^{-1}(\bigcap_{\beta\in J} B_\beta) = \bigcap_{\beta\in J} f^{-1}(B_\beta);$$
3) *show that*
$$f^{-1}(Y \setminus B_\beta) = X \setminus f^{-1}(B_\beta).$$

A function $f\colon X \to Y$ is said to be bijective if it is one-to-one and onto. In this case the inverse is denoted by $f^{-1}\colon Y \to X$. Note that f^{-1} is also bijective. If there is a bijective function f from X to Y, we call that X is isomorphic to Y as sets.

EXERCISE 1.3. *Let X be a set. Let X_α be a family of sets with indices α in a set I. Suppose that $X_\alpha = X$ for each α. Show that $\prod_{\alpha\in I} X_\alpha$ is isomorphic to the set of functions from I to X.*

A relation on a set X is a subset $\sim$ of $X \times X$. We write $x \sim y$ if $(x, y) \in \sim$. A relation on X is an *equivalence relation* if it satisfies

1) the reflexive condition: $x \sim x$ for all $x \in X$;
2) the symmetric condition: If $x \sim y$, then $y \sim x$;
3) the transitive condition: If $x \sim y$ and $y \sim z$, then $x \sim z$.

The equivalence class of x is the set

$$\{x\} = \{y \in X | x \sim y\}.$$

EXERCISE 1.4. *Let $\sim$ be an equivalence relation on X. Show that each element of X belongs to exactly one equivalence class.*

2. Monoids and Groups

A *binary operation (multiplication)* on a set X is a function $\mu\colon X \times X \to X$. We abbreviate $\mu(x, y)$ to xy or $x + y$. A *monoid* M is a set M together with a multiplication $\mu\colon M \times M \to M$ satisfying the following conditions:

1) (identity) there exists an element $1 \in M$ such that
$$1x = x1 = x$$
for any $x \in M$;
2) (associativity) the equation
$$(x_1x_2)x_3 = x_1(x_2x_3)$$
holds for any $x_1, x_2, x_3 \in M$.

A *group* is a monoid G satisfying

3) (inverse) For each $x \in G$, there exists an element $x^{-1} \in G$ such that
$$xx^{-1} = x^{-1}x = 1.$$

In other words, a group is a monoid in which every element is invertible. Note that if x is invertible, then the inverse of x is unique. A group (or monoid) G is said to *abelian* or *commutative* if $xy = yx$ for any $x, y \in G$. Let G and H be monoids (or groups). Then the Cartesian product $G \times H$ is a monoid (or group) under the multiplication defined by

$$(g, h)(g', h') = (gg', hh').$$

In additive case, we write $G \oplus H$ for $G \times H$.

A subset H of a group (monoid) is a *subgroup (submonoid)* of G if H is a group (monoid) under the binary operation of G. Let H be a subgroup (submonoid) of G and let $g \in G$. The left and right cosets of H by g are defined by

$$gH = \{gh | h \in H\} \qquad Hg = \{hg | h \in H\}.$$

EXAMPLE 1.2. Let $\mathbb{Z}^+$ be the set of non-negative integers. Then $\mathbb{Z}^+$ is a monoid under the addition $+$. $\mathbb{Z}^+$ is a submonoid of $\mathbb{Z}$. $\mathbb{Z}$ is often called the group completion of the monoid $\mathbb{Z}^+$, i.e. the "smallest group" that contains $\mathbb{Z}^+$. The set of natural numbers is a monoid under the multiplication. The "group completion" of natural numbers is the set of positive rational numbers with the multiplication.

In general, monoids and the "group completion" of monoids are very complicated and there are many research papers about these topics.

Let G and H be monoids (or groups). A homomorphism $f\colon G \to H$ is a function such that $f(1) = 1$ and

$$f(xy) = f(x)f(y)$$

for any $x, y \in G$.

Exercise 2.1. *Let G and Hbe groups and let $f\colon G \to H$ be a function such that $f(xy) = f(x)f(y)$ for any $x, y \in G$. Show that*

1) *$f(1) = 1$;*
2) *$f(x^{-1}) = f(x)^{-1}$ for any $x \in G$.*

Let G and H be monoids (or groups). The *kernel* of a homomorphism $f\colon G \to H$ is the set

$$\mathrm{Ker}(f) = \{x \in G | f(x) = 1\}.$$

Note that a homomorphism f is one-to-one (a monomorphism) if and only if

$$\mathrm{Ker}(f) = \{1\}.$$

A monoid (or group) G is called isomorphic to H if there is a bijective homomorphism $f\colon G \to H$. In this case, we write $G \cong H$ or $f\colon G \cong H$.

A subgroup K of G is normal if $gxg^{-1} \in K$ for all $g \in G$ and $x \in K$. Let G and H be groups. Then the kernel of a homomorphism $f\colon G \to H$ is a normal subgroup of G. The image of f is a subgroup of H which is not normal in general.

Exercise 2.2. *Let K be a normal subgroup of a group G. Show that*

1) *$gK = Kg$ for any $g \in G$;*
2) *the set*

$$G/K = \{gK | g \in G\}$$

is a group under the operation

$$(gK)(g'K) = (gg')K.$$

The group G/K is called the quotient group of G by K.

Let G be a group and let $g \in G$. The subgroup generated by g is the subset

$$\langle g \rangle = \{g^n | n \in \mathbb{Z}\}.$$

Proposition 1.3. *Let G be a group and let $g \in G$. Then $\langle g \rangle$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some n.*

Proof. Let $\phi\colon \mathbb{Z} \to \langle g \rangle$ be the function defined by

$$\phi(n) = g^n.$$

Then ϕ is a homomorphism of groups because

$$\phi(m+n) = g^{m+n} = g^m g^n = \phi(m)\phi(n).$$

Note that ϕ is an epimorphism, that is ϕ is onto. If $g^m \neq 1$ for any positive integer m, then ϕ is an isomorphism. Suppose that $g^m = 1$ for some positive integer m. Let

$$n = \min\{m | g^m = 1, m > 0\}.$$

Then

$$\mathrm{Ker}(\phi) = n\mathbb{Z}$$

and so

$$\langle g \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

□

If $G = \langle g \rangle$ for some g, we say that G is a *cyclic* group with generator g. A set of *generators* for a group G is a subset S of G such that each element in G is a product of powers of elements taken from S. A group G is called *finitely generated* if it is generated by a finite subset.

A *free abelian group* of rank n is the direct sum

$$\mathbb{Z}^{\oplus n} = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}.$$

THEOREM 1.4 (Decomposition Theorem). *Let G be a finitely generated abelian group. Then G is isomorphic to*

$$H_0 \oplus H_1 \oplus H_2 \oplus \cdots \oplus H_m,$$

where H_0 is a free abelian group and H_i is a cyclic group of prime power order for $1 \leq i \leq m$.

The proof can be found in any text book of algebra.

A commutator in a group G is an element

$$[g, h] = ghg^{-1}h^{-1}.$$

for some elements $g, h \in G$. The commutator subgroup $[G, G]$ is the subgroup of G generated by all commutators of G. The commutator subgroup $[G, G]$ is normal. The group $G/[G, G]$ is called the *abelianization* of the group G. Note that a group G is abelian if and only if the commutator subgroup $[G, G]$ is trivial. A group G is called perfect if $[G, G] = G$. An example of perfect groups is the alternating groups A_n for $n > 4$. Non-commutative groups are much more complicated than abelian groups.

3. G-sets

Let G be a group. A set X is called a *left G-set* if there is an operation $\mu\colon G \times X \to X$, $(g, x) \to g \cdot x$, such that

1) $1 \cdot x = x$ for all $x \in X$;
2) $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

A set X is called a *right G-set* if there is an operation $\mu\colon X \times G \to X$, $(g, x) \to x \cdot g$, such that

1) $x \cdot 1 = x$ for all $x \in X$;
2) $x \cdot (gh) = (x \cdot g) \cdot h$ for all $g, h \in G$ and $x \in X$.

EXAMPLE 1.5. Let H be a subgroup of a group G. Then the set of left cosets $\{gH | g \in G\}$ is a left G-set and the set of right cosets $\{Hg | g \in G\}$ is a right G-set.

THEOREM 1.6. *Let X be a left G-set. For any $g \in G$, the function $\theta_g\colon X \to X$ defined by*

$$x \to g \cdot x$$

is a bijective.

Proof. From the definition, we have that $\theta_g\theta_h = \theta_{gh}$ and $\theta_1 = \mathrm{id}_X$. Thus

$$\theta_g\theta_{g^{-1}} = \mathrm{id}_X = \theta_{g^{-1}}\theta_g$$

and so θ_g is a bijective.

Similarly, if X is a right G-set, then the function $\theta_g\colon X \to X$ defined by $x \to x \cdot g$ is a bijective. □

4. Categories and Functors

A category may be thought of intuitively as consisting of sets, possibly with additional structure, and functions, possibly preserving additional structure. More precisely, a category $\mathcal{C}$ consists of

1) A class of objects
2) For every ordered pair of objects X and Y, a set $\mathrm{Hom}(X, Y)$ of *morphisms* with *domain* X and *range* Y; if $f \in \mathrm{Hom}(X, Y)$, we write $f\colon X \to Y$ or $X \xrightarrow{f} Y$
3) For every ordered triple of objects X, Y and Z, a function associating to a pair of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ their *composite*

$$g \circ f\colon X \to Z$$

These satisfy the following two axioms:

Associativity. If $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, then

$$h \circ (g \circ f) = (h \circ g) \circ f\colon X \to W.$$

Identity. For every object Y there is a morphism $\mathrm{id}_Y\colon Y \to Y$ such that if $f\colon X \to Y$, then $\mathrm{id}_Y \circ f = f$, and if $h\colon Y \to Z$, then $h \circ \mathrm{id}_Y = h$.

A category is said to be *small* if the class of objects is a set. The category of sets means the category in which the objects are sets and the morphisms are functions. The category of sets is NOT small. But there are many small categories. For instance, the category of finite sets, that is in which the objects are finite sets and the morphisms are functions between finite sets. We list some examples of categories:

1) The category of sets and functions.
2) The category of pointed sets (A pointed set means a non-empty set X with a *base point* $x_0 \in X$) and pointed functions (that is the functions that preserving the base points).
3) The category of finite ordered sets and monotone functions (that is $f(x) \leq f(y)$ is $x \leq y$). This category is usually denoted by Δ. The objects in Δ are given by $\{0, 1, \cdots, n\}$ for $n \geq 0$ and the morphisms in Δ are given by monotone function from $\{0, 1, \cdots, m\}$ to $\{0, 1, \cdots, n\}$ for any m, n.
4) The category of groups and homomorphisms.
5) The category of monoids and homomorphisms.
6) The category of topological spaces and continuous functions. Topological space is a generalization of the usual spaces such as Euclidian spaces $\mathbb{R}^n$, spheres, *polyhedra*, metric spaces and etc. We will give the definition of topological space in the next chapter.

Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{C}' \subseteq \mathcal{C}$ is a category such that

a) The objects of $\mathcal{C}'$ are also objects of $\mathcal{C}$;
b) For objects X' and Y' of $\mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X', Y')$ is a subset of $\mathrm{Hom}_{\mathcal{C}}(X', Y')$ and
c) If $f'\colon X' \to Y'$ and $g'\colon Y' \to Z'$ are morphisms of $\mathcal{C}'$, their composite in $\mathcal{C}'$ equals their composite in $\mathcal{C}$.

$\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and for objects X' and Y' in $\mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X', Y') = \mathrm{Hom}_{\mathcal{C}}(X', Y')$. For example, the category of groups and homomorphisms is a subcategory of the category of sets and functions but it is not a full subcategory. The category of finite sets and functions is a full subcategory of the category of sets and functions.

Let $\mathcal{C}$ be a category. A morphism $f\colon X \to Y$ is called an equivalence if there is a morphism $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* (or *contravariant functor*) T from $\mathcal{C}$ to $\mathcal{D}$ consists of an object function which assigns to every object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$ and a morphism function which assigns to every morphism $f\colon X \to Y$ of $\mathcal{C}$ a morphism $T(f)\colon T(X) \to T(Y)$ [or $T(f)\colon T(Y) \to T(X)$] of $\mathcal{D}$ such that

a) $T(\mathrm{id}_X) = \mathrm{id}_{T(X)}$ and
b) $T(g \circ f) = T(g) \circ T(f)$ [or $T(g \circ f) = T(f) \circ T(g)$].

THEOREM 1.7. *Let T be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. Then T maps equivalences in $\mathcal{C}$ to equivalences in $\mathcal{D}$.*

PROOF. Assume that T is covariant (the argument is similar if T is contravariant). Let $f\colon X \to Y$ be an equivalence and let $f^{-1}\colon Y \to X$ be its inverse. Since $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$, $T(f^{-1}) \circ T(f) = \mathrm{id}_{T(X)}$ and $T(f) \circ T(f^{-1}) = \mathrm{id}_{T(Y)}$. Thus $T(f)$ is an equivalence. □

A topological problem on spaces is to how to classify topological spaces. In other words, roughly speaking, how to know whether a space X is homeomorphic to another space Y or not. Basic ideas in algebraic topology is to introduce various functors from the category of topological spaces to "algebraic" categories such as the category of groups, the category of abelian groups, and the category of modules and etc. Homology, fundamental group and higher homotopy groups are most important functors from the category of spaces to the category of groups.

For example, we will know that the fundamental group of $\mathbb{R}^2 \setminus \{0\}$ is $\mathbb{Z}$ but the fundamental group of $\mathbb{R} \setminus \{0\}$ is $\{0\}$. By Theorem 1.7, we have that $\mathbb{R} \setminus \{0\}$ is not homeomorphic to $\mathbb{R}^2 \setminus \{0\}$ and so $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. This is a simple example. Actually we will be able to classify all of (2-dimensional) surfaces in this course using the fundamental group.

Let $\mathcal{C}$ be any category. A *simplicial object* over $\mathcal{C}$ means a contravariant functor $X\colon \Delta \to \mathcal{C}$. A *simplicial set* means a simplicial object over the category of sets. Similarly, we have *simplicial groups, simplicial monoids, simplicial algebras* and etc. Simplicial sets and simplicial groups are combinatorial models for spaces and topological groups, respectively, in homotopy sense. By its combinatorial means, we can say that homotopy theory studies "functors". On the other hand, by its geometric means, homotopy theory studies continuous deformations of spaces and continuous maps.

CHAPTER 2

General Topology

1. Metric spaces

Let X be a set. A *metric* d for X is a function $d\colon X \times X \to \mathbb{R}$ satisfying

1) $d(x, y) = 0$ if and only if $x = y$;
2) (triangle inequality)

$$d(x, y) + d(x, z) \geq d(y, z).$$

In this case X is called a metric space with the metric d.

PROPOSITION 2.1. *If d is a metric for X, then $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.*

PROOF. By the triangle inequality, we have

$$2d(x, y) = d(x, y) + d(x, y) \geq d(y, y) = 0,$$

$$d(x, y) = d(x, y) + d(x, x) \geq d(y, x),$$

$$d(y, x) = d(y, x) + d(y, y) \geq d(x, y).$$

Thus $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$. □

EXERCISE 1.1. a) Show that each of the following is a metric for $\mathbb{R}^n$:

$$d(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{1/2} = ||x - y||; \quad d(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y; \end{cases}$$

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|; \qquad d(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

b) Show that $d(x, y) = x - y$ does not define a metric on $\mathbb{R}$.
c) Show that $d(x, y) = \min_{1 \leq i \leq n} |x_i - y_i|$ does not define a metric on $\mathbb{R}^n$.
d) Let d be a metric. Show that d' defined by

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric.

DEFINITION 2.2. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is said to be *continuous* at $x \in X$ if for any $\epsilon_x > 0$ there exists $\delta_x > 0$ such that $d_Y(f(x), f(y)) < \epsilon_x$ whenever $d_X(x, y) < \delta_x$. The function f is said to be *continuous* if it is continuous at all points $x \in X$.

EXERCISE 1.2. Let X be a metric space with metric d. Let $y \in X$.Show that the function $f\colon X \to \mathbb{R}$ defined by $f(x) = d(x, y)$ is continuous.

Definition 2.3. A subset U of a metric space (X, d) is said to be open if for any $x \in U$ there exists $\epsilon_x > 0$ such that if $y \in X$ and $d(y, x) < \epsilon_x$ then $y \in U$.

In other words U is open if for any $x \in U$ there exists an $\epsilon_x > 0$ such that the open ball

$$B_{\epsilon_x}(x) = \{y \in X | d(y, x) < \epsilon_x\} \subseteq U.$$

Exercise 1.3. a) Show that $B_\epsilon(x)$ is always an open set for any x and any $\epsilon > 0$.

b) Which of the following subsets of $\mathbb{R}^2$ (with the usual metric) are open?

$$\{(x, y)|x^2 + y^2 < 1\} \cup \{(1, 0)\}, \quad \{(x, y)|x^2 + y^2 \leq 1\},$$
$$\{(x, y)||x| < 1\}, \quad \{(x, y)|x + y < 0\},$$
$$\{(x, y)|x + y \geq 0\}, \quad \{(x, y)|x + y = 0\}.$$

Exercise 1.4. Show that if $\mathcal{U}$ is the family of open sets arising from a metric space then

i) The empty set $\emptyset$ and the whole set belong to $\mathcal{U}$;
ii) The intersection of two members of $\mathcal{U}$ belongs to $\mathcal{U}$;
iii) The union of any number of members of $\mathcal{U}$ belongs to $\mathcal{U}$.

Theorem 2.4. *A function $f: X \to Y$ between two metric spaces is continuous if and only if for any open set U in Y the set $f^{-1}(U)$ is open in X.*

Proof. Suppose that f is continuous and U is open in Y. Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(f(x)) \subseteq U$. By the definition, there exists $\delta > 0$ such that $f(y) \in B_\epsilon(f(x))$ whenever $y \in B_\delta(x)$, that is $f(B_\delta(x)) \subseteq B_\epsilon(f(x)$. Thus

$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(x)) \subseteq f^{-1}(U)$$

and so $f^{-1}(U)$ is open.

Conversely let $x \in X$; then $B_\epsilon(f(x))$ is an open subset of Y and so $f^{-1}(B_\epsilon(f(x)))$ is an open subset of X. Thus there exists $\delta > 0$ with

$$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x))).$$

In other words $d_Y(f(x), f(y)) < \epsilon$ whenever $d_X(x, y) < \delta$, that is f is continuous. □

2. Topological Spaces

Definition 2.5. Let X be a set. A *topology* $\mathcal{U}$ for X is a collection of subsets of X satisfying

i) $\emptyset$ and X are in $\mathcal{U}$;
ii) the intersection of two members of $\mathcal{U}$ is in $\mathcal{U}$;
iii) the union of any number of members of $\mathcal{U}$ is in $\mathcal{U}$.

The set X with $\mathcal{U}$ is called a *topological space*. The members $U \in \mathcal{U}$ are called the *open sets*.

Exercise 2.1. Let $\mathcal{U}$ be a topology for X. Show that the intersection of a finite number of members of $\mathcal{U}$ is in $\mathcal{U}$.

Note: The intersection of infinitely many open sets is called a *Borel set* which is not open in general.

Let X be a metric space and let $\mathcal{U}$ be the family of open sets. Then $\mathcal{U}$ is a topology. This topology is called the *metric topology*. Note that two different metrics may give rise to the same topology.

Exercise 2.2. *Let X be a metric space with metric d. Let d' be the new metric defined in Exercise 1.1. Then (X, d) and (X, d') has the same topology.*

Given a set X there may be different choices of topologies for X.

EXERCISE 2.3. *Let $X = \{a, b\}$. Show that there are four different topologies given as follows:*

$$\mathcal{U}_1 = \{\emptyset, X\}, \mathcal{U}_2 = \{\emptyset, \{a\}, X\}, \mathcal{U}_3 = \{\emptyset, \{b\}, X\}, \mathcal{U}_4 = \{\emptyset, \{a\}, \{b\}, X\}.$$

EXERCISE 2.4. *Let X be a set. Let $\mathcal{U}_1 = \{\emptyset, X\}$, let $\mathcal{U}_2 = \mathcal{S}(X)$ be the set of all subsets of X and let*

$$\mathcal{U}_3 = \{U \subseteq X | X \setminus U \quad \text{is} \quad \text{finite}\}.$$

Show that $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_3$ are topologies for X.

$\mathcal{U}_1$ is called *indiscrete topology*, $\mathcal{U}_2$ is called *discrete topology* and $\mathcal{U}_3$ is called *finite complement topology.*

Let X be a topological space and let A be a subset of X. The largest open set contained in A, this is denoted by $\overset{\circ}{A}$ and is called the *interior* of A. For example, let $X = \mathbb{R}^n$. Then the interior of the closed ball

$$D_r(x) = \{y | d(x, y) \leq r\}$$

is the open ball $B_r(x) = \{y | d(x, y) < r\}$.

EXERCISE 2.5. *Let $X = \mathbb{R}^n$ with the usual topology. Let*

$$I^n = \overbrace{[0, 1] \times \cdots \times [0, 1]}^{n}.$$

Show that

$$\overset{\circ}{I^n} = \overbrace{(0, 1) \times \cdots \times (0, 1)}^{n}.$$

Let X be a topological space. A subset $N \subseteq X$ with $x \in N$ is called a *neighborhood* of x if there is an open set U with $x \in U \subseteq N$. For example, if X is a metric space, then the closed ball $D_\epsilon(x)$ and the open ball $B_\epsilon(x)$ are neighborhoods of x.

EXERCISE 2.6. *Let X be a topological space. Prove each of the following statements.*

a) *For each point $x \in X$ there is at least one neighborhood of x.*
b) *If N is a neighborhood of x and $N \subseteq M$ then M is also a neighborhood of x.*
c) *If M and N are neighborhoods of x then so is $N \cap M$.*
d) *For each $x \in X$ and each neighborhood N of x there exists a neighborhood U of x such that $U \subseteq N$ and U is a neighborhood of each of its points.*

DEFINITION 2.6. *A subset C of a topological space X is said to closed if $X \setminus C$ is open.*

THEOREM 2.7. i) *$\emptyset$ and X are closed;*
ii) *the union of any pair of closed sets is closed;*
iii) *the intersection of any number of closed sets is closed.*

Note: The union of infinitely many closed sets is not closed in general.

EXERCISE 2.7. *Let X be a set and let $\mathcal{V}$ be a family of subsets of X satisfying*

i) *$\emptyset, X \in \mathcal{V}$;*
ii) *the union of any pair of members of $\mathcal{V}$ belongs to $\mathcal{V}$;*
iii) *the intersection of any number of members of $\mathcal{V}$ belongs to $\mathcal{V}$.*

Show that $\mathcal{U} = \{X - V | V \in \mathcal{V}\}$ is a topology for X.

Let Y be a subset of a topological space X. The set

$$\bar{Y} = \bigcap \{F | F \supseteq Y \quad F \quad \text{is} \quad \text{closed}\}$$

is called the *closure* of Y. The set $Y' = \bar{Y} \setminus Y$ is called the set of *limit points* of Y.

PROPOSITION 2.8. *Let Y be a subset of a topological space X. Then $x \in \bar{Y}$ if and only if for every neighborhood N of x, $N \cap Y \neq \emptyset$.*

PROOF. Let $x \in \bar{Y}$ and suppose that N is a neighborhood of $x$with $N \cap Y = \emptyset$. Then there is an open neighborhood U of x with $U \subseteq N$. Thus $X \setminus U$ is a closed set and $Y \subseteq X \setminus U$. It follows that $\bar{Y} \subseteq X \setminus U$ and so $x \notin U$. One gets a contradiction.

Conversely suppose that $x \notin \bar{Y}$. Since $\bar{Y}$ is closed, $X \setminus \bar{Y}$ is an (open) neighborhood of x so that $(X \setminus \bar{Y}) \cap Y = \emptyset$ is a contradiction. □

EXERCISE 2.8. *Let $X = \mathbb{R}$ with the usual topology. Find the closure of each of the following subsets of X:*

$$A = \{1, 2, 3, \cdots\}, B = \{x | x \quad \text{is} \quad \text{rational}\}, C = \{x | x \quad \text{is} \quad \text{irrational}\}.$$

EXERCISE 2.9. Prove each of the following statements.

a) If Y is a subset of a topological space X with $Y \subseteq F \subseteq X$ and F is closed then $\bar{Y} \subseteq F$.
b) Y is closed if and only if $Y = \bar{Y}$.
c) $\bar{\bar{Y}} = \bar{Y}$.
d) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
e) $X \setminus \overset{\circ}{Y} = \overline{X \setminus Y}$.
f) $\bar{Y} = Y \cup \partial Y$ where $\partial Y = \bar{Y} \cap \overline{(X \setminus Y)}$ (∂Y is called the boundary of Y).
g) Y is closed if and only if $\partial Y \subseteq Y$.
h) $\partial Y = \emptyset$ if and only if Y is both open and closed.
i) For $a < b \in \mathbb{R}$

$$\partial(a, b) = \partial[a, b] = \{a, b\}.$$

3. Continuous Functions

DEFINITION 2.9. A function $f : X \to Y$ between two topological spaces is said to be *continuous* if for every open set U of Y the preimage $f^{-1}(U)$ is open in X.

A continuous function from a topological space to a topological space is often simply called a map. The *category of topological spaces* is defined as follows: the objects are topological spaces and the morphisms are maps, that is continuous functions.

THEOREM 2.10. *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed for any closed subset C of Y.*

PROOF. Suppose that f is continuous and let C be a closed set in Y. Then $Y \setminus C$ is an open set and so $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is an open set. It follows that $f^{-1}(C)$ is a closed set. Now suppose that $f^{-1}(C)$ is closed for any closed set C and let U be an open set. Then $Y \setminus U$ is a closed set and so $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$. Thus $f^{-1}(U)$ is an open set. □

So far we have two general methods to see whether a function is continuous or not, that is by the definition or by the theorem above. If $f: X \to Y$ is a function between metric spaces, then we can also use $\epsilon - \delta$ method to test whether f is continuous or not. As we know in calculus that the compositions of continuous functions is still continuous. This is actually true in general.

THEOREM 2.11. *Let X, Y and Z be topological spaces. If $f: X \to Y$ and $g: Y \to Z$ are continuous functions then the composite $g \circ f: X \to Z$ is continuous.*

PROOF. Let U be any open set in Z. Then $g^{-1}(U)$ is an open set in Y and so $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is an open set in X. □

DEFINITION 2.12. Let X and Y be topological spaces. We say that X and Y are homeomorphic if there exist continuous functions $f: X \to Y, g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We write $X \cong Y$ and say that f and g are *homeomorphisms* between X and Y.

By the definition, a function $f: X \to Y$ is a homeomorphism if and only if

i) f is a bijective;
ii) f is continuous and
iii) f^{-1} is also continuous.

Equivalently f is a homeomorphism if and only if 1) f is a bijective, 2) f is continuous and 3) f is an open map, that is f sends open sets to open sets. Thus a homeomorphism between X and Y is a bijective between the points and the open sets of X and Y.

A very general question in topology is how to classify topological spaces under homeomorphisms. For example, we know (from complex analysis and others) that any simple closed loop is homeomorphic to the unit circle S^1. Roughly speaking topological classification of curves is known. The topological classification of (two-dimensional) surfaces is known as well. However the topological classification of 3-dimensional manifolds (we will learn manifolds later.) is quite open. The famous Poicaré conjecture is related to this problem.

EXERCISE 3.1. Give an example of spaces X, Y and a continuous bijective $f: X \to Y$ such that f^{-1} is NOT continuous. (Hint: Give a set X. Look at the discrete topology, the indiscrete topology and the identity function.)

A *pointed space* means a topological space X together with a point $x_0 \in X$. The point x_0 is called the base point of X. We often write $*$ for x_0. Let X and Y be pointed spaces with base points x_0 and y_0, respectively. A map $f: X \to Y$ is called a *pointed map* if $f(x_0) = y_0$. The category of pointed topological spaces means a category in which the objects are pointed spaces and the morphisms are pointed maps.

4. Induced Topology

DEFINITION 2.13. Let X be a topological space and let S be a subset of X. The topology on S *induced* by the topology of X is the family of the sets of the form $U \cap S$ where U is an open set in X. We call that the subset S with induced topology is a subspace of X.

Note: By this definition, an open set V in S means $V = U \cap S$ for some open set U in X. The induced topology is also called the subspace topology.

EXERCISE 4.1. *Let X be a topological space with the topology $\mathcal{U}$ and let S be a subset of X. Show that*

$$\mathcal{U} \cap S = \{U \cap S | U \in \mathcal{U}\}$$

is a topology for S.

EXAMPLE 2.14. Let S^n be the n-sphere, that is,

$$S^n = \{x = (x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i^2 = 1\} \subseteq \mathbb{R}^{n+1}$$

with the induced topology. Then S^n is a (closed) subspace of $\mathbb{R}^{n+1}$. Note that $S^n = \mathbb{R}^{n+1} \cap S^n$ is an open set in S^n but S^n is NOT open in $\mathbb{R}^{n+1}$.

PROPOSITION 2.15. *Let S be a subspace of a topological space X. Then the inclusion function $i\colon S \to X$ is continuous.*

PROOF. Let U be an open set in X. Then $i^{-1}(U) = U \cap S$ is an open set in S. □

Note: One can show that the subspace topology is the smallest topology such that the inclusion is continuous.

PROPOSITION 2.16. *Let S be a subspace of a topological space X. Then*

1) *If S is open in X, then any open set in the subspace S is open in X;*
2) *If S is closed in X, then any closed set in the subspace S is closed in X.*

PROOF. The proofs of 1) and 2) are more or less identical. We only prove assertion 2). Let V be a closed set in S. Then $S \setminus V$ is an open set in S. By the definition, there is an open set U in X such that

$$S \setminus V = U \cap S.$$

Thus $V = (X \setminus U) \cap S$. Since S and $X \setminus U$ are closed, V is closed. □

EXERCISE 4.2. Show that

1) the subspace (a, b) of $\mathbb{R}$ is homeomorphic to $\mathbb{R}$. (Hint: Use functions like $x \to \tan(\pi(cx + d))$ for suitable c and d.)
2) the subspaces $(1, \infty), (0, 1)$ of $\mathbb{R}$ are homeomorphic. (Hint: $x \to 1/x$.)
3) $S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ is homeomorphic to $\mathbb{R}^n$ with the usual topology. (Hint: Define $\phi\colon S^n \setminus \{(0, 0, \cdots, 0, 1)\} \to \mathbb{R}^n$ by

$$\phi(x_1, x_2, \cdots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}} \cdots, \frac{x_n}{1 - x_{n+1}})$$

and $\psi\colon \mathbb{R}^n \to S^n \setminus \{(0, 0, \cdots, 0, 1)\}$ by

$$\psi(x_1, \cdots, x_n) = \frac{1}{1 + \|x\|^2}(2x_1, 2x_2, \cdots, 2x_n, \|x\|^2 - 1).)$$

A map $f\colon X \to Y$ is called an *embedding* if f is one-to-one and X is homeomorphic to the image $f(X)$ with the subspace topology. The embedding problem in topology is as follows:

> Given a topological space X. Can we embed X into $\mathbb{R}^n$ for some n? If not, can we embed X into a Hilbert space? If yes, what is the minimal number n such that X can be embedded in $\mathbb{R}^n$? This number is called the embedding number of X.

This question is important (and difficult in general) because a topological space X could be very abstract but the spaces $\mathbb{R}^n$ are much easier to be understood. For instance, the circle S^1 can embed in $\mathbb{R}^2$ but S^1 can not embed in $\mathbb{R}^1$. Thus the embedding number of S^1 is 2. Well sometimes a space X could be very simple but it could have a very complicated embedding in $\mathbb{R}^n$.

A *knot* K is a subspace of $\mathbb{R}^3$ that is homeomorphic to the circle S^1. Two knots K_1 and K_2 are *similar* if there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$. The knot theory is to study the classification of knots under this relation.

5. Quotient Topology

DEFINITION 2.17. Let $f\colon X \to Y$ be a surjective function from a topological space X to a set Y. The *quotient topology* on Y with respect to f is the family

$$\mathcal{U}_f = \{U | f^{-1}(U) \quad \text{is} \quad \text{open} \quad \text{in} \quad X\}.$$

EXERCISE 5.1. Show that $\mathcal{U}_f$ above is a topology for Y.

Note: After giving the quotient topology on Y the function $f\colon X \to Y$ is continuous.

EXAMPLE 2.18 (Projective Spaces). Let set $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are defined as follows:

$$\mathbb{R}P^n = \{l | l \quad \text{is} \quad \text{a} \quad \text{line} \quad \text{in} \quad \mathbb{R}^{n+1} \quad \text{with} \quad 0 \in l\},$$

$$\mathbb{C}P^n = \{l | l \quad \text{is} \quad \text{a} \quad \text{complex} \quad \text{line} \quad \text{in} \quad \mathbb{C}^{n+1} \quad \text{with} \quad 0 \in l\}.$$

The topologies in $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are given by the quotient topology under the quotient maps $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$ and $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$, respectively.

By this example, one can see that the quotient space Y could be much more complicated than the original space X. The following theorem gives a general method to see whether a function from Y to another space is continuous or not.

THEOREM 2.19. *Let X be a topological space and let $f\colon X \to Y$ be a surjective. Suppose that Y are given the quotient topology with respect to f. Then a function $g\colon Y \to Z$ from Y to a topological space Z is continuous if and only if the composite $g \circ f$ is continuous.*

PROOF. Suppose that g is continuous. Since f is continuous, the composite $g \circ f$ is continuous.

Now suppose that the composite $g \circ f$ is continuous. Let U be any open set in Z. Then

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$$

is open in X and so $g^{-1}(U)$ is open in Y by the definition of quotient topology. □

EXERCISE 5.2. Show that
1) $\mathbb{R}P^1 \cong S^1$;
2) $\mathbb{C}P^1 \cong S^2$.

The famous Hopf fibration is the composite

$$S^3 \hookrightarrow \mathbb{C}^2 \setminus \{0\} \twoheadrightarrow \mathbb{C}P^1 \cong S^2.$$

Let A be a subspace of a space X. The space X/A is the quotient space

$$X/A = X/\sim,$$

where $\sim$ is the equivalence relation generated by

$$a \sim b$$

for any $a, b \in A$. As a set $X/A = (X \setminus A) \cup \{*\}$, where $*$ is the equivalence class of any particular choice of elements in A. The topology in X/A is given by the quotient topology. Roughly speaking X/A is the quotient space X by pinching out A to be one point.

EXERCISE 5.3. *Show that D^n/S^{n-1} is homeomorphic to S^n.*

The canonical inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1}, \mathbb{C}^n \to \mathbb{C}^{n+1}$ given by $(x_1, \cdots, x_n) \to (x_1, \cdots, x_n, 0)$ induce the maps $\mathbb{R}P^n \to \mathbb{R}P^{n+1}$ and $\mathbb{C}P^n \to \mathbb{C}P^{n+1}$, respectively. Thus $\mathbb{R}P^n$ and $\mathbb{C}P^{n+1}$ can be considered as the subspaces of $\mathbb{R}P^{n+1}$ and $\mathbb{C}P^{n+1}$, respectively, for each n.

EXERCISE 5.4. *Show that $\mathbb{R}P^{n+1}/\mathbb{R}P^n \cong S^{n+1}$ and $\mathbb{C}P^{n+1}/\mathbb{C}P^n \cong S^{2n+1}$.*

A *fibrewise topological space* means a map $f\colon X \to Y$. In the case where f is an onto, it often called a *bundle*. For each $y \in Y$, the subspace $f^{-1}(y) \subseteq X$ is called the *fibre* at y. Let $f\colon X \to Y$ be a bundle. Then

$$X = \bigcup_{y \in Y} f^{-1}(y)$$

and so X can be considered as the union of subspaces $f^{-1}(y)$ with indexes in a topological space Y. Fibre bundles and covering spaces are special bundles. We will study covering spaces in the next chapter. The category of fibrewise topological spaces is a category in which the objects are fibrewise topological spaces and the morphisms are given by the commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\psi} & Y'. \end{array}$$

In other words, the working objects for fibrewise topology are continuous maps and the "relations" between the working objects are the diagrams above. One finds surprisingly that many results in the homotopy theory of topological spaces also holds for the homotopy theory of fibrewise topological spaces. Well the latter one is much more "abstract".

6. Product Spaces, Wedges and Smash Products

Let X and Y be topological spaces with topologies $\mathcal{U}_X$ and $\mathcal{U}_Y$, respectively. Let

$$\mathcal{U}_{X \times Y} = \{\bigcup_{\alpha} U_\alpha \times V_\alpha \subseteq X \times Y | U_\alpha \in \mathcal{U}_X, V_\alpha \in \mathcal{U}_Y\},$$

that is any member in $\mathcal{U}_{X \times Y}$ is the union of Cartesian products of open sets of X and Y.

EXERCISE 6.1. Let X and Y be topological spaces. Show that $\mathcal{U}_{X \times Y}$ is a topology for $X \times Y$.

DEFINITION 2.20. Let X and Y be topological spaces. The *(Cartesian) product* $X \times Y$ is the set $X \times Y$ with the topology $\mathcal{U}_{X \times Y}$.

EXERCISE 6.2. Show that $\mathbb{R}^2$ with the usual topology is the Cartesian product $\mathbb{R}^1 \times \mathbb{R}^1$.

THEOREM 2.21. *Let $X \times Y$ be the Cartesian product of spaces X and Y. Then a set $W \subseteq X \times Y$ is open if and only if for any $w \in W$ there exist U_w and V_w such that U_w is open in X, V_w is open in Y and $w \in U_w \times V_w \subseteq W$.*

PROOF. Let W be an open set in $X \times Y$ and let $w \in W$. Then $W = \bigcup_\alpha U_\alpha \times V_\alpha$, where U_α and V_α are open in X and Y, respectively. Thus there exists an index α such that $w \in U_\alpha \times V_\alpha$. Choose $U_w = U_\alpha$ and $V_w = V_\alpha$. Conversely let w run over all elements in W we have

$$W = \bigcup_{w \in W} U_w \times V_w$$

and so W is open.

Let $\pi_X : X \times Y \to X$, $(x, y) \to x$, and $\pi_Y : X \times Y \to Y$, $(x, y) \to y$, be the coordinate projections. Since $\pi_X^{-1}(U) = U \times Y$ and $\pi_Y^{-1}(V) = X \times V$, the coordinate projections π_X and π_Y are continuous. Let $f : Z \to X$ and $g : Z \to Y$ be any maps from a space Z to X and Y, respectively. Let $\phi : Z \to X \times Y$ be the function defined by $\phi(z) = (f(z), g(z)$. Then ϕ is the unique function such that $\pi_X \circ \phi = f$ and $\pi_Y \circ \phi = g$. □

LEMMA 2.22. *The function ϕ defined above is continuous.*

PROOF. Let U and V be open sets in X and Y, respectively. Then $\phi^{-1}(U \times V) = \{z | f(z) \in U, g(z) \in V\} = f^{-1}(U) \cap g^{-1}(V)$ is an open set in Z. Now consider any open set W in $X \times Y$. Let z be any element in $\phi^{-1}(W)$ and let $w = \phi(z)$. There exist open sets U_w and V_w such that $w \in U_w \times V_w \subseteq W$. Thus $z \in \phi^{-1}(U_w \times V_w) \subseteq \phi^{-1}(W)$ and $\phi^{-1}(W)$ is a neighborhood of each of its points. It follows that $\phi^{-1}(W)$ is open. □

By using the categorical language, Lemma 2.22 shows

THEOREM 2.23. *Let X and Y be topological spaces. Then Cartesian product $X \times Y$ is the product of X and Y in the category of topological spaces.*

THEOREM 2.24. *For any $y \in Y$, the subspace $X \times \{y\} \subseteq X \times Y$ is homeomorphic to X.*

PROOF. Let $f : X \times \{y\} \to X$ be the function defined by $f(x, y) = x$. Since f is the composite

$$f : X \times \{y\} \hookrightarrow X \times Y \xrightarrow{\pi_X} X,$$

the function f is continuous. Clearly f is a bijective. It suffices to show that f is an open map, that is f sends open sets to open sets. Suppose that W is an open set in $X \times \{y\}$. Then

$$W = (\bigcup_\alpha U_\alpha \times V_\alpha) \cap X \times \{y\}$$

for some open sets U_α and V_α in X and Y, respectively. It follows that

$$f(W) = \bigcup_{\alpha, y \in V_\alpha} U_\alpha$$

is open. □

Now we look at "infinite" Cartesian products. Let $\{X_\alpha | \alpha \in J\}$ be a set of topological spaces. Recall that the Cartesian product $\prod_{\alpha \in J} X_\alpha$ of the sets X_α is the set of collections of elements (x_α), one element x_α in each X_α. Now An open set in $\prod_{\alpha \in J} X_\alpha$ is defined to be the any union of the following sets

$$U_{\alpha_1, \cdots, \alpha_n} = \{(x_\alpha) | x_{\alpha_1} \in U_{\alpha_1}, \cdots, x_{\alpha_n} \in U_{\alpha_n}\},$$

where $a_1, \cdots, a_n$ is any finite set of elements of J. This gives the topology on the product $\prod_{\alpha \in J} X_\alpha$.

PROPOSITION 2.25. *The product topology on $\prod_{\alpha\in J} X_\alpha$ is the smallest topology such that each coordinate projection*

$$\pi_\alpha\colon \prod_{\alpha'\in J} X_{\alpha'} \to X_\alpha$$

is continuous.

PROOF. Let $\mathcal{V}$ be a topology on $\prod_{\alpha'\in J} X_{\alpha'}$ such that each coordinate projection π_α is continuous. Let U_α be an open set in X_α. Then

$$\pi^{-1}(U_\alpha) = \{(x'_\alpha) | x_\alpha \in U_\alpha\} \in \mathcal{V}. \tag{1}$$

Since the product topology is given by the any union of any finite intersections of the sets of the forms 1, it follows that the product topology is smaller than $\mathcal{V}$. □

Let X and Y be pointed spaces with base points x_0 and y_0, respectively. Then the *wedge* $X\vee Y$ of X and Y is defined to be the quotient space

$$(X\coprod Y)/\{x_0, y_0\}.$$

The topology in $X\vee Y$ is given by the quotient topology under the quotient map $q\colon X\coprod Y \to X\vee Y$. This topology can be described as follows. A subset U in $X\vee Y$ is open if and only if $q^{-1}(U)$ is open. There are two cases. If $*\notin U$, then $q^{-1}(U)$ is either an open set in X that does not contain x_0 or an open set in Y that does not contain y_0. If $*\in U$, then $q^{-1}(U) = U_1\coprod U_2$ for some open set U_1 in X that contains x_0 and some open set U_2 in Y that contains y_0. Thus

$$\mathcal{U}_{X\vee Y} = \{q(U) | x_0\notin U\in\mathcal{U}_X\} \cup \{q(V) | y_0\notin V\in \mathcal{U}_Y\} \cup \{q(U_1\coprod U_2) | x_0\in U_1\in\mathcal{U}_X, y_0\in U_2\in\mathcal{U}_Y\}.$$

PROPOSITION 2.26. *Let X and Y be pointed spaces with base points x_0 and y_0, respectively. Then $X\vee Y$ is homeomorphic to the subspace $(X\times\{y_0\})\cup(\{x_0\}\times Y\}) \subseteq X\times Y$.*

PROOF. Let $Z = (X\times\{y_0\})\cup(\{x_0\}\times Y)$ be the subspace of $X\times Y$. Let $f_X\colon X\to X\times Y$ and $f_Y\colon Y\to X\times Y$ be the maps defined by $f_X(x) = (x, y_0)$ and $f_Y(y) = (x_0, y)$. The there is a unique map $\phi\colon X\vee Y\to Z$ such that $\phi\circ f_X = i_X$ and $\phi\circ f_Y = i_Y$. Clearly ϕ is bijective. It suffices to show that ϕ is an open map. If $U\in\mathcal{U}_X$ with $x_0\notin U$, then $\phi(U) = Z\cap(U\times Y)$ is open in Z. If $V\in\mathcal{U}_Y$ with $y_0\notin V$, then $\phi(V) = Z\cap(X\times V)$ is open in Z. If $U = q(U_1\coprod U_2)$ with $x_0\in U_1\in\mathcal{U}_X$ and $y_0\in U_2\in\mathcal{U}_Y$, then $\phi(U) = Z\cap(U_1\times U_2)$. Thus q is an open map. □

Let X and Y be pointed spaces. The *smash product* $X\wedge Y$ is defined by

$$(X\times Y)/((X\times\{y_0\})\cup(\{x_0\}\times Y)).$$

We write $x\wedge y$ for elements in $X\wedge Y$, where $x\in X$ and $y\in Y$.

THEOREM 2.27. *Given three pointed spaces X, Y and Z, $(X\vee Y)\wedge Z$ is homeomorphic to $(X\wedge Z)\vee(Y\wedge Z)$.*

PROOF. The function $f\colon X\times Y\times Z\to X\times Z\times Y\times Z$, defined by $f(x,y,z) = (x,z,y,z)$, is clearly continuous. Let g be the composite

$$g\colon X\times Y\times Z \xrightarrow{f} X\times Z\times Y\times Z \xrightarrow{\text{proj.}} (X\wedge Z)\times(Y\wedge Z).$$

Then

$$g((X\vee Y)\times Z) \subseteq (X\wedge Z)\vee(Y\wedge Z).$$

Moreover, the map g sends $(X \vee Y) \vee Z$ to the base point, so that g induces a map

$$\tilde{g}\colon (X \vee Y) \wedge Z \to (X \wedge Z) \vee (Y \wedge Z),$$

where $\tilde{g}((x, y_0) \wedge z) = x \wedge z$ in $X \wedge Z$ and $g((x_0, y) \wedge z) = y \wedge z$ in $Y \wedge Z$.

Conversely, let $h\colon (X \wedge Z) \vee (Y \wedge Z) \to (X \vee Y) \wedge Z$ be the map such that $h|_{X \wedge Z}$ and $h|_{Y \wedge Z}$ are the inclusions $X \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$ and $Y \wedge Z \hookrightarrow (X \wedge Y) \wedge Z$, respectively. Then $h(x \wedge z) = (x, y_0) \wedge z$ and $h(y \wedge z) = (x_0, y) \wedge z)$ so that $\tilde{g} \circ h$ and $h \circ \tilde{g}$ are identities, and hence $\tilde{g}$ is a homeomorphism. ☐

Exercise 6.3. Show that $S^n \wedge S^m \cong S^{n+m}$ for any n, m.

7. Topological Groups and Orbit Spaces

A pointed topological space X is called an H-space of there is a continuous multiplication $\mu\colon X \times X \to X$, $(x, y) \to xy$, such that $x_0 x = x x_0 = x$. The base point x_0 is often denoted as $*$ or 1. Equivalently, a pointed space X is an H-space if and only if there is a map $\mu\colon X \times X \to X$ such that $\mu|_{X \vee X} = \nabla$, where $\nabla\colon X \vee X \to X$ is the fold map defined by $\nabla(x, x_0) = x$ and $\nabla(x_0, x) = x$.

An H-space is called *associative* if diagram

$$\begin{array}{ccc} X \times X \times X & \xrightarrow{\mu \times \mathrm{id}_X} & X \times X \\ \downarrow{\scriptstyle \mathrm{id}_X \times \mu} & & \downarrow{\scriptstyle \mu} \\ X \times X & \xrightarrow{\mu} & X \end{array}$$

is commutative. An associative H-space is called a *topological monoid.* In other words, a topological monoid is monoid as a set such that the multiplication is continuous. A *topological group* G means a topological monoid such that there is a map $\nu\colon G \to G$, $x \to x^{-1}$, with $xx^{-1} = 1 = x^{-1}x$, that is the inverse is a continuous function.

Let X be a space and let G be a topological group. We say that G *acts* on X and that X is a G-space if there is map $\mu\colon G \times X \to X$, denoted by $(g, x) \to g \cdot x$, such that

i) $1 \cdot x = x$ for all $x \in X$;
ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$, that is the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \mu} & G \times X \\ \downarrow{\scriptstyle \mu_G \times \mathrm{id}_X} & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow{\mu} & X \end{array}$$

commutes.

Theorem 2.28. *Suppose that X is a G-space. Then the function $\theta_g\colon X \to X$ given by $x \to g \cdot x$ is a homeomorphism. It follows that there is a homomorphism from G to the group of homeomorphisms of X.*

Proof. The function θ_g is the composite

$$X \cong \{g\} \times X \subseteq G \times X \xrightarrow{\mu} X.$$

Thus θ_g is continuous. From the definition of G-space we see that $\theta_g \circ \theta_h = \theta_{gh}$ and $\theta_1 = \mathrm{id}_X$. Thus $\theta_g \circ \theta_{g^{-1}} = \mathrm{id}_X = \theta_{g^{-1}} \circ \theta_g$ and so θ_g is a homeomorphism. Now the function $g \to \theta_g$ is a homomorphism from G to the group of homeomorphisms of X. □

Let X be a G-space. We can define an equivalence relation $\sim$ on X by

$$x \sim y \Leftrightarrow g \cdot x = y \text{ for some } g \in G.$$

The quotient space $X/\sim$, denoted by X/G, with the quotient topology is called the *quotient space* of X by G.

Example 2.29. 1) Let $G = \mathbb{Z}/2 = \{\pm 1\}$ with discrete topology and let $X = S^n$. The G-action on X is given by $\pm 1 \cdot x = \pm x$. Then $S^n/(\mathbb{Z}/2) \cong \mathbb{R}P^n$.

2) Let $G = \mathbb{Z}$ with the discrete topology and let $X = \mathbb{R}$. The action of G on $\mathbb{R}$ is given by $n \cdot x = n + x$. Then $\mathbb{R}/\mathbb{Z} \cong S^1$.

3) Let $G = S^1 \subseteq \mathbb{C}$. Then G is a topological group under the multiplication. Let $S^{2n-1} \subseteq \mathbb{R}^{2n} = \mathbb{C}^n$ be the unit sphere. Let G act on S^{2n-1} by

$$\alpha \cdot (z_1, z_2, \cdots, z_n) = (\alpha z_1, \alpha z_2, \cdots, \alpha z_n).$$

Then $S^{2n-1}/S^1 \cong \mathbb{C}P^n$.

4) Let M_n be the set of $n \times n$-matrices over $\mathbb{R}$. Then $M_n = \mathbb{R}^{n^2}$ is a topological space. Let

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n | \det(A) \neq 0\} \subseteq M_n$$

with the subspace topology. Then $\mathrm{GL}(n, \mathbb{R})$ is a topological group, which called the *general linear group*.

5) Let $O(n)$ be the group of (real) orthogonal $n \times n$ matrices. $O(n)$ is regarded as a subspace of $\mathbb{R}^{n^2}$ with the subspace topology. For $k \leq n$ $O(k)$ is regarded as the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

with A an orthogonal $k \times k$-matrix and I_{n-k} the $(n-k) \times (n-k)$ identity matrix. Then $O(k)$ is a topological subgroup of $O(n)$. In $O(n)$ we also have the subgroup $SO(n)$ of orthogonal matrices with determinant 1, that is $SO(n)$ is the kernel of $\det \colon O(n) \to \mathbb{Z}/2$.

6) Let $U(n)$ denote the group of $n \times n$ unitary matrices regarded as a subspace of $\mathbb{C}^{n^2}$. We have the inclusions

$$U(1) \subseteq U(2) \subseteq U(3) \subseteq \cdots \subseteq U(n) \subseteq \cdots.$$

Thus $U(k)$ is a topological subgroup of $U(n)$ for $k \leq n$. We also have the subgroup $SU(n) \subseteq U(n)$ of $n \times n$ unitary matrices with determinant 1, that is $SU(n)$ is the kernel of $\det \colon U(n) \to S^1$.

Theorem 2.30. *Suppose that X is a G-space. Then the canonical projection $\pi \colon X \to X/G$ is an open mapping.*

PROOF. Let U be an open set in X. Then

$$\pi^{-1}(\pi(U)) = \{x \in X | \pi(x) \in \pi(U)\}$$
$$= \{x \in X | x = g \cdot y \text{ for some } y \in U \text{ some } g \in G\} = \bigcup_{g \in G} g \cdot U.$$

Since $\theta_g \colon X \to X$ is a homeomorphism for each $g \in G$, $g \cdot U$ is open for each g then so $\pi^{-1}(\pi(U))$ is open and hence $\pi(U)$ is open in X/G. □

EXERCISE 7.1. 1) Let X be a G-space and define the *stabilizer* of $x \in X$ to be the subspace

$$G_x = \{g \in G | g \cdot x = x\}$$

of G. Show that G_x is a topological subgroup of G.

2) Let X be a G-space and define the *orbit* of $x \in X$ to be the subspace

$$G \cdot x = \{g \cdot x | g \in G\}$$

of X. Prove that $G \cdot x$ and $G \cdot y$ are either disjoint or equal for any $x, y \in X$.

8. Compact Spaces, Hausdorff Spaces and Locally Compact Spaces

Let X be a space. A *cover* of a subset S is a collection of subsets $\{U_j | j \in J\}$ of X such that

$$S \subseteq \bigcup_{j \in J} U_j.$$

A cover is called *finite* if the indexing set J is finite. Let $\{U_j | j \in J\}$ and $\{V_k | k \in K\}$ be covers of the subset S of X. $\{U_j | j \in J\}$ is called a *subcover* of $\{V_k | k \in K\}$ if

$$\{U_j | j \in J\} \subseteq \{V_k | k \in K\}.$$

DEFINITION 2.31. Let X be a space. A subset S is called to be *compact* if every open cover of S has a finite subcover. In particular, a space X is compact if every open cover of X has a finite subcover.

EXERCISE 8.1. Show that a subset S of a space X is compact if and only if it is compact as a space given the induced topology.

EXERCISE 8.2. Show that $[0, 1] \subseteq \mathbb{R}$ is compact.

The following theorem is useful.

THEOREM 2.32. *Let $f \colon X \to Y$ be a map. If $S \subseteq X$ is a compact subspace, then $f(S)$ is compact.*

PROOF. Suppose that $\{U_j | j \in J\}$ be an open cover of $f(S)$. Then $\{f^{-1}(U_j) | j \in J\}$ is an open cover of S. Since S is compact, there exists a finite subset K of J such that

$$S \subseteq \bigcup \{f^{-1}(U_k) | k \in K\}.$$

But $f(f^{-1}(U_k)) \subseteq U_k$ and so

$$f(S) \subseteq \bigcup \{f(f^{-1}(U_k)) | k \in K\} \subseteq \{U_k | k \in K\}$$

which is a finite subcover of $\{U_j | j \in J\}$. □

THEOREM 2.33. *A closed subset of a compact space is compact.*

PROOF. Let X be a compact space and let S be a closed subset of X. Let $\{U_j\}$ be an open cover of S. Since $S \subseteq \bigcup\{U_j|j \in J\}$ we see that

$$X \subseteq \bigcup\{U_j|j \in J\} \cup (X \setminus S)$$

and so there is a finite subcover

$$X \subseteq \bigcup\{U_k|k \in K\} \cup (X \setminus S).$$

Thus

$$S \subseteq \bigcup\{U_k|k \in K\}$$

which is a finite subcover of $\{U_j|j \in J\}$. □

THEOREM 2.34. *Let X and Y be spaces. Then X and Y are compact if and only if $X \times Y$ is compact.*

PROOF. Suppose that $X \times Y$ is compact. Since $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous, X and Y are compact. Conversely assume that X and Y are compact. Let $\{W_j|j \in J\}$ be an open cover of $X \times Y$. By definition

$$W_j = \bigcup_{k \in K(j)} (U_{j,k} \times V_{j,k})$$

where $U_{j,k}$ and $V_{j,k}$ are open in X and Y, respectively. Thus

$$X \times Y \subseteq \bigcup_{j \in J, k \in K(j)} U_{j,k} \times V_{j,k}.$$

For each $x \in X$ the subspace $\{x\} \times Y$ is compact and so there is a finite subcover

$$\{x\} \times Y \subseteq \bigcup_{i=1}^{n(x)} U_i(x) \times V_i(x).$$

Let $U'(x) = \bigcap_{i=1}^{n(x)} U_i(x)$. Then $U'(x)$ is an open neighborhood of x and

$$X \subseteq \bigcup_{x \in X} U'(x)$$

Since X is compact, there are finite points $x_1, \cdots, x_m$ such that

$$X \subseteq \bigcup_{j=1}^{m} U'(x_j).$$

It follows that

$$X \times Y \subseteq \bigcup_{1 \le j \le m, 1 \le i \le n(x_j)} U'(x_j) \times V_i(x_j) \subseteq \bigcup_{1 \le j \le m, 1 \le i \le n(x_j)} U_i(x_j) \times V_i(x_j).$$

Since for each $U_i(x_j) \times V_i(x_j)$ there is an index k such that

$$U_i(x_j) \times V_i(x_j) \subseteq W_k,$$

there is a finite subcover of $\{W_j|j \in J\}$ covering $X \times Y$. □

A space X is called *Hausdorff* if for every pair of distinct points x and y there are open sets U_x and U_y such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. In other words, X is Hausdorff if for any $x \neq y$ in X there are neighborhood $N(x)$ and $N(y)$ of x and y, respectively such that $N(x) \cap N(y) = \emptyset$. Hausdorff space is also called T_2-space. In a Hausdorff space X, any point x is a closed subset. (This is not true for general topological space. For example, the indiscrete topology.)

THEOREM 2.35. *A compact subset A of a Hausdorff space X is closed.*

PROOF. We may assume that $A \neq \emptyset$ and $A \neq X$. Given $x \in X \setminus A$. For each $a \in A$, there are disjoint open sets $U_a(x)$ and $V_a(x)$ such that $a \in U_a(x)$ and $x \in V_a(x)$. Since

$$A \subseteq \bigcup_{a \in A} U_a(x)$$

and A is compact, there are finite points $a_1, \cdots, a_m$ in A such that

$$A \subseteq \bigcup_{i=1}^{m} U_{a_i}(x).$$

Now the set $V(x) = \cap_{i=1}^{m} V_{a_i}(x)$ is an open neighborhood of x with

$$A \cap V(x) \subseteq (\bigcup_{i=1}^{m} U_{a_i}(x)) \cap V(x) = \emptyset$$

and so $V(x) \subseteq X \setminus A$, which means that $X \setminus A$ is open or A is closed. □

In particular, if X can be embedded into $\mathbb{R}^n$ then X must be Hausdorff.

THEOREM 2.36 (Heine-Borel). *A subset S of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

PROOF. Suppose that S is compact. By Theorem 2.35, S is closed. Now

$$S \subseteq \bigcup_{x \in S} B_1(x)$$

and so there exist finite points $x_1, \cdots, x_m$ in S such that

$$S \subseteq \bigcup_{i=1}^{m} B_1(x_i).$$

Thus S is bounded. Conversely suppose that S is closed and bounded. There exists positive number $r >> 0$ such that

$$S \subseteq [-r, r]^n.$$

Since $[-r, r]$ is compact, $[-r, r]^n$ is compact. By Theorem 2.33, the closed subspace S is compact. □

EXERCISE 8.3. Let X and Y be spaces. Then X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff.

Thus the spaces like n-Torus $T^n = S^1 \times S^1 \times \cdots \times S^1$ are (compact) Hausdorff.

EXERCISE 8.4. Let X and Y be topological spaces. Show that

1) If X is Hausdorff, then any subspace of X is Hausdorff;
2) X and Y are Hausdorff if and only if $X \times Y$ is Hausdorff;

3) X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x,x) \in X^2 | x \in X\}$ is a closed subset of X^2;
4) $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$ are compact Hausdorff spaces;
5) Let $f\colon X \to Y$ be a map. Suppose that X is compact Hausdorff and Y is Hausdorff. Then f is a closed map. Reduce that a bijective map from a compact Hausdorff space to a Hausdorff space is a homeomorphism.

A quotient map $f\colon X \to Y$ is also called *identification* map. A quotient space may not be Hausdorff.

THEOREM 2.37. *Let $f\colon X \to Y$ be an identification map. Suppose that X is Hausdorff. If f is closed and $f^{-1}(y)$ is compact for any $y \in Y$, then Y is Hausdorff.*

PROOF. Let y_1 and y_2 be distinct points in Y. For each $x \in f^{-1}(y_1)$ and $a \in f^{-1}(y_2)$, there exist a pair of disjoint open sets $U_{x,a}$ and $V_{x,a}$ with $x \in U_{x,a}$ and $a \in V_{x,a}$. Fixed $x \in f^{-1}(y_1)$ $\{V_{x,a} | a \in f^{-1}(y_2)\}$ is an open cover of $f^{-1}(y_2)$. By the assumption, $f^{-1}(y_2)$ is compact and so there are finite points $a_1(x), \cdots, a_{m(x)}(x)$ such that

$$f^{-1}(y_2) \subseteq \bigcup_{i=1}^{m(x)} V_{x,a_i(x)}.$$

Let $V(x) = \bigcup_{i=1}^{m(x)} V_{x,a_i(x)}$ and let $U(x) = \bigcap_{i=1}^{m(x)} U_{x,a_i(x)}$. Then $U(x)$ is an open neighborhood of x and $V(x)$ is an open neighborhood of $f^{-1}(y_2)$ with $U(x) \cap V(x) = \emptyset$. Since $f^{-1}(y_1) \subseteq \bigcup_{x \in f^{-1}(y_1)} U(x)$ and $f^{-1}(y_1)$ is compact, there are finite points $x_1, \cdots, x_s \in f^{-1}(y_1)$ such that

$$f^{-1}(y_1) \subseteq \bigcup_{j=1}^{s} U(x_j).$$

Let $U = \bigcup_{j=1}^{s} U(x_j)$ and $V = \bigcap_{j=1}^{s} V(x_j)$. Then U and V are disjoint open sets with $f^{-1}(y_1) \subseteq U$ and $f^{-1}(y_2) \subseteq V$. Since f is closed, $f(X \setminus U)$ and $f(X \setminus V)$ are closed subsets in Y and so $W_1 = Y \setminus f(X \setminus U)$ and $W_2 = Y \setminus f(X \setminus V)$ are open subsets in Y with $y_1 \in W_1$ and $y_2 \in W_2$. We show that $W_1 \cap W_2 = \emptyset$. Suppose that $y \in W_1 \cap W_2$. Then $y \notin f(X \setminus U)$ and $y \notin f(X \setminus V)$. Therfore $f^{-1}(y) \cap (X \setminus U) = \emptyset$ and $f^{-1}(y) \cap (X \setminus V) = \emptyset$.It follows that $f^{-1}(y) \subseteq U \cap V = \emptyset$ and hence $W_1 \cap W_2 = \emptyset$. □

COROLLARY 2.38. *Let X be a compact Hausdorff space. Then*

i) *If G is a finite group and X is a G-space, then X/G is a compact Hausdorff space;*
ii) *If A is closed subspace of X, then X/A is compact Hausdorff.*

EXERCISE 8.5. A space is called *normal* (T_4-space) if every point in X is closed and every pair of disjoint closed sets has disjoint open neighborhood. Let G be a compact topological group and let X be a normal G-space. Show that X/G is Hausdorff.

(Hint: Let $\pi\colon X \to X/G$ be the quotient map. For each $y \in X/G$, $\pi^{-1}(y) = G \cdot x$ for some $x \in X$ with $\pi(x) = y$. Show that the orbit $G \cdot x$ is a quotient of G. Since G is compact, the orbit $G \cdot x$ is compact and so it is closed because T_4-space is Hausdorff. Let $y_1 \neq y_2$ be distinct points in X/G. Then $\pi^{-1}(y_1)$ and $\pi^{-1}(y_2)$ are disjoint closed set and so they have disjoint open neighborhood, say U and V. By Theorem 2.30, $\pi(U)$ and $\pi(V)$ are disjoint open neighborhoods of y_1 and y_2, respectively.)

For example, $\mathbb{R}P^n$ is compact Hausdorff space because $\mathbb{R}P^n$ is the quotient of S^n by the action of $\mathbb{Z}/2$. $\mathbb{C}P^n$ is a compact Hausdorff space because it is the quotient of S^{2n-1} by S^1.

A space X is called *locally compact* if every point x in X has a compact neighborhood.

EXERCISE 8.6. Let X be a locally compact Hausdorff space. Given a point $x \in X$ and a neighborhood U of x. Show that there is an open set V such that $x \in V \subseteq \bar{V} \subseteq U$ and $\bar{V}$ is compact. (Hint: Let W be a compact neighborhood of x, that is there is an open set U_1 such that $x \in U_1 \subseteq W$ and W is compact. Let $V_1 = U_1 \cap U$. Then V_1 is an open neighborhood of x and $\bar{V}_1 \setminus V_1$ is compact because it is a closed subset of the compact space W. Let $A = \bar{V}_1 \setminus V_1$. For each $y \in A$, there exist disjoint open sets $U(y)$ and $V(y)$ such that $y \in U(y)$ and $x \in V(y)$ because X is Hausdorff.Since A is compact and $A \subseteq \bigcup_{y\in A} U(y)$, there are finite points $y_1, \cdots, y_n$ such that $A \subseteq \bigcup_{i=1}^n U(y_i)$. Let

$$V = V_1 \cap \bigcap_{i=1}^{n} V(y_i).$$

Then V is an open neighborhood of x with

1) $\bar{V} \cap A = \emptyset$ (because V is disjoint with an open neighbor hood, $\bigcup_{i=1}^n U(y_i)$, of A;
2) $\bar{V} \subseteq \bar{V}_1$ because $V \subseteq V_1$ and
3) $\bar{V}$ is compact because it is a closed subset of $\bar{V}_1$.

By 1) and 2) above, we have that $\bar{V} \subseteq V_1 \subseteq U$.

THEOREM 2.39. (a) *If $p\colon X \to Y$ is a quotient map and Z is a locally compact Hausdorff space, then $p \times \mathrm{id}_Z\colon X \times Z \to Y \times Z$ is a quotient map.*
(b) *If A is a compact subspace of a space X and $p\colon X \to X/A$ is the quotient map, then for any space Z, $p \times \mathrm{id}_Z\colon X \times Z \to (X/A) \times Z$ is a quotient map.*

PROOF. Let $\pi = p \times \mathrm{id}_Z$.
(a) Let A be a subset of $Y \times Z$ such that $\pi^{-1}(A)$ is open in $X \times Z$. We show that A is open. Let $(y_0, z_0) \in Y \times Z$. Choose $x_0 \in X$ such that $p(x_0) = y_0$.

Since $\pi^{-1}(A)$ is open and Z is locally compact, there are open sets U_1 in X and V in Z such that $\bar{V}$ is compact, $U_1 \times V$ is an open neighborhood of (x_0, z_0) and $U_1 \times \bar{V} \subseteq \pi^{-1}(A)$. The point here is that $p^{-1}(p(U_1))$ is not necessarily open in X but it contains U_1. We do the following construction.

Suppose that U_i is an open neighborhood of x_0 such that $U_i \times \bar{V} \subseteq (p \times \mathrm{id}_Z)^{-1}(A)$. We construct an open set U_{i+1} of X such that

$$p^{-1}(p(U)) \times \bar{V} \subseteq U_{i+1} \times \bar{V} \subseteq \pi^{-1}(A),$$

as follows: For each point $x \in p^{-1}(p(U_i))$ the space $\{x\} \times \bar{V}$ lies in $\pi^{-1}(A)$. Using compactness of $\bar{V}$, we choose a neighborhood W_x of x such that $W_x \times \bar{V} \subseteq \pi^{-1}(A)$. Let U_{i+1} be the union of the open sets W_x; then U_{i+1} is the desired open set of X.

Finally, let U be the union of the open sets $U_1 \subseteq U_2 \subseteq \cdots$. Then $U \times V$ is a neighborhood of (x_0, z_0) and $U \times \bar{V} \subset \pi^{-1}(A)$. Since

$$U \subseteq p^{-1}(p(U)) = p^{-1}(\bigcup_{i=1}^{\infty} p(U_i)) = \bigcup_{i=1}^{infty} p^{-1}(p(U_i)) \subseteq \bigcup_{i=1}^{\infty} U_{i+1} = U,$$

we have $p^{-1}(p(U)) = U$ and so $p(U)$ is open in Y. Thus

$$p(U) \times V = \pi(U \times V) \subseteq A$$

is a neighborhood of (x_0, z_0) lying in A, as desired.
(b) Again it suffices to show that a subset U in $X/A \times Z$ is open if $\pi^{-1}(U)$ is open in $X \times Z$. As in case (a), let $(y_0, z_0) \in U$ and let $x_0 \in X$ such that $p(x_0) = y_0$.

If $x_0 \in A$, then $A \times \{z_0\} \subseteq \pi^{-1}(U)$. Since A is compact, a similar argument to that used in case (a) shows that there exist open sets $V \subseteq X$ and $W \subseteq Z$ such that

$$A \times \{z_0\} \subseteq V \times W \subseteq \pi^{-1}(U).$$

But then $(y_0, z_0) \in p(V) \times W \subseteq U$; $p(V)$ is open since $p^{-1}(p(V)) = V$ (because $A \subseteq V$), and so $p(V) \times W$ is open.

If on the other hand $x \notin A$, there certainly exit open sets $V \subseteq X$ and $W \subseteq Z$ such that $(x_0, z_0) \in V \times W \subseteq \pi^{-1}(U)$ and if $V \cap A = \emptyset$, then $p(V) \times W$ is open. However, if $V \cap A \neq \emptyset$, then $(p(A), z_0) \in U$, and we have already seen that we can then write

$$(p(A), z_0) \in p(\tilde{V}) \times \tilde{W} \subseteq U.$$

But then $(y_0, z_0) \in p(V \cup \tilde{V}) \times (W \cap \tilde{W}) \subseteq U$; $p(V \cup \tilde{V})$ is open since $A \subseteq \tilde{V}$, and so once again (x_0, z_0) is contained in an open subset of U. It follows that U is open. □

COROLLARY 2.40. *If $p\colon A \to B$ and $q\colon C \to D$ are quotient maps and if the domain of p and the range of q are locally compact Hausdorff spaces, then*

$$p \times q\colon A \times B \to C \times D$$

is a quotient map.

PROOF. We can write $p \times q$ as the composite

$$A \times B \xrightarrow{\mathrm{id}_A \times q} A \times D \xrightarrow{p \times \mathrm{id}_D} C \times D.$$

Since each of these maps is a quotient map, so is the composite $p \times q$. □

THEOREM 2.41. *If X and Y are compact and X is Hausdorff, then $(X \wedge Y) \wedge Z$ is homeomorphic to $X \wedge (Y \wedge Z)$.*

PROOF. Write p for the various quotient maps of the form $X \times Y \to X \wedge Y$, and consider the diagram

$$\begin{array}{ccc}
X \times Y \times Z & = & X \times Y \times Z \\
\downarrow{\scriptstyle p \times \mathrm{id}_Z} & & \downarrow{\scriptstyle \mathrm{id}_X \times p} \\
(X \wedge Y) \times Z & & X \times (Y \wedge Z) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
(X \wedge Y) \wedge Z & & X \wedge (Y \wedge Z).
\end{array}$$

Since X and Y are compact, $X \vee Y$ is compact. By Theorem 2.39, the map

$$p \times \mathrm{id}_Z\colon X \times Y \times Z \to (X \wedge Y) \times Z$$

is a quotient map. Since X is locally compact and Hausdorff, again by Theorem 2.39, the map $\mathrm{id}_X \times p$ is a quotient map. It follows that both $p \circ (p \times \mathrm{id}_Z)$ and $p \circ (\mathrm{id}_X \times p)$ are quotient maps. The identity map $\mathrm{id}\colon X \times Y \times Z \to X \times Y \times Z$ induces maps

$$f\colon (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z) \text{ and}$$

$$g\colon X \wedge (Y \wedge Z) \to (X \wedge Y) \wedge Z$$

that are clearly homeomorphisms. □

EXERCISE 8.7. Show that $X \wedge (Y \wedge Z)$ is homeomorphic to $X \wedge (Y \wedge Z)$ if X and Z are locally compact and Hausdorff.

9. Mapping Spaces and Compact-open Topology

Given spaces X and Y, the *mapping space* $\mathrm{Map}(X, Y)$ consists of all (continuous) maps from X to Y. The topology in $\mathrm{Map}(X, Y)$ is given by so-called *compact-open* topology that is defined as follows.

> Let K be a compact set in X and let U be an open set in Y. Let
>
> $$W_{K,U} = \{f \in \mathrm{Map}(X, Y) | f(K) \subseteq U\}.$$
>
> The compact-open topology in $\mathrm{Map}(X, Y)$ is generated by $W_{K,U}$ where K runs over all compact subsets in X and U runs over all open sets in Y. In other words, an open set in $\mathrm{Map}(X, Y)$ is a union of a finite intersection of the subsets with the form $W_{K,U}$.

If X and Y are pointed spaces. Then pointed mapping space, denoted by Y^X or $\mathrm{Map}_*(X, Y)$, is the subspace of $\mathrm{Map}(X, Y)$ consisting of all pointed (continuous) maps, that all of maps $f\colon X \to Y$ with $f(x_0) = y_0$.

EXERCISE 9.1. Let Y be a space and let X be a space with discrete topology. Show that the compact-open topology on

$$\mathrm{Map}(X, Y) = \prod_{x \in X} Y_x,$$

where Y_x is a copy of Y, is the same as the product topology.

Let $f\colon A \to X$ and $g\colon Y \to B$ be maps. Then the function $g^f\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(A, B)$ is defined by

$$g^f(\lambda) = g \circ \lambda \circ f$$

for $\lambda\colon X \to Y$. If f and g are pointed maps, then g^f induces the map $g^f\colon \mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(A, B)$ because if $\lambda \in \mathrm{Map}_*(X, Y)$, that is λ is a pointed map, then $g^f(\lambda)$ is a pointed map.

PROPOSITION 2.42. *Let $f\colon A \to X$ and $g\colon Y \to B$ be* [pointed] *maps. Then $g^f\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(A, B)$ $[g^f\colon \mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(A, B)]$ is continuous.*

PROOF. Take a sub-basic open set $W_{K,U}$ in $\mathrm{Map}(A, B)$, where K is compact in A and U is open in B. Then

$$(g^f)^{-1}(W_{K,U}) = \{\lambda\colon X \to Y | g \circ \lambda f(K) \subseteq U\}$$
$$= \{\lambda\colon X \to Y | \lambda(f(K)) \subseteq g^{-1}(U)\} = W_{f(K), g^{-1}(U)}$$

because $f(K)$ is compact in X and $g^{-1}(U)$ is open in Y. Thus g^f is continuous. □

Let X and Y be pointed spaces. Then both $\mathrm{Map}_*(X,Y)$ and $\mathrm{Map}(X,Y)$ are pointed spaces, where the base-point is the constant map $c\colon X \to Y$, $c(x) = y_0$. Let $i\colon \{x_0\} \to X$ be the inclusion, then we have the sequence

$$\mathrm{Map}_*(X,Y) \hookrightarrow \mathrm{Map}(X,Y) \xrightarrow{\mathrm{id}_Y^i} \mathrm{Map}(\{x_0\},Y) \cong Y.$$

This sequence is called the canonical fibration for mapping spaces. Observe that $\lambda \in \mathrm{Map}_*(X,Y)$ is and only if $\mathrm{id}_Y^i(\lambda)$ is the base-point. As sets, one can see that $\mathrm{Map}(X,Y)$ is isomorphic to $\mathrm{Map}_*(X,Y) \times Y$. But as spaces $\mathrm{Map}(X,Y)$ is quite different from $\mathrm{Map}_*(X,Y) \times Y$ in general. The pointed mapping space $\mathrm{Map}_*(S^1,Y)$ is denoted by ΩY, which is called the *loop space* of Y. The mapping space $\mathrm{Map}(S^1,Y)$ is often denoted by ΛY and is call the *free loop* space of Y in many references. We will see that ΩY is actually an H-space, while ΛY is not in general. It was found in physics that some problems related to so-called *n-body* problem in physics are related to the homology of ΛY for certain spaces Y. There are many machines in algebraic topology for computing the homology of ΩY, but the determination of the homology of ΛY for many interesting spaces Y remains as interesting problems and have been studied by people.

PROPOSITION 2.43. (a) *If Z is a subspace of Y, then $\mathrm{Map}(X,Z)$ is a subspace of $\mathrm{Map}(X,Y)$;*
(b) *If Z is a pointed subspace of Y, then $\mathrm{Map}_*(X,Z)$ is a subspace of $\mathrm{Map}_*(X,Y)$.*

PROOF. The proofs of assertions (a) and (b) are similar. So we only prove assertion (a). We have to show that a set is open in $\mathrm{Map}(X,Z)$ if and only if it is the intersection with $\mathrm{Map}(X,Z)$ of a set that is open in $\mathrm{Map}(X,Y)$. Let $j\colon Z \to Y$ be the inclusion. Then $j^{\mathrm{id}_X}\colon \mathrm{Map}(X,Z) \to \mathrm{Map}(X,Y)$ is continuous, so that if $U \subseteq \mathrm{Map}(X,Y)$ is open, $U \cap \mathrm{Map}(X,Z) = (j^{\mathrm{id}_X})^{-1}(U)$ is open in $\mathrm{Map}(X,Z)$. To prove the converse, it sufficient to consider an open set in $\mathrm{Map}(X,Z)$ of the form $W_{K,U}$, where $K \subseteq X$ is compact and $U \subseteq Z$ is open. But $U = V \cap Z$ for some open set V in Y and

$$\begin{aligned} W_{K,V} \cap \mathrm{Map}(X,Z) &= \{f\colon X \to Y | f(K) \subseteq V \text{ and } f(X) \subseteq Z\} \\ &= \{f\colon X \to Z | f(K) \subseteq V \cap Z = U\} = W_{K,U}. \end{aligned}$$

That is an open set in $\mathrm{Map}(X,Z)$ is the intersection with $\mathrm{Map}(X,Z)$ of an open set in $\mathrm{Map}(X,Y)$. □

Given spaces X and Y, the *evaluation map*

$$e\colon \mathrm{Map}(X,Y) \times X \to Y$$

is defined by

$$e(\lambda, x) = \lambda(x)$$

for $x \in X$ and $\lambda\colon X \to Y$. If X and Y are pointed spaces, the restriction of e gives the evaluation map $e\colon \mathrm{Map}_*(X,Y) \times X \to Y$. If λ is the constant map or x is the base point x_0, then $e(\lambda, x) = y_0$. That is $e(\mathrm{Map}_*(X,Y) \vee X) = y_0$ and so e induces the *evaluation map*

$$e\colon \mathrm{Map}_*(X,Y) \wedge X \to Y.$$

THEOREM 2.44. *Let X and Y be pointed spaces. If X is locally compact Hausdorff, then the evaluation maps*

$$e\colon \mathrm{Map}(X,Y) \times X \to Y \text{ and}$$
$$e\colon \mathrm{Map}_*(X,Y) \wedge X \to Y$$

are continuous.

PROOF. Let U be an open set in Y and that $e(\lambda, x) = \lambda(x) \in U$. Then $x \in \lambda^{-1}(U)$ which is open in X. Since X is locally compact and Hausdorff, there exists an open set V in X such that $x \in V \subseteq \bar{V} \subseteq \lambda^{-1}(U)$, and $\bar{V}$ is compact. Consider $W_{\bar{V},U} \times V \subseteq \mathrm{Map}(X,Y) \times X$; this contains (λ, x) and if (λ', x') is another point in it, then

$$e(\lambda', x') = \lambda'(x') \in \lambda'(\bar{V}) \subseteq U.$$

Thus $W_{\bar{V},U} \times V \subseteq e^{-1}(U)$ and so $e^{-1}(U)$ is open or $e\colon \mathrm{Map}(X,Y) \times X \to Y$ is continuous. It follows that the restriction

$$e\colon \mathrm{Map}_*(X,Y) \times X \to Y$$

is continuous and so $e\colon \mathrm{Map}_*(X,Y) \wedge X \to Y$ is continuous. □

Note: The evaluation $e\colon \mathrm{Map}(X,Y) \times X \to Y$ may NOT be continuous in general. This is somewhat "not-so-good" in the category of topological spaces. Norman Steenrod then introduced "compact generated topological spaces" as a convenient category of topological spaces [**22**]. We just give the definition of compactly generated space. A space X is called *compactly generated* if X is Hausdorff and each subset A of X with the property that $A \cap C$ is closed for every compact subset C of X is itself closed. A locally compact Hausdorff space is compactly generated.

THEOREM 2.45. *Let X, Y and Z be pointed spaces. Suppose that X and Y are Hausdorff. Then*

(a) $\mathrm{Map}(X \coprod Y, Z) \cong \mathrm{Map}(X,Z) \times \mathrm{Map}(Y,Z)$;
(b) $\mathrm{Map}_*(X \vee Y, Z) \cong \mathrm{Map}_*(X,Z) \times \mathrm{Map}_*(Y,Z)$.

PROOF. We only prove assertion (b). Let x_0 and y_0 are base points of X and Y respectively, and define

$$i_X\colon X \to X \vee Y, \quad i_Y\colon Y \to X \vee Y$$

by $i_X(x) = (x, y_0)$ and $i_Y(y) = (x_0, y)$. Then i_X and i_Y are continuous. Define a function

$$\theta\colon \mathrm{Map}_*(X,Z) \times \mathrm{Map}_*(Y,Z) \to \mathrm{Map}_*(X \vee Y, Z \vee Z)$$

by $\theta(\lambda, \mu) = \lambda \vee \mu$ for $\lambda\colon X \to Z$ and $\mu\colon Y \to Z$. Consider the composites

$$\phi\colon Z^{X\vee Y} \xrightarrow{\Delta} Z^{X\vee Y} \times Z^{X\vee Y} \xrightarrow{\mathrm{id}_Z^{i_X} \times \mathrm{id}_Z^{i_Y}} Z^X \times Z^Y \quad \text{and}$$

$$\psi\colon Z^X \times Z^Y \xrightarrow{\theta} (Z \vee Z)^{X \vee Y} \xrightarrow{\nabla} Z^{X\vee Y},$$

where Δ is the diagonal map and $\nabla Z \vee Z \to Z$ is the fold map, that is $\nabla(z, z_0) = \nabla(z_0, z) = z$ for $z \in Z$. Given $\nu\colon X \vee Y \to Z$, $\phi(\nu = (\nu \circ i_X, \nu \circ i_Y)$ and given $\lambda\colon X \to Z$ and $\mu\colon Y \to Z$, $\psi(\lambda, \mu) = \nabla(\lambda \vee \mu)$. Thus $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions, and the only point that remains in showing ϕ is a homeomorphism is to show that θ is continuous.

To do so, consider the set $W_{K,U}$, where $K \subseteq X \vee Y$ is compact and $U \subseteq Z \vee Z$ is open. Now

$$\theta^{-1}(W_{K,U}) = \{(\lambda,\mu) | (\lambda \vee \mu)(K) \subseteq U\}$$

$$= \{(\lambda,\mu) | \lambda(K \cap X) \subseteq U \cap (Z \times \{z_0\} \text{ and } \mu(K \cap Y) \subseteq U \cap (\{z_0\} \times Z)\}.$$

Clearly $U_1 = U \cap (Z \times \{z_0\})$ and $U_2 = U \cap (\{z_0\} \times Z)$ are open. But since X and Y are Hausdorff, so is $X \times Y$ and hence is $X \vee Y$; thus K, X, Y are closed in $X \vee Y$, so that $K \cap X$ and $K \cap Y$ are closed and hence compact. That is,

$$\theta^{-1}(W_{K,U}) = W_{K\cap X, U_1} \times W_{K \cap Y, U_2}$$

so that θ is continuous and hence ϕ is a homeomorphism. □

Let X be a topological space and let $\mathcal{S}$ be a family of subsets of X. $\mathcal{S}$ is called a *sub-base* of open sets if any member in $\mathcal{S}$ is open and any open set in X is a union of finite intersections of members in $\mathcal{S}$. In other words if $\mathcal{S}$ is a sub-base of open sets then the topology on X is generated by $\mathcal{S}$. We are going to give a result involving $(Y \times Z)^X$ and $Y^X \times Z^X$. We need the following lemma.

LEMMA 2.46. *Let X be a Hausdorff space and let $\mathcal{S}$ be a sub-base of open sets for a space Y. Then the sets of the form $W_{K,U}$ for $K \subseteq X$ compact and $U \in \mathcal{S}$, form a sub-base of open sets for* $\mathrm{Map}(X, Y)$.

PROOF. Let $K \subseteq X$ be compact, $V \subseteq Y$ be open and let $\lambda \in W_{K,V}$. Then $V = \bigcup_\alpha V_\alpha$, where V_α is a finite intersection of members in $\mathcal{S}$, and so

$$K \subseteq \bigcup_\alpha \lambda^{-1}(V_\alpha);$$

hence, since K is compact, a finite collection of the sets $\lambda^{-1}(V_\alpha)$, say $\lambda^{-1}(V_1), \cdots, \lambda^{-1}(V_n)$, suffice to cover K. Given $x \in K$, there exists r such that $x \in \lambda^{-1}(V_r)$. Since K is a compact Hausdorff space and $K \cap \lambda^{-1}(V_r)$ is an open neighborhood of x, there exists an open set A_x in K such that

$$x \in A_x \subseteq \bar{A}_x \subseteq K \cap \lambda^{-1}(V_r).$$

Again, a finite collection of the open sets A_x will cover K, and their closures are each contained in just one set of the form $\lambda^{-1}(V_r)$. Thus by taking suitable unions of $\bar{A}_x$'s, we can write $K = \bigcup_{r=1}^n K_r$, where $K_r \subseteq \lambda^{-1}(V_r)$ and K_r is closed and so compact. It follows that

$$\lambda \in \bigcap_{r=1}^n W_{K_r,V} \subseteq W_{K,V},$$

since if $\mu(K_r) \subseteq V_r$, for each r, then $\mu(K) \subseteq \bigcup_{r=1}^n V_r \subseteq V$. But if, say, $V_r = \bigcap_{s=1}^m U_s$ for $U_s \in \mathcal{S}$, then $W_{K_r,V_r} = \bigcap_{s=1}^m W_{K_r,U_s}$. Hence λ is contained in a finite intersection of sets of the form W_{K_r,U_s} for $U_s \in \mathcal{S}$ and this intersection is contained in $W_{K,V}$. □

THEOREM 2.47. *Let X, Y and Z be pointed spaces. Suppose that X is Hausdorff. Then*

$$\mathrm{Map}(X, Y \times Z) \cong \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \quad \text{and}$$

$$\mathrm{Map}_*(X, Y \times Z) \cong \mathrm{Map}_*(X, Y) \times \mathrm{Map}_*(X, Z).$$

PROOF. We only prove that

$$\mathrm{Map}(X, Y \times Z) \cong \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z).$$

Let $p_Y : Y \times Z \to Y$ and $p_Z : Y \times Z \to Z$ be coordinate projections. Define a function

$$\theta\colon \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \to \mathrm{Map}(X \times X, Y \times Z)$$

by $\theta(\lambda, \mu) = \lambda \times \mu$ for $\lambda\colon X \to Y$ and $\mu\colon X \to Z$. Consider the composites

$$\phi\colon \mathrm{Map}(X, Y \times Z) \xrightarrow{\Delta} \mathrm{Map}(X, Y \times Z) \times \mathrm{Map}(X, Y \times Z) \xrightarrow{p_Y^{\mathrm{id}_X} \times p_Z^{\mathrm{id}_X}} \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z)$$

$$\psi\colon \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Z) \xrightarrow{\theta} \mathrm{Map}(X \times X, Y \times Z) \xrightarrow{\mathrm{id}^\Delta} \mathrm{Map}(X, Y \times Z),$$

where Δ is a diagonal map. If $\nu\colon X \to Y \times Z$, then $\phi(\nu) = (p_Y \circ \nu, p_Z \circ \nu)$ and if $\lambda\colon X \to Y$ and $\mu\colon X \to Z$, then $\psi(\lambda, \mu) = (\lambda \times \mu) \circ \Delta$. Thus $\phi \circ \psi$ and $\psi \circ \phi$ are identity functions, and it remains only to prove that θ is continuous.

Since X is Hausdorff, by Lemma 2.46, it sufficient to consider sets of the form $W_{K,U\times V}$, where $K\subseteq X\times X$ is compact and $U\subseteq Y$, $V\subseteq Z$ are open. Then

$$\theta^{-1}(W_{K,U\times V})=\{(\lambda,\mu)|(\lambda\times\mu)(K)\subseteq U\times V\}=\{(\lambda,\mu)|K\subseteq\lambda^{-1}(U)\times\mu^{-1}(V)\}.$$

But if $p_1,p_2\colon X\times X\to X$ be the first and the second coordinate projections, then $p_1(K)$ and $p_2(K)$ are compact, and $K\subseteq\lambda^{-1}(U)\times\mu^{-1}(V)$ if and only if $p_1(K)\times p_2(K)\subseteq\lambda^{-1}(U)\times\mu^{-1}(V)$. Hence

$$\theta^{-1}(W_{K,U\times V})=W_{p_1(K),V}\times W_{p_2(K),V}$$

and so θ is continuous. □

At this point we posses rules for manipulating mapping spaces analogous to the index laws $a^{b+c}=a^b\cdot a^c$ and $(a\cdot b)^c=a^c\cdot b^c$ for real numbers, and it remains to investigate what rule, if any, corresponds to the index law $a^{b\cdot c}=(a^b)^c$. Now we define the 'association map'.

Given spaces X, Y and Z, the *(unreduced) association map* is the function $\alpha\colon \mathrm{Map}(X\times Y,Z)\to\mathrm{Map}(X,\mathrm{Map}(Y,Z))$ defined by

$$[\alpha(\lambda)(x)](y)=\lambda(x,y)$$

for $x\in X$, $y\in Y$ and $\lambda\colon X\times Y\to Z$.

To justify this definition, we have to show that $\alpha(\lambda)$ is an element in $\mathrm{Map}(X,\mathrm{Map}(Y,Z))$. For a fixed x, the function $\alpha(\lambda)(x)\colon Y\to Z$ is continuous because it is the composite

$$Y\cong\{x\}\times Y\subseteq X\times Y\xrightarrow{\lambda}Z.$$

Thus at least $\alpha(\lambda)$ is a function from X to $\mathrm{Map}(Y,Z)$.

PROPOSITION 2.48. *The function $\alpha(\lambda)\colon X\to\mathrm{Map}(Y,Z)$ is continuous.*

PROOF. Consider $W_{K,U}$, where $K\subseteq Y$ is compact and $U\subseteq Z$ is open. If $x\in X$ is a point such that $\alpha(\lambda)(x)\in W_{K,U}$, then $\lambda(\{x\}\times K)\subseteq U$ or $\{x\}\times K\subseteq(\lambda)^{-1}(U)$. Since $\lambda^{-1}(U)$ is open and K is compact, there is an open set V in X such that

$$\{x\}\times K\subseteq V\times K\subseteq\lambda^{-1}(U).$$

That is

$$x\in V\subseteq(\alpha(\lambda))^{-1}(W_{K,U})$$

and so $\alpha(\lambda)$ is continuous. □

Thus the function $\alpha\colon \mathrm{Map}(X\times Y,Z)\to\mathrm{Map}(X,\mathrm{Map}(Y,Z))$ is well-defined. Now we consider the pointed case. Let X, Y and Z be pointed spaces. Let $p\colon X\times Y\to X\wedge Y$ be the quotient map. Then we have the map

$$\mathrm{id}_Z^p\colon \mathrm{Map}_*(X\wedge Y,Z)\to\mathrm{Map}_*(X\times Y,Z)\subseteq\mathrm{Map}(X\times Y,Z).$$

Clearly α maps the image of id_Z^p into the subspace

$$\mathrm{Map}_*(X,\mathrm{Map}_*(Y,Z))\subseteq\mathrm{Map}(X,\mathrm{Map}_*(Y,Z))\subseteq\mathrm{Map}(X,\mathrm{Map}(Y,Z))$$

because if $\lambda\colon X\wedge Y\to Z$, then $\lambda\circ p\colon X\times Y\to Z$ has the property that

$$\lambda\circ p|_{X\vee Y}\colon X\vee Y\to Z$$

is the constant map and so $\alpha(\lambda)(x_0)(y)=\alpha(x)(y_0)=z_0$ for any x,y. Thus the association map $\alpha\colon \mathrm{Map}(X\times Y,Z)\to\mathrm{Map}(X,\mathrm{Map}(Y,Z))$ induces the *reduced association map*

$$\bar{\alpha}\colon \mathrm{Map}_*(X\wedge Y,Z)\to\mathrm{Map}_*(X,\mathrm{Map}_*(Y,Z))$$

with

$$[\bar{\alpha}(\lambda)(x)](y) = \lambda(x \wedge y)$$

for $x \in X$, $y \in Y$ and $\lambda\colon X \wedge Y \to Z$.

PROPOSITION 2.49. *If X is Hausdorff, then the association map*

$$\alpha\colon \operatorname{Map}(X \times Y, Z) \to \operatorname{Map}(X, \operatorname{Map}(Y, Z))$$

is continuous and therefore the reduced association map

$$\bar{\alpha}\colon Z^{X \wedge Y} \to (Z^Y)^X$$

is continuous.

PROOF. By Lemma 2.46, it suffices to consider $\alpha^{-1}(W_{K,U})$, where $K \subseteq X$ is compact and $U \subseteq \operatorname{Map}(Y, Z)$ is of the form $W_{L,V}$ for $L \subseteq Y$ compact and $V \subseteq Z$ open. Now

$$\alpha^{-1}(W_{K,U} = \{\lambda | (\alpha(\lambda)(K) \subseteq W_{L,V}\} = \{\lambda | \lambda(K \times L) \subseteq V\} = W_{K \times L, V}.$$

Thus α is continuous. □

THEOREM 2.50. (a) *For all spaces X, Y and Z, the functions*

$$\alpha\colon \operatorname{Map}(X \times Y, Z) \to \operatorname{Map}(X, \operatorname{Map}(Y, Z)) \quad \text{and}$$
$$\bar{\alpha}\colon Z^{X \wedge Y} \to (Z^Y)^X$$

are one-to-one.

(b) *If Y is locally compact Hausdorff, then both α and $\bar{\alpha}$ are onto.*
(c) *If both X and Y are locally compact Hausdorff, then α is a homeomorphism.*
(d) *If both X and Y are compact and Hausdorff, then $\bar{\alpha}$ is homeomorphism.*

PROOF. (a) We only show that $\bar{\alpha}$ is one-to-one. Let $\lambda, \mu\colon X \wedge Y \to Z$ such that $\alpha(\lambda) = \alpha(\mu)$. Then for any $x \in X$ and $y \in Y$, we have

$$\lambda(x \wedge y) = [\alpha(\lambda)(x)](y) = [\alpha(\mu)(x)](y) = \mu(x \wedge y),$$

so that $\lambda = \mu$.
(b) Let $\lambda\colon X \to \operatorname{Map}(Y, Z)$ be a map. Let $\mu\colon X \times Y \to Z$ be the composite

$$X \times Y \xrightarrow{\lambda \times \mathrm{id}_Y} \operatorname{Map}(Y, Z) \times Y \xrightarrow{e} Z,$$

where e is the evaluation map. By Theorem 2.44, the evaluation e is continuous and so is μ. Clearly $\alpha(\mu) = \lambda$ and so α is onto. Now given a pointed map $\lambda'\colon X \to \operatorname{Map}_*(Y, Z)$, let $\mu'\colon X \wedge Y \to Z$ be the composite

$$X \wedge Y \xrightarrow{\lambda' \wedge \mathrm{id}_Y} \operatorname{Map}_*(Y, Z) \wedge Y \xrightarrow{e} Z,$$

where e is the evaluation. Again by Theorem 2.44 e is continuous and so is μ'. Clearly $\bar{}(\mu') = \lambda'$ and so $\bar{\alpha}$ is onto.
(c) Certainly α is continuous, one-to-one and onto, so we have only to show that the inverse to α is continuous. Let θ be the composite

$$\theta\colon \operatorname{Map}(X, \operatorname{Map}(Y, Z)) \times X \times Y \xrightarrow{e \times \mathrm{id}_Y} \operatorname{Map}(Y, Z) \times Y \xrightarrow{e} Z,$$

where e are evaluations. By Theorem 2.44, θ is continuous. By Proposition 2.48, the function

$$\alpha(\theta)\colon \operatorname{Map}(X, \operatorname{Map}(Y, Z) \to \operatorname{Map}(X \times Y, Z)$$

is continuous. Clearly $\alpha(\theta)$ is the inverse of the association map α.

(d) By Theorem 2.41, there is a homeomorphism

$$(Z^Y)^X \wedge (X \wedge Y) \cong ((Z^Y)^X \wedge X) \wedge Y.$$

Let ψ be the composite

$$(Z^Y)^X \wedge (X \wedge Y) \cong ((Z^Y)^X \wedge X) \wedge Y \xrightarrow{e \wedge \mathrm{id}_Y} Z^Y \wedge Y \xrightarrow{e} Z.$$

Then ψ is continuous and

$$\bar{\alpha}(\psi)\colon (Z^Y)^X \to Z^{X \wedge Y}$$

is the inverse to the reduced association $\bar{\alpha}$. □

Let X be a pointed space. The *n-fold loop space* $\Omega^n(X)$ of X is defined by

$$\Omega^n(X) = \mathrm{Map}_*(S^n, X).$$

EXERCISE 9.2. Let X and Y be pointed spaces. Show that $\Omega^n(X \times Y) \cong \Omega^n(X) \times \Omega^n(Y)$ and $\Omega^{n+m}(X) \cong \Omega^m(\Omega^n(X))$.

10. Manifolds and Configuration Spaces

A Hausdorff space M is called an *n-manifold* if each point of M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$.

For example, $\mathbb{R}^n$ and the n-sphere S^n is an n-manifold. A 2-dimensional manifold is called a *surface*. The objects traditionally called 'surfaces in 3-space' can be made into manifolds in a standard way. The compact surfaces have been classified as spheres or projective planes with various numbers of handles attached.

EXERCISE 10.1. Show that the real projective space $\mathbb{R}P^n$ is an n-manifold and the complex projective space $\mathbb{C}P^n$ is a $2n$-manifold.

By the definition of manifold, the closed n-disk D^n is not an n-manifold because it has the 'boundary' S^{n-1}. D^n is an example of 'manifolds with boundary'. We give the definition of manifold with boundary as follows.

A Hausdorff space M is called an *n-manifold with boundary* ($n \geq 1$) if each point in M has a neighborhood homeomorphic to an open set in the half space

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_n \geq 0\}.$$

Manifold is one of models that we can do calculus 'locally'. By means of calculus, we need local coordinate systems. Let $x \in M$. By the definition, there is a an open neighborhood $U(x)$ of x and a homeomorphism ϕ_x from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The collection $\{(U(x), \phi_x) | x \in M\}$ has the property that 1) $\{U(x) | x \in M\}$ is an open cover and 2) ϕ_x is a homeomorphism from $U(x)$ onto an open set in $\mathbb{R}^n_+$. The subspace $\phi_x(U_x)$ in $\mathbb{R}^n_+$ plays a role as a local coordinate system. The collection $\{(U(x), \phi_x) | x \in M\}$ is somewhat too large and we may like less local coordinate systems. This can be done as follows.

Let M be a space. A *chart* of M is a pair (U, ϕ) such that 1) U is an open set in M and 2) ϕ is a homeomorphism from U onto an open set in $\mathbb{R}^n_+$. An *atlas* for M means a collection of charts $\{(U_\alpha, \phi_\alpha) | \alpha \in J\}$ such that $\{U_\alpha | \alpha \in J\}$ is an open cover of M.

PROPOSITION 2.51. *A Hausdorff space M is a manifold (with boundary) if and only if M has an atlas.*

PROOF. Suppose that M is a manifold. Then the collection $\{(U(x), \phi_x)|x \in M\}$ is an atlas. Conversely suppose that M has an atlas. For any $x \in M$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x that is homeomorphic to an open set in $\mathbb{R}^n_+$. Thus M is a manifold. $\square$

We define a subset ∂M as follows: $x \in \partial M$ if there is a chart (U_α, ϕ_α) such that $x \in U_\alpha$ and $\phi_\alpha(x) \in \mathbb{R}^{n-1} = \{x \in \mathbb{R}^n | x_n = 0\}$. ∂M is called the boundary of M. For example the boundary of D^n is S^{n-1}.

PROPOSITION 2.52. *Let M be a n-manifold with boundary. Then ∂M is an $(n-1)$-manifold without boundary.*

PROOF. Let $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ be an atlas for M. Let $J' \subseteq J$ be the set of indices such that $U_\alpha \cap \partial M \neq \emptyset$ if $\alpha \in J'$. Then Clearly

$$\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M} | \alpha \in J'\}$$

can be made into an atlas for ∂M. $\square$

DEFINITION 2.53. A Hausdorff space M is called a *differential manifold of class C^k* if there is an atlas of M

$$\{(U_\alpha, \phi_\alpha | \alpha \in J\}$$

such that

For any $\alpha, \beta \in J$, the composites

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$

is differentiable of class C^k.

The atlas $\{(U_\alpha, \phi_\alpha | \alpha \in J\}$ is called a *differential atlas of class C^k* on M.

Two differential atlases of class C^k $\{(U_\alpha, \phi_\alpha)|\alpha \in I\}$ and $\{(V_\beta, \psi_\beta)|\beta \in J\}$ are called *equivalent* if

$$\{(U_\alpha, \phi_\alpha)|\alpha \in I\} \cup \{(V_\beta, \psi_\beta)|\beta \in J\}$$

is again a differential atlas of class C^k (this is an equivalence relation). A *differential structure of class C^k* on M is an equivalence class of differential atlases of class C^k on M. Thus a differential manifold of class C^k means a manifold with a differential structure of class C^k. A *smooth* manifold means a differential manifold of class C^∞.

Note: A general manifold is also called *topological manifold*. Kervaire and Milnor [**11**] have shown that the topological sphere S^7 has 28 distinct oriented smooth structures.

Let M be a smooth manifold and let $\{(U_\alpha, \phi_\alpha)|\alpha \in J\}$ be a C^∞-atlas for M. For $\alpha, \beta \in J$, the function

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n_+$$

is a smooth map from an open set in $\mathbb{R}^n_+$ to an open set in $\mathbb{R}^n_+$. The Jacobian matrix

$$M_{\alpha\beta}(x) = \left(\left. \frac{\partial(\phi_\alpha \circ \phi_\beta^{-1})_i}{\partial x_j} \right|_{\phi_\beta(x)} \right)$$

is invertible for any $x \in U_\alpha \cap U_\beta$. A smooth manifold M is called *oriented* if there is an C^∞-atlas $\{(U_\alpha \phi_\alpha | \alpha \in J\}$ for M such that the determinant of the Jacobian

$$\det(M_{\alpha\beta}(x)) > 0$$

for any $\alpha, \beta \in J$ and $x \in U_\alpha \cap U_\beta$. For example $\mathbb{R}P^n$ is oriented if and only if n is odd. On the other hand $\mathbb{C}P^n$ is oriented for any n.

DEFINITION 2.54. let M and N be smooth manifolds of dimensions m and n respectively. A map $f\colon M \to N$ is called *smooth* if for some smooth atlases $\{(U_\alpha, \phi_\alpha|\alpha \in I\}$ for M and $\{(V_\beta, \psi_\beta)\beta \in J\}$ for N the functions

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}|_{\phi_\alpha(f^{-1}(V_\beta)\cap U_\alpha)}\colon \phi_\alpha(f^{-1}(V_\beta) \cap U_\alpha) \to \mathbb{R}^n_+$$

are of class C^∞.

PROPOSITION 2.55. *If $f\colon M \to N$ is smooth with respect to atlases*

$$\{(U_\alpha, \phi_\alpha|\alpha \in I\}, \quad \{(V_\beta, \phi_\beta|\beta \in J\}$$

for M, N then it is smooth with respect to equivalent atlases

$$\{(U'_\delta, \theta_\delta|\alpha \in I'\}, \quad \{(V'_\gamma, \eta_\gamma|\beta \in J'\}$$

PROOF. Since f is smooth with respect with the atlases

$$\{(U_\alpha, \phi_\alpha|\alpha \in I\}, \ \{(V_\beta, \phi_\beta|\beta \in J\},$$

f is smooth with respect to the smooth atlases

$$\{(U_\alpha, \phi_\alpha|\alpha \in I\} \cup \{(U'_\delta, \theta_\delta|\alpha \in I'\}, \quad \{(V_\beta, \phi_\beta|\beta \in J\} \cup \{(V'_\gamma, \eta_\gamma|\beta \in J'\}$$

by look at the local coordinate systems. Thus f is smooth with respect to the atlases

$$\{(U'_\delta, \theta_\delta|\alpha \in I'\}, \ \{(V'_\gamma, \eta_\gamma|\beta \in J'\}.$$

□

Thus the definition of smooth maps between two smooth manifolds is independent of choice of atlas.

Let M be a m-manifold. The *(ordered) configuration space* $F(M, n)$ is defined by

$$F(M, n) = \{(x_1, \cdots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}.$$

In other words, the configuration space $F(M, n)$ is the subspace of the Cartesian product M^n by removing the 'flat' diagonals. The symmetric group Σ_n acts on $F(M, n)$ by permuting coordinates. The *(unordered) configuration space* $B(M, n)$ is the quotient of $F(M, n)$ by Σ_n, that is

$$B(M, n) = F(M, n)/\Sigma_n.$$

Clearly both $F(M, n)$ and $B(M, n)$ are mn-manifolds. configuration spaces are arisen from many areas in mathematics and physics. In geometry and physics, the diagonals play as singularities in many cases and so we have to remove them, then this gives the configuration space. In combinatorics, the homology of configuration spaces is related to 'subspace arrangements'. The determination of the homology of $F(M, n)$ and $B(M, n)$ still remains open for general manifold M though it is known for many cases. The fundamental groups of configuration spaces are interesting as well. A typical example is that the fundamental group of $F(\mathbb{R}^2, n)$ is the pure braid group K_n and the fundamental group of $B(\mathbb{R}^2, n)$ is the Artin braid group B_n. The braid groups are important in group theory, low dimensional topology and mathematical physics. In homotopy theory, configuration spaces are used to construct various combinatorial models for mapping spaces. (As we have seen that mapping spaces are quite complicated, the construction means that we construct certain 'simpler spaces' that has the same homotopy groups and homology groups of a mapping space. So if one needs to know the homotopy groups and homology groups of a complicated mapping space, one may look at these simpler spaces.)

CHAPTER 3

Elementary Homotopy Theory

1. Homotopy Sets

1.1. Homotopy Relative to a Subspace. The problem of classifying topological spaces and continuous maps up to topological equivalence (homeomorphism) does not seem to be amenable to attack directly by computable algebraic functors. Many of the computable functors, because they are computable, are invariant under continuous deformation. Therefore they cannot distinguish between spaces (or maps) that can be continuously deformed from one to the other; the most that can be hoped for from such functors is that they characterize the space (or map) up to continuous deformation.

The intuitive concept of a continuous deformation will be made precise in this section in the concept of homotopy. This leads to the homotopy category which is fundamental for algebraic topology. Its objects are topological spaces and its morphisms are equivalence classes of continuous maps (two maps being equivalent of one can be continuously deformed into the other).

Roughly speaking two continuous maps $f_0, f_1\colon X \to Y$ are said to be homotopic if there is an intermediate family of maps $f_t\colon X \to Y$ for $0 \leq t \leq 1$ which vary continuously with respect to t. Let $I = [0,1]$.

Definition 3.1. Let $f, g\colon X \to Y$ be two maps. We say that f is homotopic to g if there is a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for any $x \in X$. The map F is called a *homotopy* between f and g. We write $f \simeq g$ or $F\colon f \simeq g$.

For each $0 \leq t \leq 1$, we denote $F(x,t)$ by $F_t(x)$. So gives a family of maps $F_t\colon X \to Y$. Just keep in mind that F_t is continuous in t as a map from I to $\mathrm{Map}(X,Y)$. A map $f\colon X \to Y$ is called *null homotopic* if f is homotopic to a constant map.

Definition 3.2. Suppose that A is a subset of X and that $f, g\colon X \to Y$ are maps. We say that f is *homotopic to g relative* to A, denoted $f \simeq g \operatorname{rel} A$ or $F\colon f \simeq g \operatorname{rel} A$, if there is a homotopy $F\colon X \times I \to Y$ such that

1) $F(x,0) = f(x)$ for any $x \in X$;
2) $F(x,1) = g(x)$ for any $x \in X$ and
3) $F(a,t) = f(a)$ for any $a \in A$ and $t \in I$.

We say that f is *null homotopic relative to* A if f is homotopic to a constant map relative to A.

Note that $f(a) = g(a)$ for all $a \in A$ if $f \simeq g \operatorname{rel} A$. When $A = \emptyset$, $f \simeq g \operatorname{rel} A$ is equivalent to $f \simeq g\colon X \to Y$. Given two maps $f, g\colon X \to Y$ such that $f(a) = g(a)$ for $a \in A$. The question whether f is homotopic to g relative to A is in fact an 'extension question' by the following diagram

$$\begin{array}{ccc} X\times\{0\}\cup X\times\{1\}\cup A\times I & \hookrightarrow & X\times I \\ \Big\| & & \Big\downarrow{\scriptstyle F} \\ X\times\{0\}\cup X\times\{1\}\cup A\times I & \xrightarrow{\ \phi\ } & Y, \end{array}$$

where $\phi|_{X\times\{0\}} = f$, $\phi|_{X\times\{1\}} = g$ and $\phi(a,t) = f(a) = g(a)$ for $a \in A$ and $t \in I$. In other words such an extension F exists if and only if f is homotopic to g relative to A.

THEOREM 3.3. *Homotopy relative to A is an equivalence relation in the set of maps from X to Y.*

PROOF. *Reflexivity.* For $f\colon X \to Y$, define $F\colon X\times I \to Y$ by $F(x,t) = f(x)$. Thus $f \simeq a \operatorname{rel} A$.
Symmetry. Given $F\colon f \simeq g \operatorname{rel} A$, define $F'\colon g \simeq f \operatorname{rel} A$ by

$$F'(x,t) = F(x, 1-t).$$

Transitivity. Given $F\colon f \simeq g \operatorname{rel} A$ and $G\colon g \simeq h \operatorname{rel} A$, define $H\colon f \simeq h \operatorname{rel} A$ by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1. \end{cases}$$

Note that H is continuous because its restriction to each of closed sets $X\times[0,1/2]$ and $X\times[1/2,1]$ is continuous. □

It follows that the set of maps from X to Y is partitioned into disjoint equivalence classes by the relation of homotopy relative to A. These equivalence classes are called *homotopy classes relative to* A. We use the notation $[X,Y]_A$ to denote this set of homotopy classes. Given $f\colon X \to Y$, we use $[f]_A$ to denote the element in $[X,Y]_A$ determined by f. For (unpointed) spaces X and Y, the notation $[X,Y]$ usually means $[X,Y]_\emptyset$.

THEOREM 3.4. *Let A and B be subspaces of X and Y respectively. Let $f_0, f_1\colon X \to Y$ be homotopic relative to A and $g_0, g_1\colon Y \to Z$ be homotopic relative to B such that $f_1(A) \subseteq B$. Then* $g_0\circ f_0 \simeq g_1\circ f_1 \operatorname{rel} A$.

PROOF. Let $F\colon f_0 \simeq f_1 \operatorname{rel} A$ and $G\colon g_0 \simeq g_1 \operatorname{rel} B$. Then the composite

$$X\times I \xrightarrow{F} Y \xrightarrow{g_0} Z$$

is a homotopy relative to A from $g_0\circ f_0$ to $g_0\circ f_1$, and the composite

$$X\times I \xrightarrow{f_1\times \mathrm{id}_I} Y\times I \xrightarrow{G} Z$$

is a homotopy relative to $f_1^{-1}(B)$ from $g_0\circ f_1$ to $g_1\circ f_1$. Since $A \subseteq f_1^{-1}(B)$, we have shown that$g_0\circ f_0 \simeq g_0\circ f_1 \operatorname{rel} A$ and $g_0\circ f_1 \simeq g_1\circ f_1 \operatorname{rel} A$. The result follows from Theorem 3.3. □

1.2. Pointed Homotopy. A *pointed space* means a topological space X with a fixed choice of base-point x_0. Let X and Y be pointed spaces with base-points x_0 and y_0, respectively. A *pointed map* $f\colon X \to Y$ means a continuous map $f\colon X \to Y$ such that $f(x_0) = y_0$.

Let X and Y be pointed spaces and let $f, g\colon X \to Y$ be pointed maps. f is called (pointed) *homotopic* to g is $f \simeq g \operatorname{rel} x_0$, where x_0 is the base point. If there is no confusion, we simply denote $f \simeq g$ for $f \simeq g \operatorname{rel} x_0$ (in pointed case). For pointed spaces, the notation $[X, Y]$ means the set of equivalence classes of pointed maps from X to Y by the relation of homotopy relative to the base point x_0. For any pointed map f, $[f]$ means the homotopy class determined by f. The homotopy category of pointed spaces means the category in which objects are pointed spaces and morphisms are homotopy classes $[f]$. The composition in the homotopy category of pointed spaces is defined by $[f] \circ [g] = [f \circ g]$. Theorem 3.4 shows that this is a well-defined composition operation.

Definition 3.5. Let X be a pointed space. The *n-homotopy group* $\pi_n(X)$ is defined by

$$\pi_n(X) = [S^n, X]$$

for $n \geq 0$.

Note: $\pi_0(X)$ is NOT a group in general. $\pi_1(X)$ is also called the *fundamental group* of X. We will show that the fundamental group $\pi_1(X)$ is a group for any X (but non-commutative in general). We will also show that $\pi_n(X)$ is an abelian group for $n \geq 2$.

Theorem 3.6. *A pointed map $f\colon Y_1 \to Y_2$ gives rise to a function*

$$f_*\colon [X, Y_1] \to [X, Y_2]$$

for any pointed space X with the following properties:

1) *If $f'\colon Y_1 \to Y_2$ is another map, and $f' \simeq f$, then $f_* = f'_*$;*
2) *If $\mathrm{id}\colon Y \to Y$ is the identity map, then $\mathrm{id}_*\colon [X, Y] \to [X, Y]$ is the identity function;*
3) *If $g\colon Y_2 \to Y_3$ is another map, then*

$$(g \circ f)_* = g_* \circ f_*.$$

Proof. Let $[\lambda] \in [X, Y_1]$ be the homotopy class of a map $\lambda\colon X \to Y$. Define

$$f_*([\lambda]) = [f \circ \lambda] \in [X, Y_2].$$

The function f_* is well-defined because if $\lambda'\colon X \to Y_1$ is another map with $[\lambda'] = [\lambda]$, that is $\lambda' \simeq \lambda$, then $[f \circ \lambda'] = [f \circ \lambda]$ by Theorem 3.4. Properties 1 to 3 follow immediately from the definition and Theorem 3.4. □

Let $f\colon X_1 \to X_2$ be any pointed map. Define

$$f^*\colon [X_2, Y] \to [X_1, Y]$$

by

$$f^*([\lambda]) = [\lambda \circ f]$$

for any pointed map $\lambda\colon X_2 \to Y$. By the similar arguments, we have

Theorem 3.7. *A pointed map $f\colon X_1 \to X_2$ gives rise to a function*

$$f^*\colon [X_2, Y] \to [X_1, Y]$$

for any pointed space Y with the following properties:

1) *If $f'\colon X_1 \to X_2$ is another map, and $f' \simeq f$, then $f^* = f'^*$;*

2) *If* $\mathrm{id}\colon X \to X$ *is the identity map, then* $\mathrm{id}^*\colon [X,Y] \to [X,Y]$ *is the identity function;*
3) *If* $g\colon X_2 \to X_3$ *is another map, then*

$$(g \circ f)^* = f^* \circ g^*.$$

Let X and Y be pointed spaces with base points x_0 and y_0, respectively. Then the *wedge* $X \vee Y$ of X and Y is defined to be the quotient space

$$(X \coprod Y)/\{x_0, y_0\}.$$

The topology in $X \vee Y$ is given by the quotient topology under the quotient map $q\colon X \coprod Y \to X \vee Y$. Note from general topology that $X \vee Y \cong (X \times \{y_0\}) \cup (\{x_0\} \times Y\})$ as a subspace of $X \times Y$.

Let X and Y be pointed spaces. The *smash product* $X \wedge Y$ is defined by

$$(X \times Y)/((X \times \{y_0\}) \cup (\{x_0\} \times Y)).$$

We write $x \wedge y$ for elements in $X \wedge Y$, where $x \in X$ and $y \in Y$.

Theorem 3.8. *Let* X, Y *and* Z *be pointed spaces. Then*
1) $[X \vee Y, Z] \cong [X,Z] \times [Y,Z]$ *and*
2) $[X, Y \times Z] \cong [X,Y] \times [X,Z]$.

Proof. 1) Let

$$\theta\colon [X \vee Y, Z] \to [X,Z] \times [Y,Z]$$

be defined by

$$\theta([\lambda]) = (i_X^*([\lambda], i_Y^*([\lambda]))$$

for any $[\lambda] \in [X \vee Y \to Z]$, where $i_X\colon X \to X \vee Y$ and $i_Y\colon Y \to X \vee Y$ be the inclusions. We first show that θ is onto. For any $([\lambda_1],[\lambda_2]) \in [X,Z] \times [Y,Z]$, where $\lambda_1\colon X \to Z$ and $\lambda_2\colon Y \to Z$ are pointed maps. Then there is a unique pointed map $\lambda\colon X \vee Y \to Z$ such that $\lambda|_X = \lambda_1$ and $\lambda|_Y = \lambda_2$ and so

$$\theta([\lambda]) = ([\lambda_1],[\lambda_2])$$

or θ is onto. Now we show that θ is one-to-one. Let $\lambda, \lambda'\colon X \vee Y \to Z$ such that $\theta([\lambda]) = \theta([\lambda'])$. Then $i_X^*([\lambda]) = i_X^*([\lambda'])$ and $i_Y^*([\lambda]) = i_Y^*([\lambda'])$ that is there are pointed homotopies

$$F\colon \lambda|_X \simeq \lambda'_X \quad \text{and}$$

$$G\colon \lambda|_Y \simeq \lambda'|_Y$$

and so the map $H\colon (X \vee Y) \times I \to Z$ defined by

$$H(x,t) = \begin{cases} F(x,t) & \text{for} \quad x \in X \\ G(x,t) & \text{for} \quad x \in Y \end{cases}$$

is a homotopy from λ to λ'.

2). Let

$$\theta\colon [X, Y \times Z] \to [X,Y] \times [X,Z]$$

be the function defined by

$$\theta([\lambda]) = (p_{Y*}([\lambda]), p_{Z*}(\lambda]))$$

for any $\lambda\colon X \to Y \times Z$. Similar arguments show that θ is one-to-one and onto. □

COROLLARY 3.9. *Let X and Y be a pointed space. Then*

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$$

for each $n \geq 0$.

Given spaces X and Y, the *mapping space* $\mathrm{Map}(X, Y)$ consists of all (continuous) maps from X to Y. The topology in $\mathrm{Map}(X, Y)$ is given by so-called *compact-open* topology that is defined as follows.

> Let K be a compact set in X and let U be an open set in Y. Let
>
> $$W_{K,U} = \{f \in \mathrm{Map}(X, Y) | f(K) \subseteq U\}.$$
>
> The compact-open topology in $\mathrm{Map}(X, Y)$ is generated by $W_{K,U}$ where K runs over all compact subsets in X and U runs over all open sets in Y. In other words, an open set in $\mathrm{Map}(X, Y)$ is a union of a finite intersection of the subsets with the form $W_{K,U}$.

If X and Y are pointed spaces. Then pointed mapping space, denoted by Y^X or $\mathrm{Map}_*(X, Y)$, is the subspace of $\mathrm{Map}(X, Y)$ consisting of all pointed (continuous) maps, that all of maps $f\colon X \to Y$ with $f(x_0) = y_0$.

THEOREM 3.10. *Let X, Y and Z be pointed spaces. If Y is locally compact and Hausdorff, then the association map*

$$\bar{\alpha}\colon \mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$

induces and one-to-one correspondence

$$\bar{\alpha}_*\colon [X \wedge Y, Z] \xrightarrow{\cong} [X, \mathrm{Map}_*(Y, Z)].$$

PROOF. Let $p\colon X \times Y \to X \wedge Y$ be the quotient map. Suppose that $F\colon (X \wedge Y) \times I$ be a pointed homotopy between maps $f, g\colon X \wedge Y \to Z$. Then the map

$$F \circ (p \times \mathrm{id}_I)\colon X \times Y \times I \to Z$$

sends $X \times \{y_0\} \times I$ and $\{x_0\} \times Y \times I$ to z_0 and so induces a map $F'\colon (X \times I) \wedge Y \to Z$. Then $\alpha(F')\colon X \times I \to Z^Y$ sends $x_0 \times I$ to the base-point and is clearly a homotopy between $\bar{\alpha}_*(f)$ and $\bar{\alpha}_*(g)$. Thus $\bar{\alpha}_*\colon [X \wedge Y, Z] \to [X, Z^Y]$ is a well-defined function (though $\bar{\alpha}$ may not be continuous in general).

By assertion (b) of Theorem 2.50, the function

$$\bar{\alpha}\colon Z^{X \wedge Y} \to (Z^Y)^X$$

is onto and so $\bar{\alpha}_*\colon [X \wedge Y, Z] \to [X, Z^Y]$ is onto.

Now we show that $\bar{\alpha}_*$ is one-to-one. Let $f, g\colon X \wedge Y \to Z$ such that $\bar{\alpha}(f) \simeq \bar{\alpha}(g)$, that is there is a pointed homotopy $F\colon X \times I \to Z^Y$ such that $F_0 = f$ and $F_1 = g$. By assertion (b) of Theorem 2.50, there is a map $F'\colon (X \times I) \wedge Y \to Z$ such that $\bar{\alpha}(F') = F$. Let $q\colon X \times Y \times I \to (X \times I) \wedge Y$ be the quotient map defined by $q(x, y, t) = (x, t) \wedge y$. Then $F' \circ q$ sends $X \times \{y_0\} \times I$ and $\{x_0\} \times Y \times I$ to z_0. BY Theorem 2.39, the map $(X \times Y) \times I \to (X \wedge Y) \times I$ is a quotient map because I is locally compact and Hausdorff. Thus $F \circ q$ induces a map $F''\colon (X \wedge Y) \times I \to Z$, which is clearly a pointed homotopy between f and g. □

COROLLARY 3.11. *Let X be a pointed space. Then*

$$\pi_n(X) \cong \pi_0(\Omega^n X) \cong \pi_1(\Omega^{n-1} X)$$

for any $n \geq 1$.

1.3. Path Connected Components. Let X be a topological space. A *path* in X means a continuous map $\lambda\colon I \to X$. $\lambda(0)$ is called the *initial point* and $\lambda(1)$ is called the it final or it terminal point. Clearly, a path in X is a homotopy from one point space to X. Given a space X, define an equivalence relation by $x \simeq y$ if there is a path in X joining x and y. Let x be any point in X. The *path-connected component* of X that contains x is defined to be the subspace

$$\{y \in X | y \simeq x\} \subseteq X.$$

A space X is called *path-connected* if X has only one path-connected component. In other words, X is path-connected if for any two points x, y in X there is a path joining x and y. By Theorem 3.3, $\simeq$ is an equivalence relation on X and so X is a disjoint union of its path-connected components. Let $X/\simeq$ be the set of equivalence classes of X by $\simeq$.

EXERCISE 1.1. Let $f\colon X \to Y$ be a map. If X is path-connected, then the image $f(X)$ is path-connected.

EXERCISE 1.2. Let X be a non-empty space and let x_0 be any point in X which is regarded as the base point. Then

$$\pi_0(X) \cong (X/\simeq).$$

In particular, X is path-connected if and only if $\pi_0(X)$ is the one-point set $\{0\}$.

EXERCISE 1.3. Let X and Y be topological spaces. Then X and Y are path-connected if and only if $X \times Y$ is path-connected. (Hint: $\pi_0(X \times Y) \cong \pi_0(X) \times \pi_0(Y)$.)

2. Homotopy Equivalences and Contractible Spaces

A map $f\colon X \to Y$ is called an *homotopy equivalence* if there is a map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. The map g is called a homotopy inverse of f. A space X is called *homotopy equivalent* to Y if there is a homotopy equivalence between X and Y. In this case, we call that X has the same homotopy type of Y, denoted by $X \simeq Y$.

DEFINITION 3.12. *A space X is called contractible if the identity map is homotopic to some constant map from X to itself.*

PROPOSITION 3.13. *Any two maps of an arbitrary space to a contractible space are homotopic.*

PROOF. Let Y be a contractible space and suppose that $\mathrm{id}_Y \simeq c$, where $c\colon Y \to Y$ is a constant map. Let $f_0, f_1\colon X \to Y$ be any two maps. Then

$$f_0 = \mathrm{id}_Y \circ f_0 \simeq c \circ f_0 = c \circ f_1 \simeq \mathrm{id}_Y \circ f_1 = f_1$$

and so $f_0 \simeq f_1$. □

COROLLARY 3.14. *If Y is a contractible space, then any two constant maps of Y to itself are homotopic, and the identity map is homotopic to any constant map of Y to itself.*

EXERCISE 2.1. Show that any vector space V over $\mathbb{R}$ is contractible. (Hint: Check that the map $F\colon V \times I \to V$, $(x, t) \to (1 - t)x$, is a homotopy between the identity map and a constant map.)

THEOREM 3.15. *A space is contractible if and only if it has the same homotopy type as a one-point space.*

Proof. Assume that X is contractible and let F be a homotopy between the identity map and a constant map $c\colon X \to X$, $x \to x_0$. Let P be the one-point space $\{x_0\}$ and let $f\colon X \twoheadrightarrow P$ and $j\colon P \subseteq X$. Then $f \circ j = \mathrm{id}_P$ and $F\colon \mathrm{id}_X \simeq j \circ f$. Thus f is a homotopy equivalence from X to P.

Conversely, if X has the same homotopy type as a one-point space P, let $f\colon X \to P$ be a homotopy equivalence with homotopy inverse $g\colon P \to X$. Then $\mathrm{id}_X \simeq g \circ f$. Because $g \circ f$ is a constant map, X is contractible. □

Corollary 3.16. *Any two contractible spaces have the same homotopy type, and any continuous map between contractible spaces is a homotopy equivalence.*

Proof. Let X and Y be two contractible spaces. Let P be a one-point space. Then $X \simeq P \simeq Y$ and so $X \simeq Y$. The second part follows from Proposition 3.13. □

Theorem 3.17. *Let p_0 be any point of S^n and let $f\colon S^n \to Y$. The following are equivalent:*

(a) *f is null homotopic;*
(b) *f can be continuously extended over D^{n+1};*
(c) *f is null homotopic relative to p_0.*

Proof. $(a) \Rightarrow (b)$. Let $F\colon f \simeq c$, where c is the constant map of S^n to $y_0 \in Y$. Define an extension $\tilde{f}$ of f over E^{n+1} by

$$\tilde{f}(x) = \begin{cases} y_0 & 0 \leq \|x\| \leq 1/2 \\ F(x/\|x\|, 2-2\|x\|) & 1/2 \leq \|x\| \leq 1. \end{cases}$$

Since $F(x,1) = y_0$ for all $x \in S^n$, the map $\tilde{f}$ is well-defined. $\tilde{f}$ is continuous because its restriction to each of the closed sets $\{x \in E^{n+1} | 0 \leq \|x\| \leq 1/2\}$ and $\{x \in E^{n+1} | 1/2 \leq \|x\| \leq 1\}$ is continuous. Since $F(x,0) = f(x)$ for $x \in S^n$, $\tilde{f}|_{S^n} = f$ and $\tilde{f}$ is a continuous extension of f to D^{n+1}.

$(b) \Rightarrow (c)$. If f has the continuous extension $\tilde{f}\colon E^{n+1} \to Y$, define $F\colon S^n \times I \to Y$ by

$$F(x,t) = \tilde{f}((1-t)x + tp_0).$$

Then $F(x,0) = \tilde{f}(x) = f(x)$ and $F(x,1) = \tilde{f}(p_0)$ for $x \in S^n$. Since $F(p_0,t) = \tilde{f}(p_0)$ for $t \in I$, F is a homotopy relative to $\{p_0\}$ from f to a constant map.

$(c) \Rightarrow (a)$. This is obvious. □

Exercise 2.2. Show that any continuous map from S^n to a contractible space has a continuous extension over E^{n+1}.

Exercise 2.3. The *comb space* Y is defined by

$$Y = \{(x,y) \in \mathbb{R}^2 | 0 \leq y \leq 1, x = 0, 1/n \text{ or } y = 0, 0 \leq x \leq 1\}.$$

Show that the identity map of Y is homotopic to the constant map to $(0,1) \in Y$. (Hint: By Proposition 3.13, it suffices to show that Y is contractible. Let $F\colon Y \times I \to Y$ be defined by $F((x,y),t) = (x,(1-t)y)$. Then F is a homotopy from id_Y to the projection of Y to the x-axis. Since the latter map is homotopic to a constant map, Y is contractible.)

3. Retraction, Deformation and Homotopy Extension Property

This section is concerned mainly with inclusion maps. We consider whether such a map has a left inverse, a right inverse and a two-sided inverse in either the category of spaces or the homotopy category.

3.1. Retraction.

Definition 3.18. *A subspace A of X is called a retract of X if the inclusion $i\colon A \to X$ has a left inverse, that is, there is a map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. A subspace A is called a weak retract of X if $i\colon A \subseteq X$ has a left homotopy inverse, that is there is a map $r\colon X \to A$ such that $r \circ i \simeq \mathrm{id}_A$.*

Example 3.19. Let A be the comb space of exercise 2.3 and let $X = I^2$. Then A is a weak retract of X because both A and X are contractible and so the inclusion $i\colon A \subseteq X$ is a homotopy equivalence (in particular i has a left inverse. We show that A is not a retract of X. Suppose that there were a retraction $r\colon X \to A$. Let $x_0 = (0,1) \in A$. Then $r(x_0) = x_0$. Let $U = \{y \mid \|y - x_0\| < 1/2\} \cap A = B1/2(x_0) \cap A$ be the open neighborhood of x_0. There is an open neighborhood V of x_0 in I^2 such that $r(V) \subseteq U$. Let ϵ be a small positive number such that $B_\epsilon(x_0) \cap I^2 \subseteq V$. Since $B_\epsilon(x_0) \cap I^2$ is path-connected, the image $r(B_\epsilon(x_0) \cap I^2) \subseteq U$ is path connected in U. Let $m \neq n$ be positive integers such that $1/m, 1/n < \epsilon$. Then $(1/m, 1), (1/n, 1) \in r(B_\epsilon(x_0) \cap I^2)$ because r is a retraction and so there is a path λ in $r(B_\epsilon(x_0) \cap I^2) \subseteq U$ joining them. This contracts to that $(1/m, 1)$ and $(1/n, 1)$ lie in different path-connected components of U.

Exercise 3.1. Show that a subspace $i\colon A \subseteq X$ is a weak retract if and only if $i^*\colon [X, A]_\emptyset \to [A, A]_\emptyset$ is onto.

Despite the fact that, in general, a weak retract need not be a retract, these concepts do coincide when A is a suitable subspace of X. This occurs frequently enough to warrant special consideration and will prove of use later.

3.2. Homotopy Extension Property.

Definition 3.20. *Let (X, A) be a pair of spaces (that is A is a subspace of X) and Y be a space. (X, A) is said to have the homotopy extension property with respect to Y if, given maps $g\colon X \to Y$ and $G\colon A \times I \to Y$ such that $g(a) = G(a, 0)$ for $x \in A$, there is a map $F\colon X \times I \to Y$ such that $F(x, 0) = g(x)$ for $x \in X$ and $F|_{A \times I} = G$. In other words, the following commutative diagram holds*

$$\begin{array}{ccc} X \times 0 \cup A \times I & \xrightarrow{g \cup G} & Y \\ \big\downarrow & & \big\| \\ X \times I & \overset{F}{\dashrightarrow} & Y, \end{array}$$

for any $g\colon X \to Y$ and $G\colon A \times I \to Y$ such that $g|_A = G_{A \times 0}$.

Proposition 3.21. *Suppose that (X, A) has the homotopy extension property with respect to Y and $f_0, f_1\colon A \to Y$ are homotopic. If f_0 has an extension to X, then so is f_1.*

Proof. Let $G\colon A \times I \to Y$ be the homotopy from f_0 to f_1 and let $g\colon X \to Y$ be the extension of f_0. By the definition, there is a map $F\colon X \times I \to Y$ such that $F|_{X \times 0 \cup A \times I} = g \cup G$. Then $F_1\colon X \to Y$ is an extension of f_1. □

3.3. Cofibration. Of particular importance is the case when (X, A) has the homotopy extension property with respect to any space. More generally, we have the following important concept:

DEFINITION 3.22. *A map $f\colon X' \to X$ is called a cofibration if for any space Y and any given maps $g\colon X \to Y$ and $G\colon X' \times I \to Y$ such that*

$$G(x', 0) = g(f(x'))$$

for any $x' \in X'$, there exists a map $F\colon X \times I \to Y$ such that $F(x, 0) = g(x)$ and $F(f(x'), t) = G(x', t)$ for any $x \in X$, $x' \in X'$ and $t \in I$.

The existence of F is equivalent to the existence of a map represented by the dotted arrow which makes the following diagram commutative:

$$\begin{array}{ccccc} X' \times I & \xrightarrow{f \times \mathrm{id}_I} & X \times I & \hookleftarrow & X \times 0 \\ \downarrow{\scriptstyle G} & & \vdots & & \downarrow{\scriptstyle g} \\ Y & = & Y & = & Y. \end{array}$$

Thus an inclusion $i\colon A \subseteq X$ is a cofibration if and only if (X, A) has the homotopy extension property with respect to any space Y.

PROPOSITION 3.23. *An inclusion $i\colon A \subseteq X$ is a cofibration if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.*

PROOF. Suppose that $i\colon A \subseteq X$ is a cofibration. Then the identity map of $X \times 0 \cup A \times I$ can be extended to $X \times I$ and so $X \times 0 \cup A \times I$ is a retract of $X \times I$.

Conversely, let

$$r\colon X \times I \to X \times 0 \cup A \times I$$

be a retraction. Let Y be any space and let $g\colon X \to Y$ and $G\colon A \times I \to Y$ be maps such that $G|_{A\times 0} = g|_A$. Then the composite

$$X \times I \xrightarrow{r} X \times 0 \cup A \times I \xrightarrow{g\cup G} Y$$

is an extension of $g \cup G$. □

EXERCISE 3.2. Let $A \subseteq B \subseteq X$ be subspaces. Suppose that $A \subseteq B$ and $B \subseteq X$ are co-fibrations. Show that $A \subseteq X$ is a cofibration.

EXERCISE 3.3. Show that $S^n \subseteq D^{n+1}$ is a cofibration.

3.4. A Relation between Retract and Weak Retract.

THEOREM 3.24. *If (X, A) has the homotopy extension property with respect to A, then A is a weak retract of X if and only if A is a retract of X.*

PROOF. We show that any weak retraction $r\colon X \to A$ is, in fact, homotopic to a retraction. Let $G\colon A \times I \to A$ be the homotopy from $r \circ i$ to id_A. Because (X, A) has the homotopy extension property with respect to A, there is an extension $F\colon X \times I \to A$ such that $F(x, 0) = r(x)$ and $F(a, t) = G(a, t)$. Then $F_1\colon X \to A$ is a retraction. □

3.5. Deformation. Given $X' \subseteq X$, a *deformation* D of X' in X is a homotopy $D\colon X' \times I \to X$ such that $D(x', 0) = x'$ for $x' \in X'$. If, moreover, $D(X' \times 1)$ is contained in a subspace A of X, D is said to be a *deformation of* X' *into* A and X' is said to be *deformable in* X *into* A. A space X is said to be *deformable* into a subspace A if it is deformable in itself into A. Thus a space X is contractible if and only if it is deformable into one of its points.

Exercise 3.4. Show that a space X is deformable into a subspace A if and only if the inclusion $i\colon A \subseteq X$ has a right homotopy inverse.

Note that an inclusion $i\colon A \subseteq X$ never has a right inverse in the category of topological spaces except the trivial case $A = X$.

A subspace $A \subseteq X$ is called a *weak deformation retract* of X if the inclusion $i\colon A \subseteq X$ is a homotopy equivalence.

Exercise 3.5. Show that A is a weak deformation retract of X if and only if A is a weak retract of X and X is deformable into A.

A is called a *deformation retract* if there is a retraction r of X to A such that $i \circ r \simeq \mathrm{id}_X$. A is called a *strong deformation retract* of X if there is a retraction r of X to A such that $i \circ r \simeq \mathrm{id}_X \ \mathrm{rel}\, A$.

Exercise 3.6. Suppose that X is deformable into a retract A. Show that A is a deformation retraction of X.

Proposition 3.25. *If (X, A) has the homotopy extension property with respect to A, then A is a weak deformation retract of X if and only if A is a deformation retract of X.*

Proof. Since (X, A) has the extension property with respect to A and A is a weak retract of X, A is a retract of X. Let $r\colon X \to A$ be a retraction. Since $i\colon A \subseteq X$ is a homotopy equivalence, i has a right homotopy inverse and so r is a right homotopy inverse of i. Thus A is a deformation retract of X. □

Proposition 3.26. *If $(X \times I, (X \times 0) \cup (A \times I) \cup (X \times 1))$ has the homotopy extension property with respect to X and A is closed in X, then A is a deformation retract of X if and only if A is a strong deformation retract of X.*

Proof. $\Leftarrow$ is obvious by definition.

$\Rightarrow$ Let $r\colon X \to A$ be a retract and let $F\colon X \times I \to X$ be a homotopy from id_X to $i \circ r$, where $i\colon A \subseteq X$. A homotopy

$$G\colon ((X \times 0) \cup (A \times I) \cup (X \times 1)) \times I \to X$$

is defined by the equations

$$\begin{aligned} G((x,0),t') &= x & & x \in X, t' \in I \\ G((a,t),t') &= F(a,(1-t')t) & & a \in A, t, t' \in I \\ G((x,1),t') &= F(r(x),1-t') & & x \in X, t' \in I. \end{aligned}$$

G is well-defined, because for $a \in A$

$$G((a,0)t') = a = F(a,0)$$

by the first two equations and

$$G((a,1),t') = F(a,1-t') = F(r(a),1-t')$$

by the last two equations. G is continuous because its restriction to each of the closed sets $X \times 0 \times I$, $A \times I \times I$ and $X \times 1 \times I$ is continuous. Furthermore

$$G|_{((X\times 0)\cup(A\times I)\cup(X\times 1))\times 0} = F_{(X\times 0)\cup(A\times I)\cup(X\times 1)}$$

[because $F(x,0) = x$ and since r is a retraction, $F(r(x),1) = ir(r(x)) = F(x,1)$.] Thus G restrict to $((X \times 0) \cup (A \times I) \cup (X \times 1)) \times 0$ can be extended to $(X \times I) \times 0$. From the homotopy extension property in the hypothesis, G restrict to $((X \times 0) \cup (A \times I) \cup (X \times 1)) \times 1$ can be extended to $(X \times I) \times 1$. Let $G' \colon (X \times I) \times 1 \to X$ be such an extension, and define $H \colon X \times I \to X$ by $H(x,t) = G'((x,t),1)$. Then we have

$$\begin{array}{lr} H(x,0) = G'((x,0),1) = G((x,0),1) = x & x \in X \\ H(x,1) = G((x,1),1) = F(r(x),0) = r(x) & x \in X \\ H(xa,t) = G((a,t),1) = F(a,0) = a & a \in A, t \in I \end{array}$$

and so H is a homotopy relative to A from id_X to $i \circ r$, or A is a strong deformation retract of X. □

Proposition 3.27 (First Criteria for Homotopy Equivalence). *If A is contractible and the pair (X, A) satisfies the homotopy extension property with respect to X, then the quotient map $q \colon X \to X/A$ is a homotopy equivalence.*

Proof. Let a_0 be a point in A. Since A is contractible, there exists a map $F \colon A \times I \to A$ such that $F_0 = \mathrm{id}_A$ and $F_1(a) = a_0$ for $a \in A$. By the homotopy extension property, there exists an extension $\tilde{F} \colon X \times I \to X$ such that $\tilde{F}_0 = \mathrm{id}_X$ and $\tilde{F}|_{A\times I} = F$, that is, there is a commutative diagram

$$\begin{array}{ccc} X \times 0 \cup A \times I & \xrightarrow{\mathrm{id}_X \cup F} & X \\ \downarrow & & \| \\ X \times I & \xrightarrow{\tilde{F}} & X. \end{array}$$

Since $\tilde{F}(A \times I) = F(A \times I) \subseteq A$, the composite $q \circ \tilde{F}$ factors through the quotient $q \times \mathrm{id}_I \colon X \to X/A$, that is there exists maps $\bar{F}_t$ such that the diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\tilde{F}} & X \\ \downarrow{\scriptstyle q \times \mathrm{id}_I} & & \downarrow{\scriptstyle q} \\ X/A \times I & \xrightarrow{\bar{F}} & X/A \end{array}$$

commutes.

Note that, for $a \in A$, $\tilde{F}(a,1) = F(a,1) = a_0$. The map $\tilde{F}_1 \colon X \to X$ factors through the quotient map $q \colon X \to X/A$, that is, there exists a map $g \colon X/A \to X$ such that $\tilde{F}_1 = g \circ q$.

Now $g \circ q = \tilde{F}_1 \simeq \tilde{F}_0 = \mathrm{id}_X$ by $\tilde{F}$. From the above commutative diagram

$$q \circ g = \bar{F}_1 \simeq \bar{F}_0 = \mathrm{id}_{X/A}$$

by $\bar{F}$ and hence the result. □

Let (X, A) be a pair of spaces and let $f\colon A \to Y$ be a map. The *adjunction space* $Y \cup_f X$ is defined to the quotient space $X \coprod Y/\sim$, where $\sim$ is generated by

$$x \sim f(x)$$

for $x \in A$. Roughly speaking, the space $Y \cup_f X$ is obtained by gluing the subspace A of X to Y via the map f.

Let (X, A) and (Y, B) be pairs of spaces. Write $(X, A) \times (Y, B)$ for $(X \times Y, X \times B \cup A \times Y)$.

PROPOSITION 3.28 (Second Criteria for Homotopy Equivalence). *Let (X, A) be a pair of spaces and let $f, g\colon A \to Y$ be maps. Suppose that*

(1). $f \simeq g\colon A \to Y$,
(2). *A is closed in X, and*
(3). *$(X, A) \times (I, 0)$ and $(X, A) \times (I, 0) \times (I, \{0, 1\})$ have the homotopy extension property with any spaces.*

Then $Y \cup_f X \simeq Y \cup_g X$ rel Y.

PROOF. Let $F\colon A \times I \to Y$ be a a homotopy from f to g. Consider the space $Y \cup_F (X \times I)$. Note that $Y \cup_f X$ and $Y \cup_g X$ are subspaces of $Y \cup_F (X \times I)$ as the quotients of $X \times 0 \coprod Y$ and $X \times 1 \coprod Y$ via F, respectively.

Observe that the inclusion $X \times 0 \cup A \times I \hookrightarrow X \times I$ is a homotopy equivalence. By Propositions 3.25 and 3.26, $X \times 0 \cup A \times I$ is a strong deformation retract of $X \times I$. Let G_t be a strong deformation retraction of $X \times I$ onto $X \times 0 \cup A \times I$. Then the homotopy

$$G_t \cup \mathrm{id}_Y : Y \cup_F (X \times I) \longrightarrow Y \cup_F (X \times I)$$

is a strong deformation retraction of $Y \cup_F (X \times I)$ onto $Y \cup_f X$. Thus $Y \cup_f X \simeq Y \cup_F (X \times I)$ rel Y. Similarly $Y \cup_g X$ is a strong deformation retract of $Y \cup_F (X \times I)$. It follows that

$$Y \cup_f X \simeq Y \cup_F (X \times I) \simeq Y \cup_g X \ \mathrm{rel}\, Y$$

and hence the result. □

Remark. It was proved in Steenrod's paper, *A conventional category of topological spaces*, Michigan Math. J. **14**(1967), 133-152, that if (X, A) and (Y, B) have the homotopy extension property with respect to any spaces, then $(X, A) \times (Y, B)$ has the homotopy extension property with respect to any spaces. In particular, if (X, A) has the homotopy extension property with respect to any spaces, then both $(X, A) \times (I, 0)$ and $(X, A) \times (I, 0) \times (I, \{0, 1\})$ have the homotopy extension property with any spaces.

4. H-spaces and Co-H-spaces

In this section, a space X means a pointed space. The notation $[X, Y]$ means the set of pointed homotopy classes of pointed maps from X to Y.

4.1. H-spaces. An *H-space* consists of a pointed space P together with a continuous multiplication $\mu\colon P \times P \to P$ for which the (unique) constant map $c\colon P \to P$ is a *homotopy identity*, that is, the following diagram

$$\begin{array}{ccccc} P & \xrightarrow{(c,\mathrm{id}_P)} & P \times P & \xleftarrow{(\mathrm{id}_P, c)} & P \\ \| & & \downarrow{\scriptstyle \mu} & & \| \\ P & = & P & = & P \end{array}$$

commutes up to homotopy.

Exercise 4.1. Let P be a pointed space and let $\mu\colon P \times P \to P$ be a map. Then μ has a homotopy identity if and only if there is a homotopy commutative diagram

$$\begin{array}{ccc} P \times P & \xrightarrow{\mu} & P \\ \uparrow & & \| \\ P \vee P & \xrightarrow{\nabla} & P, \end{array}$$

where ∇ is the fold map defined by $\nabla(x, x_0) = x$ and $\nabla(x_0, y) = y$.

An H-space P is called *homotopy associative* if the diagram

$$\begin{array}{ccc} P \times P \times P & \xrightarrow{\mu \times \mathrm{id}_P} & P \times P \\ \downarrow{\scriptstyle \mathrm{id}_P \times \mu} & & \downarrow{\scriptstyle \mu} \\ P \times P & \xrightarrow{\mu} & P \end{array}$$

commutes up to homotopy. An H-space P is called *homotopy commutative* if the diagram

$$\begin{array}{ccc} P \times P & \xrightarrow{T} & P \times P \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ P & = & P \end{array}$$

commutes up to homotopy, where $T(x, y) = (y, x)$. A map $\nu\colon P \to P$ is called a *homotopy inverse* if the diagram

$$\begin{array}{ccccc} P \times P & \xrightarrow{\mu} & P & \xleftarrow{\mu} & P \times P \\ \uparrow{\scriptstyle (\nu, \mathrm{id}_P)} & & \| & & \uparrow{\scriptstyle (\mathrm{id}_P, \nu)} \\ P & \xrightarrow{c} & P & \xleftarrow{c} & P \end{array}$$

commutes up to homotopy.

An H-space P is called an H-group if μ is homotopy associative with a homotopy inverse.
Note: We have been to call a space X is an H-space if there is a multiplication $\mu\colon X \times X \to X$ such that μ has a strict identity. In general, a multiplication $\mu\colon X \times X \to X$ that has a homotopy identity may not have a strict identity. But under certain conditions, homotopy identity $\Rightarrow$ strict identity.

Proposition 3.29. *Let X be a pointed space with a base point x_0. Let $\mu\colon X \times X \to X$ be a multiplication with a homotopy identity, that is, X is an H-space. Suppose that $X \vee X \subseteq X \times X$ is a cofibration. Then there is a multiplication $\mu'\colon X \times X \to X$ such that μ' has a strict identity.*

Proof. Let $\nabla\colon X \vee X \to X$ be the fold map. Since $\mu|_{X\vee X}\colon X \vee X \to X$ is homotopic to ∇ and $X \vee X \hookrightarrow X \times X$ has homotopy extension property with respect to X, ∇ has an extension $\mu'\colon X \times X \to X$. $\square$

Note: It is known that if $\{x_0\} \to X$ is a cofibration, then $X \vee X \to X \times X$ is a cofibration. A base-point x_0 of X is called *non-degenerate* if the inclusion $\{x_0\} \to X$ is a cofibration.
Note: In homotopy theory, there are (were) many questions about H-spaces. We list few of them:

1) Suppose that P is a homotopy associative H-space. Do there exist a space Q and a multiplication μ' on Q such that Q is a topological monoid under μ' and $Q \simeq P$? Suppose that P is path-connected. The answer of this question is: Yes if and only if P is homotopy equivalent to a loop space ΩX for some X. James Stasheff studied this question in 1960's and produced a method to test whether a space is homotopy equivalent to a loop space. His methods has been applied to Quantum Groups in 1980's.
2) Since one knows that S^1, S^3 and S^7 are H-spaces, people asked for which n S^n is an H-space? The answer was given by Adams in 1950's that S^n is an H-space if and only if $n = 1, 3, 7$.
3) We will show that the double loop spaces are homotopy associative and homotopy commutative H-spaces. One has been to ask whether a double loop space is homotopy equivalent to a (strict) commutative topological group. The answer, was given by Milnor in 1950's, is that if a path connected space X is homotopy equivalent to a commutative topological space if and only if X is a product of the spaces Y with the property that Y has at most one possible nontrivial commutative homotopy group, that is there is an integer n such that $\pi_i(Y) = 0$ for $i \neq n$ and $\pi_n(Y)$ is commutative.

Let P and Q be H-spaces. A (pointed) map $f\colon P \to Q$ is called an H-map if the diagram

$$\begin{array}{ccc} P \times P & \xrightarrow{\mu_P} & P \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ Q \times Q & \xrightarrow{\mu_Q} & Q \end{array}$$

commutes up to homotopy. If this diagram commutes strictly, we call f is a homomorphism. Clearly a homomorphism is an H-map. On the other hand, an H-map may not be a homomorphism in general.

Problem 3.30. *Let X and Y be H-spaces and let $f\colon X \to Y$ be an H-map. Under what conditions on X, Y and f such that there exists a homomorphism $g\colon X \to Y$ with $g \simeq f$?*

This problem has not been studied much and is related to a problem, so-called Freyd conjecture, in homotopy theory.

Now we give some basic properties of H-spaces.

Proposition 3.31. *Let P be an H-space. Let Q be a space and let $f\colon Q \to P$ be a pointed map. Suppose that f has a left pointed homotopy inverse. Then Q is an H-space.*

Proof. Let $r\colon Q \to P$ be a left pointed homotopy inverse of f, that is $r \circ f \simeq \mathrm{id}_Q$. Define a multiplication $\mu_Q\colon Q \times Q \to Q$ by the composite

$$Q \times Q \xrightarrow{f\times f} P \times P \xrightarrow{\mu_P} P \xrightarrow{r} Q.$$

Since there is a homotopy commutative diagram

$$\begin{array}{ccccc}
Q \times Q & \xrightarrow{f \times f} & P \times P & \xrightarrow{\mu} & P \\
\uparrow & & \uparrow & & \| \\
Q \vee Q & \xrightarrow{f \vee f} & P \vee P & \xrightarrow{\nabla} & P \\
\| & & \downarrow{\scriptstyle r \vee r} & & \downarrow{\scriptstyle r} \\
Q \vee Q & = & Q \vee Q & \xrightarrow{\nabla} & Q,
\end{array}$$

μ_Q has a homotopy identity and so Q is an H-space. □

Theorem 3.32. *If P is a homotopy associative H-space (H-group), then $[X, P]$ is a monoid (group) for any X. Furthermore if P is homotopy commutative, then $[X, P]$ is commutative.*

Proof. The multiplication $\mu\colon P \times P \to P$ induces a function

$$\mu_*\colon [X, P] \times [X, P] \cong [X, P \times P] \to [X, P]$$

for any X. This makes $[X, P]$ to be an H-set. Since μ is homotopy associative, μ_* is associative and so $[X, P]$ is a monoid. Furthermore if μ has a homotopy inverse, then μ_* has in inverse and so $[X, P]$ is a group. □

Lemma 3.33. *Let $f_0, f_1\colon A \to X$ and $g_0, g_1\colon Y \to B$ be pointed maps. Suppose that $f_0 \simeq f_1$ and $g_0 \simeq g_1$ under pointed homotopies. Assume that A is Hausdorff. Then $g_0^{f_0} \simeq g_1^{f_1}\colon \mathrm{Map}_*(X, Y) \to \mathrm{Map}_*(A, B)$.*

Proof. First we show that

$$\mathrm{id}_Y^{f_0} \simeq \mathrm{id}_Y^{f_1}\colon Y^X \to Y^A$$

Let $F\colon A \times I \to X$ be a homotopy from f_0 to f_1. Then F induces a map

$$\phi\colon \mathrm{Map}(X, Y) \xrightarrow{\mathrm{id}_Y^F} \mathrm{Map}(A \times I, Y) \cong \mathrm{Map}(I \times A, Y) \xrightarrow{\alpha} \mathrm{Map}(I, \mathrm{Map}(A, Y)),$$

where the association α is continuous because I is Hausdorff. Since I is locally compact Hausdorff, the association map

$$\alpha\colon \mathrm{Map}(\mathrm{Map}(X, Y) \times I, \mathrm{Map}(A, Y)) \to \mathrm{Map}(\mathrm{Map}(X, Y), \mathrm{Map}(I, \mathrm{Map}(A, Y)))$$

is onto-to-onto and onto. Thus $\alpha^{-1}(\phi)$ defines a map

$$F' = \alpha^{-1}(\phi)\colon \operatorname{Map}(X, Y) \times I \to \operatorname{Map}(A, Y).$$

The map F' is given as follows:

$$F'(\lambda, t)(a) = \lambda \circ F(a, t)$$

for $\lambda\colon X \to Y$ and $t \in I$. Clearly, F' maps $\operatorname{Map}_*(X, Y) \times I$ into $\operatorname{Map}_*(A, Y)$ with $F'_0 = \mathrm{id}_Y^{f_0}$, $F'_1 = \mathrm{id}_Y^{f_1}$ and $F'(*, t) = *$. Thus $\mathrm{id}_Y^{f_0} \simeq \mathrm{id}_Y^{f_1}$.

Now we show that $g_0^{\mathrm{id}_A} \simeq g_1^{\mathrm{id}_A}\colon Y^A \to B^A$. Let $G\colon g_0 \simeq g_1$ be a pointed homotopy. Consider the map

$$G^{\mathrm{id}_A}\colon \operatorname{Map}(A, Y) \times \operatorname{Map}(A, I) \cong \operatorname{Map}(A, Y \times I) \to \operatorname{Map}(A, B).$$

Let a_0 be the base-point of A. The constant map $A \to \{a_0\}$ induces a map

$$\theta\colon I = \operatorname{Map}(\{a_0\}, I) \to \operatorname{Map}(A, I).$$

Note that $\theta(t)$ is just the constant map from A to $t \in I$ for each t. Let G' be the composite

$$G'\colon \operatorname{Map}(A, Y) \times I \xrightarrow{\mathrm{id} \times \theta} \operatorname{Map}(A, Y) \times \operatorname{Map}(A, I) \xrightarrow{G^{\mathrm{id}_A}} \operatorname{Map}(A, B).$$

Then

$$G'(\lambda, t)(a) = G(a, t).$$

Clearly G' maps $\operatorname{Map}_*(A, Y) \times I$ into $\operatorname{Map}_*(A, B)$, $G'_0 = g_0^{\mathrm{id}_A}$, $G'_1 = g_1^{\mathrm{id}_A}$ and $G'(*, t) = *$. Thus $g_0^{\mathrm{id}_A} \simeq g_1^{\mathrm{id}_A}$ and so

$$g_0^{f_0} = g_0^{\mathrm{id}_A} \circ \mathrm{id}_Y^{f_0} \simeq g_1^{\mathrm{id}_A} \circ \mathrm{id}_Y^{f_1} = g_1^{f_1}$$

and therefore we have the result. $\square$

THEOREM 3.34. *Let P be an H-space (H-group) and let X be a pointed Hausdorff space. Then $\operatorname{Map}_*(X, P)$ is an H-space (H-group). In particular, $\Omega^n P$ is an H-space for each $n \geq 0$.*

PROOF. The multiplication on $\operatorname{Map}_*(X, P)$ is defined by

$$\mu = \mu_P^{\mathrm{id}_X}\colon P^X \times P^X \cong (P \times P)^X \to P^X.$$

The assertion follows from Lemma 3.33 $\square$

4.2. co-H-space. A pointed space X is called a *co-H-space* if there is a comultiplication $\mu'\colon X \to X \vee X$ such that μ' has a homotopy co-identity, that is there is a homotopy commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{(\mathrm{id}_X, c)} & X \vee X & \xrightarrow{(c, \mathrm{id}_X)} & X \\
\| & & \uparrow \mu' & & \| \\
X & = & X & = & X,
\end{array}$$

where c is the constant map. μ' is called *homotopy coassociative* if there is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu'} & X \vee X \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu' \vee \mathrm{id}_X} \\ X \vee X & \xrightarrow{\mathrm{id}_X \vee \mu'} & X \vee X \vee X. \end{array}$$

A *homotopy inverse* is a map $\nu \colon X \to X$ such that the diagram

$$\begin{array}{ccccc} X \vee X & \xleftarrow{\mu'} & X & \xrightarrow{\mu'} & X \vee X \\ \downarrow{\scriptstyle (\mathrm{id}, \nu)} & & \| & & \downarrow{\scriptstyle (\nu, \mathrm{id})} \\ X & \xleftarrow{c} & X & \xrightarrow{c} & X. \end{array}$$

μ' is called *homotopy cocommutative* if there is a homotopy commutative diagram

$$\begin{array}{ccc} X & = & X \\ \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu'} \\ X \vee X & \xrightarrow{T} & X \vee X, \end{array}$$

where $T(x,y) = (y,x)$. An co-H-space X is called an *co-H-group* if μ is homotopy coassociative with a homotopy inverse.

Let X and Y be co-H-spaces. A map $f \colon X \to Y$ is called a *co-H-map* if there is a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu'_X} & X \vee X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \vee f} \\ Y & \xrightarrow{\mu'_Y} & Y \vee Y. \end{array}$$

THEOREM 3.35. *Let X be a pointed Hausdorff space. Suppose that X is a co-H-space (co-H-group) with a comultiplication $\mu' \colon X \to X \vee X$. Then $\mathrm{Map}_*(X,Y)$ is an H-space (H-group) for any Y. In particular, $[X,Y] = \pi_0(\mathrm{Map}_*(X,Y))$ is a monoid (group).*

PROOF. The multiplication on Y^X is defined by the composite

$$\mu = \mathrm{id}_Y^{\mu'_X} \colon Y^X \times Y^X \cong Y^{X \vee X} \to Y^X.$$

The assertion follows from Lemma 3.33. □

EXERCISE 4.2. Let S^1 be identified with $I/\partial I = [0,1]/\{0,1\}$. Show that S^1 is a co-H-group under the comultiplication μ' defined by

$$\mu'(t) = \begin{cases} (2t, *) & 0 \leq t \leq 1/2 \\ (*, 2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

and a homotopy inverse ν defined by $\nu(t) = 1 - t$.

By this exercise, we have the following important theorem.

THEOREM 3.36. *Any loop space ΩX is an H-group. In particular,*

$$\pi_n(X) = \pi_0(\Omega^n X) = \pi_0(\Omega(\Omega^{n-1}(X))$$

is a group for $n \geq 1$.

By the definition, the multiplication on ΩX is induced by the comultiplication $\mu' \colon S^1 \to S^1 \vee S^1$. In other words, $\mu \colon \Omega X \times \Omega X \to \Omega X$ is given by

$$\mu(\lambda, \lambda')(t) = \begin{cases} \lambda(2t) & 0 \leq t \leq 1/2 \\ \lambda'(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

LEMMA 3.37. *Let $f_0, f_1 \colon X \to A$ and $g_0, g_1 \colon Y \to B$ be pointed maps. Suppose that $f_0 \simeq f_1$ and $g_0 \simeq g_1$ under pointed homotopies. Then*

$$f_0 \wedge g_0 \simeq f_1 \wedge g_1 \colon X \wedge Y \to A \wedge B.$$

PROOF. Let $F \colon X \times I \to A$ be a pointed homotopy from f_0 to f_1. Then we have the map

$$F' \colon X \times Y \times I \cong (X \times I) \times Y \xrightarrow{F \times \mathrm{id}_Y} A \times Y \xrightarrow{p} A \wedge Y,$$

where p is the quotient map. Clearly F' factors through $(X \wedge Y) \times I$. Since I is locally compact Hausdorff, the map $p \times \mathrm{id}_I \colon X \times Y \times I \to (X \wedge Y) \times I$ is a quotient map and so F' induces a (pointed) homotopy

$$F'' \colon (X \wedge Y) \times I \to A \wedge Y$$

with $F''_0 = f_0 \wedge \mathrm{id}_Y$ and $F''_1 = f_1 \wedge \mathrm{id}_Y$. Thus $f_0 \wedge \mathrm{id}_Y \simeq f_1 \wedge \mathrm{id}_Y$. Similarly, $\mathrm{id}_A \wedge g_0 \simeq \mathrm{id}_A \wedge g_1$. Thus

$$f_0 \wedge g_0 = (\mathrm{id}_A \wedge g_0) \circ (f_0 \wedge \mathrm{id}_Y) \simeq (\mathrm{id}_A \wedge g_1) \circ (f_1 \wedge \mathrm{id}_Y) = f_1 \wedge g_1$$

and hence the result. □

THEOREM 3.38. *Let X and Y be pointed spaces. Suppose that X is a co-H-space (co-H-group). Then so is $X \wedge Y$.*

PROOF. The comultiplication μ' is defined by

$$\mu' \colon X \wedge Y \xrightarrow{\mu'_X \wedge \mathrm{id}_Y} (X \vee X) \wedge Y \cong (X \wedge Y) \vee (X \wedge Y).$$

By Lemma 3.37, $X \wedge Y$ is a co-H-space (co-H-group if X is). □

Let X be a pointed space. The *n-fold suspension* of X is defined by

$$\Sigma^n X := S^n \wedge X.$$

Note that

$$\Sigma^n X = (S^1 \wedge S^1 \wedge \cdots \wedge S^1) \wedge X = S^1 \wedge \Sigma^{n-1} X$$

if $n \geq 1$ by Theorem 2.41. Thus we have

THEOREM 3.39. *Let X be a pointed space. Then $\Sigma^n X$ is a co-H-group for each $n \geq 1$.*

Now we want to show that $\pi_n(X)$ is abelian for $n \geq 2$.

LEMMA 3.40. *Let S be an H-set. Suppose that there is a function*

$$\phi\colon S \times S \to S$$

such that

1) $\phi(x,1) = x = \phi(1,x)$ *for any* $x \in S$ *and*
2) $\phi(x_1x_2, y_1y_2) = \phi(x_1,y_1)\phi(x_2,y_2)$ *for any* $x_1,x_2,y_1,y_2 \in S$.

Then S is a commutative monoid and $\phi(x,y) = xy$ for any $x,y \in S$.

PROOF. Let $x * y$ denote $\phi(x,y)$. Since

$$xy = (x*1)(1*y) = (x1)*(1y) = x*y,$$

we have $xy = \phi(x,y)$ for any x,y. Since

$$xy = x*y = (1x)*(y1) = (1*y)(x*1) = yx,$$

S is commutative. Since

$$x(yz) = x(y*z) = (x*1)(y*z) = (xy)*(1z) = (xy)z$$

S is associative. Thus S is a commutative monoid. □

Suppose that X is a co-H-space and Y is an H-space. Then there are two multiplications on $[X,Y]$, one is induced by the comultiplication $X \to X \vee X$ and another is induced by the multiplication $Y \times Y \to Y$.

THEOREM 3.41. *Let X be a co-H-space and let Y be an H-space. Then the two multiplication on $[X,Y]$ induced by μ'_X and μ_Y agree and are both associative and commutative.*

COROLLARY 3.42. *Suppose that Y is an H-space. Let X be any pointed space. Then $[\Sigma X, Y]$ is an abelian group. In particular,*

1) $\pi_1(Y)$ *is abelian;*
2) $\pi_n(Z) = [S^1, \Omega^{n-1}Z]$ *is an abelian group for any pointed space Z.*

Note. One of differential geometers through the internet has been to asked whether $S^1 \vee S^1$ is homotopy equivalent to a topological group. We will see that $\pi_1(S^1 \vee S^1)$ is a free group of rank 2, that is, two generators. In particular, $\pi_1(S^1 \vee S^1)$ is not abelian and so $S^1 \vee S^1$ is not an H-space or $S^1 \vee S^1$ is not homotopy equivalent to a topological group.

EXERCISE 4.3. Let μ_1 and μ_2 be two multiplications on Y such that Y is an H-space under μ_1 and μ_2. Show that

$$\Omega\mu_1 \simeq \Omega\mu_2\colon \Omega(Y \times Y) \to \Omega Y.$$

PROOF OF THEOREM 3.41. The multiplication on $[X,Y]$ induced by μ'_X is given as follows: For $[f],[g] \in [X,Y]$, $[f][g]$ is the homotopy class represented by the composite

$$X \xrightarrow{\mu'} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y,$$

where ∇ is the fold map. The multiplication $[X,Y]$ induced by μ_Y is given as follows: For $[f],[g] \in [X,Y]$, $[f]*[g]$ is the homotopy class represented by the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{\mu} Y.$$

By Lemma 3.40, it suffices to show that

$$([f] * [g])([f'] * [g']) = ([f][f']) * ([g][g'])$$

for any f, f', g, g'. This follows from the following commutative diagram because the right column represents $(ff') * (gg')$ and left column represents $(f * g)(f' * g')$.

$$\begin{array}{ccccc}
X & & = & & X \\
\downarrow{\scriptstyle \mu'} & & & & \downarrow{\scriptstyle \Delta} \\
X \vee X & & & & X \times X \\
\downarrow{\scriptstyle \Delta \vee \Delta} & & & & \downarrow{\scriptstyle \mu' \times \mu'} \\
(X \times X) \vee (X \times X) & \xrightarrow[\cong]{\phi} & (X \times * \times X \times *) \cup (* \times X \times * \times X) & \hookrightarrow & (X \vee X) \times (X \vee X) \\
\downarrow{\scriptstyle (f, g, f', g')} & & \downarrow{\scriptstyle (f, f', g, g')} & & \downarrow{\scriptstyle (f, f', g, g')} \\
(Y \times Y) \vee (Y \times Y) & \xrightarrow[\cong]{\phi} & (Y \times * \times Y \times *) \cup (* \times Y \times * \times Y) & \hookrightarrow & (Y \vee Y) \times (Y \vee Y) \\
\downarrow{\scriptstyle \mu \vee \mu} & & & & \downarrow{\scriptstyle \nabla \times \nabla} \\
Y \vee Y & & & & Y \times Y \\
\downarrow{\scriptstyle \nabla} & & & & \downarrow{\scriptstyle \mu} \\
Y & & = & & Y,
\end{array}$$

where $(Z \times Z) \vee (Z \times Z)$ is considered as the subspace of $Z \times Z \times Z \times Z$ by

$$(Z \times Z) \vee (Z \times Z) = (Z \times Z \times * \times *) \cup (* \times * \times Z \times Z)$$

and the map ϕ switches the middle two coordinates which sends (z_1, z_2, z_3, z_4) to (z_1, z_3, z_2, z_4).

Clearly the middle two squares commute. For checking the top square, let $\mu'(x) = (x', x'') \in X \vee X \subseteq X \times X$. Then

$$\begin{array}{rcl}
(\mu' \times \mu') \circ \Delta(x) & = & (\mu' \times \mu')(x, x) \\
& = & (x', x'', x', x'') \\
& = & \phi(x', x', x'', x'') \\
& = & \phi \circ \Delta(x', x'') \\
& = & \phi \circ \Delta \circ \mu'(x)
\end{array}$$

and so the top square commutes. For checking the bottom square, we have

$$\begin{array}{rcl} \mu\circ(\nabla\times\nabla)\circ\phi(y_1,y_2,*,*) & = & \mu\circ(\nabla\times\nabla)(y_1,*,y_2,*) \\ & = & \mu(y_1,y_2) \\ & = & \nabla(\mu(y_1,y_2),*) \\ & = & \nabla\circ(\mu\vee\mu)(y_1,y_2,*,*). \end{array}$$

Similarly

$$\mu\circ(\nabla\times\nabla)\circ\phi(*,*,y_1,y_2)=\nabla\circ(\mu\vee\mu)(*,*,y_1,y_2).$$

Thus the bottom square commutes and hence the result. □

Note. According to the proof, the identity

$$(f*g)(f'*g')=(ff')*(gg')$$

strictly holds in the mapping space $\mathrm{Map}_*(X,Y)$ because the above diagram commutes strictly. Recall that if $*\to Y$ is a cofibration, then there is a multiplication on Y with strict identity. In this case, the multiplication on $\mathrm{Map}_*(X,Y)$ induced from the multiplication on Y also has the strict identity. Suppose that the multiplication on $\mathrm{Map}_*(X,Y)$ induced from the comultiplication on X has the strict identity. Then $\mathrm{Map}_*(X,Y)$ is strictly associative and commutative by Lemma 3.40. Unfortunately the comultiplication on X never has strict identity unless X is a point. This means that the multiplication on $\mathrm{Map}_*(X,Y)$ induced from the comultiplication on X **does not** have a good chance to have strict identity even if in the simply case $X=S^1$ and $Y=\Omega Z$, that is $\mathrm{Map}_*(X,Y)=\Omega^2Z$. In homotopy theory, it was known that for any path-connected finite complex Z that is not homotopy equivalent to a point or a wedge of circles, Ω^2Z is never homotopy equivalent to a commutative topological group. There are still a lot of mysteries on double loop spaces Ω^2Z although there have been a lot of theories on double loop spaces.

4.3. The James Construction. Let X be a pointed (Hausdorff) space with the base-point $*$. The *James Construction* $J_n(X)$ is defined by

$$J_n(X)=X^n/\sim,$$

where $\sim$ is the equivalence relation generated by

$$(x_1,\cdots,x_{i-1},*,x_i,\cdots,x_{n-1})\sim(x_1,\cdots,x_{j-1},*,x_j,\cdots,x_{n-1})$$

for any $1\leq i,j\leq n$ and any $x_s\in X$. The elements in $J_n(X)$ is written as a word

$$w=x_1x_2\cdots x_n,$$

where we just keep in mind that, for example,

$$*x_1x_2=x_1*x_2=x_1x_2*$$

in $J_3(X)$.

Let $q_n\colon X^n\to J_n(X)$ be the quotient map. The inclusion $X^{n-1}\hookrightarrow X^n$, $(x_1,\cdots,x_{n-1})\to(x_1,\cdots,x_{n-1},*)$, induces a map $i_n\colon J_{n-1}(X)\hookrightarrow J_n(X)$ such that the diagram

$$\begin{array}{ccc} X^{n-1} & \hookrightarrow & X^n \\ \downarrow{\scriptstyle q_{n-1}} & & \downarrow{\scriptstyle q_n} \\ J_{n-1}X & \overset{i_n}{\hookrightarrow} & J_n(X) \end{array}$$

commutes. We show that i_n is a closed map. Let C be a closed set in $J_{n-1}(X)$. Then $q_{n-1}^{-1}(C)$ is a closed set in X^{n-1} and so

$$E_i = \{(x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_n) \in X^n | (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in q_{n-1}^{-1}(C)\}$$

is a closed set in X^n because E^i is the intersection of $X^{i-1} \times * \times X^{n-i}$ and $\pi_i^{-1}(q_{n-1}^{-1}(C))$, where

$$\pi_i(x_1, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

is the coordinate projection. Since

$$q_n^{-1}(i_n(C)) = \bigcup_{i=1}^{n} E_i,$$

$q_n^{-1}(C)$ is a closed set in X^n and so $i_n(C)$ is a closed set in $J_n(X)$. Thus i_n maps $J_{n-1}(X)$ homeomorphically onto the closed subspace $i_n(J_{n-1}(X))$ in $J_n(X)$ and so we may identify $J_{n-1}(X)$ as a closed subspace of $J_n(X)$. This gives a tower of closed spaces

$$J_1(X) \subseteq J_2(X) \subseteq J_3(X) \subseteq \cdots .$$

Define

$$J(X) = \bigcup_{n=1}^{\infty} J_n(X)$$

with so-called *weak topology*, that is, C is a closed set in $J(X)$ if and only if $C \cap J_n(X)$ is closed in $J_n(X)$ for each n. This makes $J(X)$ to be a topological spaces and each $J_n(X)$ is a closed subspace of $J(X)$.

An exact definition of weak topology is as follows. Let X be a space and let $\{A_\alpha\}$ be a family of closed set in whose union is X. We say X has the *weak topology* with respect to $\{A_\alpha\}$ if it satisfies the following condition: a subset C of X, whose intersection with each of A_α is closed, is itself closed. Let X be a set, and let $\{A_\alpha\}$ be a family of topological spaces, each a subset of X. We shall say that $\{A_\alpha\}$ is a *coherent family* (of topological spaces) on X if

1) $X = \bigcup_\alpha A_\alpha$;
2) $A_\alpha \cap A_\beta$ is a closed set of A_α for each α, β;
3) for every α, β, the topologies induced on $A_\alpha \cap A_\beta$ by A_α and A_β coincide.

Let A_α be a coherent family on X. Define a subset C of X to be closed if $C \cap A_\alpha$ is closed for each α. Then

1) X is a topological space (that is the completements of the closes sets form a topology on X);
2) Each A_α is a closed subspace of X;
3) X has the weak topology with respect to $\{A_\alpha\}$.

LEMMA 3.43. *Suppose X has the weak topology with respect to $\{A_\alpha\}$. Let U be a subset in X. Then U is open if and only if $U \cap A_\alpha$ is open for each α.*

PROOF. If U is open, clearly $U \cap A_\alpha$ is open because A_α is a subspace. Conversely, assume that $U \cap A_\alpha$ is open for each α. Then

$$(X \setminus U) \cap A_\alpha = A_\alpha \setminus (U \cap A_\alpha)$$

is closed in A_α for each α and so $X \setminus U$ is closed or U is open. □

LEMMA 3.44. *Suppose X has the weak topology with respect to $\{A_\alpha|\alpha \in I\}$ and Y has the weak topology with respect to $\{B_\beta|\beta \in J\}$. Then $X \times Y$ has the weak topology with respect to*

$$\{A_\alpha \times B_\beta|\alpha \in I, \beta \in J\}.$$

PROOF. Let C be a subset in $X \times Y$ such that $C \cap (A_\alpha \times B_\beta)$ is closed for any α, β. Let $U = X \times Y \setminus C$. We show that U is open. Since

$$U \cap A_\alpha \times B_\beta = A_\alpha \times B_\beta \setminus (C \cap A_\alpha \times B_\beta),$$

$U \cap A_\alpha \times B_\beta$ is open for any α and β. Let $\phi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ be the coordinate projections. Given any $\alpha \in I$, then

$$\pi_X(U) \cap A_\alpha = \bigcup_{\beta \in J} \pi_X(U \cap A_\alpha \times B_\beta).$$

Since $U \cap A_\alpha \times B_\beta$ is open in $A_\alpha \times B_\beta$ and $\pi_X|_{A_\alpha \times B_\beta}$ is the first coordinate projection,

$$\pi_X(U \cap A_\alpha \times B_\beta)$$

is open in A_α for each β and so the union

$$\pi_X(U) \cap A_\alpha$$

is open in A_α for any given α. It follows that $\pi_X(U)$ is open in X because X has the weak topology. Similarly, $\pi_Y(U)$ is open in Y. Thus U is open in $X \times Y$ and hence the result. $\square$

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THEOREM 3.45. *Let X be a pointed locally compact Hausdorff space. Then $J(X)$ is a topological monoid.*

PROOF. The composite

$$X^n \times X^m = X^{n+m} \xrightarrow{q_{n+m}} J_{n+m}(X)$$

factors through $J_n(X) \times J_m(X)$, that is there is a map $\mu_{n,m} \colon J_n(X) \times J_m(X) \to J_{n+m}(X)$ such that the diagram

$$\begin{array}{ccc} X^n \times X^m & = & X^{n+m} \\ \downarrow{\scriptstyle q_n \times q_m} & & \downarrow{\scriptstyle q_{n+m}} \\ J_n(X) \times J_m(X) & \xrightarrow{\mu_{n,m}} & J_{n+m}(X), \end{array}$$

where $q_m \times q_n$ is a quotient map because X is locally compact Hausdorff. By writing down the elements, we have

$$\mu_{n,m}(x_1x_2\cdots x_n, y_1y_2\cdots y_m) = x_1x_2\cdots x_ny_1y_2\cdots y_m$$

and so $\mu_{m,n}$ induces a unique function $\mu\colon J(X)\times J(X)\to J(X)$ such that the diagram

$$\begin{array}{ccc} J_n(X)\times J_m(X) & \xrightarrow{\mu_{n,m}} & J_{n+m}(X) \\ \big\downarrow & & \big\downarrow \\ J(X)\times J(X) & \xrightarrow{\mu} & J(X) \end{array}$$

commutes for any n,m. We show that μ is continuous. Let C be any closed set in $J(X)$. For any n,m, $C\cap J_{n+m}(X)$ is closed and so

$$\mu^{-1}(C)\cap(J_n(X)\times J_m(X))=\mu^{-1}_{n,m}(C\cap J_{n+m}(X))$$

is closed. By Lemma 3.44, $J(X)\times J(X)$ has the weak topology with respect to $\{J_m(X)\times J_n(X)\}$. Thus C is closed and hence μ is continuous. Clearly μ has the identity $*=1$ and is associative. Thus $J(X)$ is a topological monoid. □

We write $X^{(n)}$ for the n-fold self smash of X.

THEOREM 3.46. *There is an homeomorphism*

$$J_n(X)/J_{n-1}(X)\cong X^{(n)}$$

for each n.

PROOF. Let $q_n\colon X^n\to J_n(X)$ and $p_n\colon X^n\to X^{(n)}$ be the quotient maps. Then p_n factors $J_n(X)$, that is, there is a function $p'_n\colon J_n(X)\to X^{(n)}$ such that $p_n=p'_n\circ q_n$. It follows that p'_n is quotient map. Since $p'_n(J_{n-1}(X))=*$, p'_n induces a quotient map

$$p''_n\colon J_n(X)/J_{n-1}(X)\to X^{(n)}.$$

The map p''_n is a homeomorphism because it is one-to-one, onto and a quotient map. □

One of applications of the James construction to H-spaces is as follows.

THEOREM 3.47. *Let X be a pointed space. Then X is an H-space with a strict identity if and only if X is a (pointed) retract of a topological monoid.*

PROOF. Suppose that X is a retract of a topological monoid M. Let $j\colon X\to M$ be the inclusion with $j(*)=1$ and let $r\colon M\to X$ be a retraction with $r(1)=*$. Define a multiplication on X by

$$X\times X\xhookrightarrow{j\times j}M\times M\xrightarrow{\mu}M\xrightarrow{r}X.$$

Then $*x=x*=x$ for $x\in X$. Conversely, suppose that there is a multiplication $\mu\colon X\times X\to X$ with a strict identity. We write $x\cdot y$ for $\mu(x,y)$. Define a map

$$\phi_n\colon X^n\to X$$

by

$$\phi_n(x_1,x_2,\cdots,x_n)=((\cdots((x_1\cdot x_2)\cdot x_3)\cdots)\cdot x_n).$$

Since $* = 1$ is a strict identity for μ, the map ϕ_n factors through the quotient $J_n(X)$, that is there is a map $\phi'_n \colon J_n(X) \to X$ such that the diagram

$$\begin{array}{ccc} X^n & \xrightarrow{\phi_n} & X \\ \downarrow{\scriptstyle q_n} & & \| \\ J_n(X) & \xrightarrow{\phi'_n} & X \end{array}$$

commutes. Clearly

$$\phi'_n|_{J_{n-1}(X)} = \phi'_{n-1} \colon J_{n-1}(X) \to X$$

and so ϕ'_n induces a unique function $\phi' \colon J(X) \to X$ such that

$$\phi'_n = \phi'|_{J_n(X)}.$$

We show that ϕ' is continuous. Let C be a closed set in X. Then

$$\phi'^{-1}(C) \cap J_n(X) = {\phi'_n}^{-1}(C)$$

is a closed set in $J_n(X)$ for each n. Thus $\phi'(C)$ is closed because $J(X)$ has the weak topology with respect to $\{J_n(X)\}$. Since

$$\phi'|_{J_1(X)} = \phi'_1 \colon J_1(X) = X \to X$$

is the identity map, the map ϕ' is a retraction and hence the result because $J(X)$ is a topological monoid. □

Note. If $*$ is non-degenerate, that is $* \to X$ is a cofibration, then X is an H-space with a homotopy identity if and only if X is an H-space with a strict identity. (See Proposition 3.29) Thus suppose that $*$ is non-degenerate, then X is an H-space if and only if X is a retract of a topological monoid.
Note. It is known that $J(X) \simeq \Omega\Sigma X$ if X is a path-connected CW-complex. (For this reason, $J(X)$ is known as a 'combinatorial model' for loop suspensions. For instance, $J(S^1) \simeq \Omega S^2$.) Thus suppose that X is a path-connected CW-complex with a non-degenerate base-point, then X is an H-space is and only if X is a retract of a loop space.

5. Barratt-Puppe Exact Sequences

Let S and T be pointed sets with basepoints s_0 and t_0, respectively and let $f \colon S \to T$ be a pointed function. Denote by $\mathrm{Ker}(f) = f^{-1}(t_0)$ the pre-image of the basepoint. A sequence of pointed sets

$$\cdots \longrightarrow S_{n+1} \xrightarrow{f_{n+1}} S_n \xrightarrow{f_n} S_{n-1} \longrightarrow \cdots$$

is called *exact* if each function f_n is pointed with $\mathrm{Ker}(f_n) = \mathrm{Im}(f_{n+1})$.

Let X be a pointed space with basepoint x_0. The *(reduced) cone* CX is defined by

$$CX = X \times I/(x_0 \times I \cap X \times 1).$$

Note that X can identified as the subspace of CX consisting of $(x,0)$ for $x \in X$. Let $f \colon X \to Y$ be a pointed map. The *reduced mapping cone* $Y \cup_f CX$ is defined by

$$Y \cup_f CX = CX \coprod Y/(x,0) \sim f(x)$$

with quotient topology. In other words, the mapping cone if the adjunction space via the map $f \colon X = X \times 0 \to Y$ by attaching the cone CX to Y.

Recall that for pointed spaces X and Y we write $[X, Y]$ for the set of homotopy classes of pointed maps under pointed homotopy.

LEMMA 3.48. *Let $f\colon X \to Y$ be any pointed map and let $j\colon Y \to Y \cup_f CX$ be the inclusion map. Then the sequence*

$$[X, Z] \xleftarrow{f^*} [Y, Z] \xleftarrow{j^*} [Y \cup_f CX, Z]$$

is exact for any pointed space Z.

PROOF. First we show that $\mathrm{Im}(j^*) \subseteq \mathrm{Ker}(f^*)$, that is the composite

$$X \xrightarrow{f} Y \xrightarrow{j} Y \cup_f CX \xrightarrow{g} Z$$

is null homotopic (relative to the basepoint). It suffices to check that the composite $j \circ f\colon X \to Y \cup_f CX$ is null homotopic. This follows from the commutative diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\text{proj.}} & CX \\ \uparrow & & \downarrow \\ X & \xrightarrow{j \circ f} & Y \cup_f CX, \end{array}$$

namely the composite of the right map with the top map is a homotopy from $j \circ f$ to the constant map relative to the basepoint.

Next we show that $\mathrm{Ker}(f^*) \subseteq \mathrm{Im}(j^*)$. Let $g\colon Y \to Z$ be any pointed map such that the composite $g \circ f\colon X \to Z$ is null homotopic relative to the basepoint. Then there is a homotopy $F\colon X \times I \to Z$ such that $F_0 = g \circ f$, $F_t(x_0) = z_0$ for each t and $F_1(x) = z_0$ for any $x \in X$. The map F factors through the quotient $q\colon X \times I \to CX$ and there is a unique pointed map $\bar{F}\colon CX \to Z$ such that $F = \bar{F} \circ q$. Now the map

$$g \coprod \bar{F}\colon Y \coprod CX \longrightarrow Z$$

factors the quotient $Y \cup_f CX$ and so

$$g \cup \bar{F}\colon Y \cup_f CX \longrightarrow Z$$

is a well-defined map with the property that $g \cup \bar{F}|_Y = g$, that is, $g = (g \cup \bar{F}) \circ j$. Thus $[g] \in \mathrm{Im}(j^*)$. This finishes the proof. □

Starting from any pointed map $f\colon X \to Y$, there is a sequence

$$X \xrightarrow{f} Y \xhookrightarrow{j} Y \cup_f CX \xhookrightarrow{k} (Y \cup_f CX) \cup_j CY \xhookrightarrow{l} ((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX) \cdots.$$

LEMMA 3.49. *There is a commutative diagram up to homotopy*

$$\begin{array}{ccc} (Y \cup_f CX) \cup_j CY & \xhookrightarrow{l} & ((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX) \\ \downarrow \simeq & & \downarrow \simeq \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y, \end{array}$$

$\Sigma f([x,t]) = [f(x),t]$.

PROOF. There is

$$\begin{array}{ccc}
(Y \cup_f CX) \cup_j CY & \overset{l}{\hookrightarrow} & ((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX) \\
\downarrow \simeq & & \downarrow \simeq \\
((Y \cup_f CX) \cup_j CY)/CY & \overset{\bar{l} = l/Cj}{\hookrightarrow} & (((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX))/C(Y \cup_f CX)).
\end{array}$$

Now

$$\begin{aligned}
(Y \cup_f CX) \cup_j CY)/CY &\cong (Y \cup_f CX)/Y \\
&\cong CX/X \\
&= \Sigma X
\end{aligned}$$

$$\begin{aligned}
(((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX))/C(Y \cup_f CX)) &\cong (((Y \cup_f CX) \cup_j CY)/(Y \cup_f CX) \\
&\cong CY/Y \\
&= \Sigma Y.
\end{aligned}$$

Let

$$q\colon (Y \cup_f CX) \cup_j CY \twoheadrightarrow \Sigma X$$

$$q'\colon ((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX) \longrightarrow \Sigma Y$$

be the quotient maps. Regard $(Y \cup_f CX) \cup_j CY$ as $CX \cup CY$ with identification $[x,0] \sim [f(x),0]$. Define

$$\theta\colon \Sigma X \longrightarrow (Y \cup_f CX) \cup_j CY = CX \cup CY$$

by

$$\theta([x,t]) = \begin{cases} [x, 2(t-1/2)] \in CX & \text{if} \quad 1/2 \leq t \leq 1 \\ [f(x), 1-2t] \in CY & \text{if} \quad 0 \leq t \leq 1/2. \end{cases}$$

Then there is a homotopy commutative diagram

$$
\begin{array}{ccc}
\Sigma X & = & \Sigma X \\
\| & & \uparrow {\scriptstyle \simeq\, q} \\
\Sigma X & \xrightarrow[\simeq]{\theta} & (Y \cup_f CX) \cup_j CY \\
\downarrow {\scriptstyle \nu'} & & \downarrow {\scriptstyle l} \\
 & & ((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX) \\
 & & \downarrow {\scriptstyle \simeq\, q'} \\
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y,
\end{array}
$$

where $\nu'([x,t]) = [x, 1-t]$. Thus

$$[q'] \circ [l] \circ [\theta] = [\Sigma f] \circ [\nu']$$

with $[\theta] = [q]^{-1}$. It follows that

$$[q'] \circ [l] = [q'] \circ [l] \circ [\theta] \circ [q] = [\Sigma f] \circ [\nu'],$$

that is there is a commutative diagram up to homotopy

$$
\begin{array}{ccc}
(Y \cup_f CX) \cup_j CY & \overset{l}{\hookrightarrow} & ((Y \cup_f CX) \cup_j CY) \cup_k C(Y \cup_f CX) \\
\downarrow {\scriptstyle \simeq\, \nu' \circ q} & & \downarrow {\scriptstyle \simeq\, q'} \\
\Sigma X & \xrightarrow{\Sigma f} & \Sigma Y
\end{array}
\tag{2}
$$

and hence the result. □

By the above lemmas, we obtain the following important general theorem:

THEOREM 3.50 (Barratt-Puppe). *Let $f\colon X \to Y$ be any pointed map. Then there is a long exact sequence*

$$[X,Z] \xleftarrow{f^*} [Y,Z] \xleftarrow{j^*} [Y \cup_f CX, Z] \longleftarrow [\Sigma X, Z] \xleftarrow{\Sigma f^*} [\Sigma Y, Z] \xleftarrow{\Sigma j^*} [\Sigma Y \cup_f CX, Z] \longleftarrow \cdots$$

for any pointed spaces Z. □

REMARK 3.51. We give some remarks:

(1). The map $[Y \cup_f CX, Z] \longleftarrow [\Sigma X, Z]$ is induced by the composite

$$Y \cup_f CX \xrightarrow{\text{pinch}} Y \cup_f CX/Y \cong \Sigma X \xrightarrow{\nu'} \Sigma X$$

according to Diagram (9).

(2). In the Barratt-Puppe exact sequence, the functions are group homomorphisms except the first three terms.
(3). The map $[\Sigma X, Z] \to [Y \cup_f CX, Z]$ admits the action of the group $[\Sigma X, Z]$ on $[Y \cup_f CX, Z]$, see the book R. M. Switzer, *Algebraic Topology-Homotopy and Homology*, Proposition 2.48 for details.
(4). If $f \colon X \to Y$ is the inclusion of closed subspace such th at f is a cofibration, then $Y \cup_f CX \simeq Y/X$ and so one can replace $Y \cup_f CX$ by Y/X.
(5). For any pointed map $f \colon X \to Y$, there is a dual version of the Barratt-Puppe exact sequence given by

$$\cdots \longrightarrow [Z, \Omega P_f] \longrightarrow [Z, \Omega X] \xrightarrow{\Omega f_*} [Z, \Omega Y] \longrightarrow [Z, P_f] \longrightarrow [Z, X] \xrightarrow{f_*} [Z, Y]$$

for any pointed space Z, where

$$P_f = \{(x, \lambda) \in X \times PY \mid f(x) = \lambda(1)\}$$

and $PY = \{\lambda \colon I \to Y \mid \lambda(0) = y_0\}$ the space of paths starting from the basepoint.
(6). From the view of the Barratt-Puppe exact sequence, the mapping cone $Y \cup_f CX$ can be thought as the *cokernel* in homotopy sense, which is also called the *homotopy cofibre* of $f \colon X \to Y$. Similarly, the *mapping path-space* P_f can be thought as the *kernel* in homotopy sense, which is also called the *homotopy fibre* of $f \colon X \to Y$.

Example 3.52. Let $f \colon S^1 \to S^1$ be given by $f(z) = z^2$. Then $\mathbb{R}\mathrm{P}^2 = S^1 \cup_f CS^1 = S^1 \cup_2 D^2$. It follows that for any pointed space Z there is an exact sequence

$$[S^1, Z] = \pi_1(Z) \xleftarrow{2} [S^1, Z] = \pi_1(Z) \longleftarrow [\mathbb{R}\mathrm{P}^2, Z] \longleftarrow \pi_2(Z) \xleftarrow{2} \pi_2(Z) \longleftarrow \cdots$$

and so there is a short exact sequence of groups

$$\pi_n(Z)/2\pi_n(Z) \hookrightarrow [\Sigma^{n-2}\mathbb{R}\mathrm{P}^2, Z] \twoheadrightarrow \mathrm{Ker}(2 \colon \pi_{n-1}(Z) \to \pi_{n-1}(Z))$$

for $n \geq 3$. If $n = 2$, the above sequence is only the short exact sequence of sets because $[\mathbb{R}\mathrm{P}^2, Z]$ is only a set in general.
For instance, if $Z = S^2$, we will know that $\pi_1(S^2) = 0$ and $\pi_2(S^2) = \mathbb{Z}$. From the above short exact sequence, we have

$$[\mathbb{R}\mathrm{P}^2, S^2] \cong \mathbb{Z}/2$$

as sets. This means that there are only two homotopy classes for $\mathbb{R}\mathrm{P}^2$ to S^2. One can check that the pinch map $q \colon \mathbb{R}\mathrm{P}^2 \to S^2 = \mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1$ is not homotopic to the constant map. Hence any pointed map $f \colon \mathbb{R}\mathrm{P}^2 \to S^2$ is either homotopic to q or null homotopic relative to the basepoint.

CHAPTER 4

The Fundamental Groups and Covering Spaces

1. The fundamental Group

1.1. The fundamental Groupoid. Let λ and μ be two paths in X with $\lambda(1) = \mu(0)$. Then the product $\lambda * \mu$ is defined by

$$(\lambda * \mu)(t) = \begin{cases} \lambda(2t) & 0 \le t \le 1/2 \\ \mu(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Two paths λ and λ' are briefly said to be *homotopic*, denoted by $\lambda \simeq \lambda'$, if they are homotopic relative to $\partial I = \{0,1\}$. Note that if $\lambda \simeq \lambda'$, then $\lambda(0) = \lambda'(0)$ and $\lambda(1) = \lambda'(1)$.

LEMMA 4.1. *Let $\lambda_0, \lambda_1, \mu_0, \mu_1$ are paths in X with $\lambda_0(1) = \mu_0(0)$ and $\lambda_1(1) = \mu_1(0)$. If $\lambda_0 \simeq \lambda_1$ and $\mu_0 \simeq \mu_1$, then $\lambda_0 * \mu_0 \simeq \lambda_1 * \mu_1$.*

PROOF. Let $F\colon lambda_0 \simeq \lambda_1$ and $G\colon \mu_0 \simeq \mu_1$ be the homotopies relative to ∂I. Then $H\colon I \times I \to X$ defined by

$$H(t,s) = \begin{cases} F(2t,s) & 0 \le t \le 1/2 \\ G(2t-1,s) & 1/2 \le t \le 1 \end{cases}$$

is a homotopy relative to ∂I between $\lambda_0 * \mu_0$ and $\lambda_1 * \mu_1$. □

LEMMA 4.2. *Suppose that $\lambda_0, \lambda_1, \lambda_2$ are paths in X with $\lambda_0(1) = \lambda_1(0)$ and $\lambda_1(1) = \lambda_2(0)$. Then $(\lambda_0 * \lambda_1) * \lambda_2 \simeq \lambda_0 * (\lambda_1 * \lambda_2)$.*

PROOF. The map $F\colon I \times I \to X$ defined by

$$F(t,s) = \begin{cases} \lambda_0((4t)/(1+s)) & 0 \le t \le (s+1)/4, \\ \lambda_1(4t-s-1) & (s+1)/4 \le t \le (s+2)/4, \\ \lambda_2((4t-s-2)/(2-s)) & (s+2)/4 \le t \le 1; \end{cases}$$

is a homotopy relative to ∂I between $(\lambda_0 * \lambda_1) * \lambda_2$ and $\lambda_0 * (\lambda_1 * \lambda_2)$. □

For each $x \in X$, we define $\epsilon_x\colon I \to X$ as the constant path with $\epsilon_x(t) = x$ for any t.

LEMMA 4.3. *Let λ be in path in X with $\lambda(0) = x$ and $\lambda(1) = y$. Then $\epsilon_x * \lambda \simeq \lambda$ and $\lambda * \epsilon_y \simeq \lambda$.*

PROOF. The map $F\colon I \times I \to X$ defined by

$$F(t,s) = \begin{cases} x & 0 \le t \le (1-s)/t, \\ \lambda((2t-1+s)/(1+s)) & (1-s)/2 \le t \le 1; \end{cases}$$

is a homotopy relative to ∂I between $\epsilon_x * \lambda$ and λ. The map $G\colon I \times I \to X$ defined by

$$G(t,s) = \begin{cases} \lambda(\frac{2}{1+s}t) & 0 \le t \le \frac{1+s}{2} \\ y & \frac{1+s}{2} \le t \le 1. \end{cases}$$

is a homotopy relative to ∂I between $\lambda * \epsilon_y$ and λ. □

Given a path λ in X, the inverse λ^{-1} is defined by $\lambda^{-1}(t) = \lambda(1-t)$.

LEMMA 4.4. *Let λ be a path in X with $\lambda(0) = x$ and $\lambda(1) = y$. Then $\lambda * \lambda^{-1} \simeq \epsilon_x$ and $\lambda^{-1} * \lambda \simeq \epsilon_y$.*

PROOF. The map $F\colon I \times I \to X$ defined by

$$F(t,s) = \begin{cases} \lambda(2t(1-s)) & 0 \le t \le 1/2, \\ \lambda((2-2t)(1-s)) & 1/2 \le t \le 1; \end{cases}$$

is a homotopy relative to ∂I between $\lambda * \lambda^{-1}$ and ϵ_x. Similarly $\lambda^{-1} * \lambda \simeq \epsilon_y$. □

A category is called *small* if the class of objects is a set. A *groupoid* is a small category in which every morphism is an equivalence. Let X be a space. Let category $\mathcal{P}(X)$ is defined by:

> the objects in $\mathcal{P}(X)$ are points in X and morphisms from x to y are path classes from x to y. The composite operation is defined by $[\mu] \circ [\lambda] = [\lambda * \mu]$ for a path λ from x to y and a path μ from y to z.

By the lemmas above, we have

THEOREM 4.5. *Let X be a space. Then $\mathcal{P}(X)$ is a groupoid.*

1.2. Change of Base. Let X be a space with $x \in X$. Consider x is the basepoint of X. Then $\pi_1(X,x) = \pi_1(X)$ is called the *fundamental group* of X with base point x. Recall that $\pi_1(X,x)$ is a group, where the multiplication is given by the path multiplication. Note that the fundamental group depends on the choice of the base point x.

THEOREM 4.6. *Let $x, y \in X$. If there is a path in X from x to y, then the groups $\pi_1(X,x)$ and $\pi_1(X,y)$ are isomorphic.*

PROOF. Let λ be a path from x to y, that is $\lambda(0) = x$ and $\lambda(1) = y$. Define a function

$$\chi_\lambda\colon \pi_1(X,x) \to \pi_1(X,y)$$

by

$$\chi_\lambda([\mu]) = [\lambda^{-1} * \mu * \lambda].$$

This is a homomorphism of groups because

$$\begin{aligned} \chi_\lambda([\mu][\mu']) &= [\lambda^{-1} * \mu * \mu' * \lambda] = [\lambda^{-1} * \mu * \lambda * \lambda^{-1} * \mu' * \lambda] \\ &= [\lambda^{-1} * \mu\lambda][\lambda^{-1} * \mu * \lambda] = \chi_\lambda([\mu])\chi_\lambda([\mu']). \end{aligned}$$

λ^{-1} is path from y to x and so

$$\chi_{\lambda^{-1}}\colon \pi_1(X,y) \to \pi_1(X,x).$$

For $\mu \in \pi_1(X,x)$, we have

$$\chi_{\lambda^{-1}} \circ \chi_\lambda([\mu]) == [\lambda * \lambda^{-1}\mu * \lambda * \lambda^{-1}] = [\mu]$$

and so $\chi_{\lambda^{-1}} \circ \chi_\lambda = \mathrm{id}$. Similarly $\chi_\lambda \circ \chi_{\lambda^{-1}} = \mathrm{id}$. Thus χ_λ is an isomorphism of groups. □

Let $f\colon X \to Y$ be a map. Then f induces a homomorphism of groups

$$f_*\colon \pi_1(X,x) = [S^1, X] \to \pi_1(Y, f(x)) = [S^1, Y].$$

If $f \simeq g \operatorname{rel} x$, then

$$f_* = g_*\colon \pi_1(X,x) \to \pi_1(Y,y),$$

where $y = f(x) = g(x)$. If $X \simeq Y$ relative the base-point, then $\pi_1(X) \cong \pi_1(Y)$.

EXERCISE 1.1. *Prove that if there is a path in X from x_0 to x_1, then $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic.*

1.3. The fundamental Group of a Circle. The map $\mathrm{e}\colon \mathbb{R} \to S^1$ is defined by

$$\mathrm{e}(t) = \exp^{2\pi i t}.$$

Then e is continuous, $\mathrm{e}(t_1 + t_2) = \mathrm{e}(t_1)\,\mathrm{e}(t_2)$ and $\mathrm{e}(t_1) = \mathrm{e}(t_2)$ if and only if $t_1 - t_2$ is an integer. It follows that $\mathrm{e}|_{(-1/2,1/2)}$ is a homeomorphism of the open interval $(-1/2, 1/2)$ onto $S^1 \setminus \{\exp(\pi i)\}$. Let

$$\log\colon S^1 \setminus \{\exp(\pi i)\} \to (-1/2, 1/2)$$

be the inverse of $\mathrm{e}|_{(-1/2,1/2)}$.

A subset $X \subseteq \mathbb{R}^n$ is called *starlike* from a point x_0 if, whenever $x \in X$, the closed segment $[x_0, x]$ from x_0 to x lies in X.

LEMMA 4.7. *Let X be compact and starlike from $x_0 \in X$. Given any continuous map $f\colon X \to S^1$ and any $t_0 \in \mathbb{R}$ such that $\mathrm{e}(t_0) = f(x_0)$, there exists a continuous map $\tilde{f}\colon X \to \mathbb{R}$ such that $\tilde{f}(x_0) = t_0$ and $\mathrm{e} \circ \tilde{f}(x) = f(x)$ for all $x \in X$.*

PROOF. Clearly we can translate X so that it is starlike from the origin; hence there is no loss of generality in assuming $x_0 = 0$. Since X is compact, f is uniformly continuous and there exists $\epsilon > 0$ such that if $\|x - x'\| < \epsilon$, then $\|f(x) - f(x')\| < 2$ [that is, $f(x)$ and $f(x')$ are not antipodes in S^1]. Since X is bounded, there exists a positive integer n such that $\|x\|/n < \epsilon$ for all $x \in X$. Then for each $0 \le j < n$ and all $x \in X$

$$\|\frac{(j+1)x}{n} - \frac{jx}{n}\| = \frac{\|x\|}{n} < \epsilon$$

and so

$$\|f\left(\frac{(j+1)x}{n}\right) - f\left(\frac{jx}{n}\right)\| < 2.$$

It follows that the quotient $f((j+1)x/n)/f(jx/n)$ is a point of $S^1 \setminus \{\exp(\pi i)\}$. Let $g_j\colon X \to S^1 \setminus \{\exp(\pi i)\}$ for $0 \le j < n$ be the map defined by

$$g_j(x) = \frac{f((j+1)x/n)}{f(jx/n)}.$$

Then for all $x \in X$, we see that

$$f(x) = f(0)g_0(x)g_1(x)\cdots g_{n-1}(x).$$

We define $\tilde{f}\colon X \to \mathbb{R}$ by

$$\tilde{f}(x) = t_0 + \log(g_0(x)) + \log(g_1(x)) + \cdots + \log(g_{n-1}(x)).$$

Since f' is the sum of $n+1$ continuous functions from X to $\mathbb{R}$, it is continuous. Clearly $\tilde{f}(0) = t_0$ and $\mathrm{e} \circ \tilde{f} = f$. □

LEMMA 4.8. *Let X be a connected space and let $\tilde{f}, \tilde{g}\colon X \to \mathbb{R}$ be maps such that $\mathrm{e} \circ \tilde{f} = \mathrm{e} \circ \tilde{g}$ and $\tilde{f}(x_0) = \tilde{g}(x_0)$ for some $x_0 \in X$. Then $\tilde{f} = \tilde{g}$.*

PROOF. Let $h = \tilde{f} - \tilde{g}\colon X \to \mathbb{R}$. Since $\mathrm{e} \circ \tilde{f} = \mathrm{e} \circ \tilde{g}$, $\mathrm{e} \circ h$ is the constant map of X to $1 \in S^1$. Thus h is a continuous map from X to $\mathbb{R}$, taking only integral values. Because X is connected, h is constant, and since $h(x_0) = 0$, $h(x) = 0$ for all $x \in X$. □

Let $\alpha\colon I \to S^1$ be a closed path at 1. Because I is starlike from 0 and $\alpha(0) = 1 = \mathrm{e}(0)$, it follows from Lemmas 4.7 and 4.8 there exists a unique lifting $\tilde{\alpha}\colon I \to \mathbb{R}$ such that $\tilde{\alpha}(0) = 0$ and $\mathrm{e} \circ \tilde{\alpha} = \alpha$. Because $\mathrm{e}(\tilde{\alpha}(1)) = \alpha(1) = 1$, it follows that $\tilde{\alpha}(1)$ is an integer. We define the degree of α by

$$\deg(\alpha) = \tilde{\alpha}(1).$$

Lemma 4.9. *Let α and β be homotopic closed paths in S^1 at 1. Then* $\deg(\alpha) = \deg(\beta)$.

Proof. Let $F\colon I \times I \to S^1$ be a homotopy relative to ∂I from α to β. Because $I \times I$ is a starlike set of $\mathbb{R}^2$ from $(0,0)$, it follows that there is a (unique) lifting $\tilde{F}\colon I \times I \to \mathbb{R}$ such that $\tilde{F}(0,0) = 0$ and $\mathrm{e} \circ \tilde{F} = F$. Since F is a homotopy relative to ∂I, $F(0,t) = F(1,t) = 1$ for all $t \in I$. Thus $\tilde{F}(0,t)$ and $\tilde{F}(1,t)$ take on only integral values for all $t \in I$. It follows that $\tilde{F}(0,t)$ must be constant and $\tilde{F}(1,t)$ must be constant. Because $\tilde{F}(0,0) = 0$, $\tilde{F}(0,t) = 0$ for all t. Let $\tilde{\alpha}, \tilde{\beta}\colon I \to \mathbb{R}$ be the maps defined by $\tilde{\alpha}(t) = \tilde{F}(t,0)$ and $\tilde{\beta}(t) = \tilde{F}(t,1)$. Then $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $\mathrm{e} \circ \tilde{\alpha} = \alpha$ and $\mathrm{e} \circ \tilde{\beta} = \beta$. Thus

$$\deg(\alpha) = \tilde{\alpha}(1) = \tilde{F}(1,0) = \tilde{F}(1,t) = \tilde{F}(1,1) = \tilde{\beta}(1) = \deg(\beta).$$

□

It follows that there is a well-defined function deg from $\pi_1(S^1, 1)$ to $\mathbb{Z}$ defined by

$$\deg([\alpha]) = \deg(\alpha).$$

Theorem 4.10. *The function* deg *is an isomorphism of groups*

$$\deg\colon \pi_1(S^1, 1) \cong \mathbb{Z}.$$

Proof. To prove that deg is a homomorphism, let α and β be two closed paths in S^1 at 1 and let $\alpha\beta$ be the closed path which is their pointwise product in the group multiplication of S^1. We know from Theorem 3.41 that $[\alpha] * [\beta] = [\alpha\beta]$. Let $\tilde{\alpha}, \tilde{\beta}\colon I \to \mathbb{R}$ be such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = 0$, $\mathrm{e} \circ \tilde{\alpha} = \alpha$ and $\mathrm{e} \circ \tilde{\beta} = \beta$. Let $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}\colon I \to \mathbb{R}$. Then $\tilde{\gamma}(0) = 0$ and $\mathrm{e}(\tilde{\gamma}) = \alpha\beta$. Thus

$$\deg([\alpha] * [\beta]) = \deg([\alpha\beta]) = \tilde{\gamma}(1) = \tilde{\alpha}(1) + \tilde{\beta}(1) = \deg([\alpha]) + \deg([\beta]).$$

The map deg is an epimorphism: For any integer n, let $\tilde{\alpha}\colon I \to \mathbb{R}$ be the path defined by $\tilde{\alpha}(t) = tn$ and let $\alpha = \mathrm{e} \circ \tilde{\alpha}\colon I \to S^1$. Then clearly $\deg([\alpha]) = n$.

The map deg is a monomorphism: If $\deg([\alpha]) = 0$, then there is a path $\tilde{\alpha}\colon I \to \mathbb{R}$ with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = 0$ and $\mathrm{e} \circ \tilde{\alpha} = \alpha$. Since $\mathbb{R}$ is contractible, $\tilde{\alpha} \simeq \epsilon_0$ and

$$\alpha = \mathrm{e} \circ \tilde{\alpha} \simeq \mathrm{e} \circ \epsilon_0 = \epsilon_1.$$

□

Exercise 1.2. Show that the map $f\colon S^1 \to S^1$, $z \to z^n$ is of degree n.

Corollary 4.11. *The fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$.*

Theorem 4.12 (The Fundamental Theorem of Algebra). *Every non-constant complex polynomial has a root.*

Proof. We may assume without loss of generality that our polynomial has the form

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

with $n \geq 1$. Assume that p has no zero.

Let S_r be the circle $|z| = r$ of radius r. Choose $r >> 0$ such that

$$r^n > |a_1| r^{n-1} + |a_2| r^{n-2}| + \cdots + |a_{n-1}| r + |a_n|.$$

Let $F\colon S_r \times I \to \mathbb{C}$ be the map defined by

$$F(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_n).$$

Since

$$|F(z,t)| \geq |z|^n - t(|a_1||z|^{n-1} + \cdots + |a_n|) > 0$$

for $|z| = r$ and $0 \leq t \leq 1$, the image of F lies in $\mathbb{C} \setminus \{0\}$. Let

$$G\colon S^1 \times I \to S^1$$

be the composite

$$S^1 \xrightarrow{rz} S_r \xrightarrow{\frac{F(z,t)}{F(r,t)}} \mathbb{C} \setminus \{0\} \xrightarrow{\frac{z}{|z|}} S^1.$$

Then $G(1,t) = (F(r,t)/F(r,t))/|F(r,t)/F(r,t)| = 1$ for $t \in I$, $G(z,0) = z^n$ and

$$G(z,1) = \frac{p(rz)}{p(r)} \frac{|p(r)|}{|p(rz)|}.$$

Thus $f(z) = (|p(r)|/(p(r)|p(rz)|))p(rz) \simeq z^n$ is of degree n.

Let $H\colon S^1 \times I \to S^1$ be the map defined by

$$H(z,t) = \frac{p(rzt}{p(rt)} \frac{|p(rt)|}{|p(rzt)|},$$

where H is well-defined (and so it is continuous) because $p(z)$ is never zero. Then $H(1,t) = 1$ for all t, $H(z,0) = 1$ and $H(z,1) = f(z)$. It follows that $f(z)$ is of degree 0, which is a contradiction (unless $n = 0$).

THEOREM 4.13 (Brouwer Fixed Point Theorem). *Any continuous map $f\colon D^2 \to D^2$ has a fixed point, that is a point x such that $f(x) = x$.*

PROOF. Suppose that $x \neq f(x)$ for all $x \in D^2$. Then we may define a map $\phi\colon D^2 \to S^1$ by setting $\phi(x)$ to be the point on S^1 obtained from the intersection of the line segment from $f(x)$ to x extended to meet S^1. Let $i\colon S^1 \to D^2$ be the inclusion. Then $\phi \circ i = \mathrm{id}_{S^1}$. Thus there is a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = \pi_1(S^1) & = & \mathbb{Z} = \pi_1(S^1) \\ \downarrow i_* & & \uparrow \phi_* \\ 0 = \pi_1(D^2) & = & 0 = \pi_1(D^2), \end{array}$$

which is impossible. This contradiction proves the result. □

EXERCISE 1.3. Show that $\pi_n(S^1) = 0$ for $n > 1$. (Hint: Let $q\colon I^n \to S^n = I^n/\partial I^n$ the pinch map. Let $f\colon S^n \to S^1$ be any map. Consider $f \circ q\colon I^n \to S^1$. Since I^n is starlike, there is a unique lifting $\alpha\colon I^n \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\mathrm{e} \circ \alpha = f \circ q$. Since

$$\mathrm{e} \circ \alpha(x) = f \circ q(x) = f(*) = 1$$

for $x \in \partial I^n$, $\mathrm{e} \circ \alpha|_{\partial I^n}$ is the constant map and so $\alpha|_{\partial I^n}$ is a continuous map from ∂I^n to integers. It follows that $\alpha|_{\partial I^n}$ is a constant map because $\partial I^n \cong S^{n-1}$ is path-connected when $n > 1$. Since

$\alpha(0) = 0$, $\alpha(x) = 0$ for $x \in \partial I^n$ and so α induces a map $\bar{\alpha}\colon S^n = I^n/\partial I^n \to \mathbb{R}$. Since $\mathrm{e}\circ\alpha = f \circ q$, we have $\mathrm{e}\circ\bar{\alpha} = f$. Since $\mathbb{R}$ is contractible, $\bar{\alpha} \simeq \epsilon_0$ and so

$$f = \mathrm{e}\circ\alpha \simeq \mathrm{e}\circ\epsilon_0 = \epsilon.$$

This show that any map from S^n to S^1 is null homotopic and so $\pi_n(S^1) = 0$.)

1.4. Simply Connected Spaces. A space X is said to be *n-connected* for $n \geq 0$ if every continuous map $f\colon S^k \to X$ for $k \leq n$ has a continuous extension over E^{k+1}. A 1-connected space is also said to *simply connected*. Note that if $0 \leq m \leq n$, an n-connected space is m-connected. It follows from Theorem 3.17 that a space X is n-connected if and only if it is path-connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. By Exercise 1.1, X is n-connected if and only if it is path-connected and $\pi_k(X, x_0) = 0$ for $1 \leq k \leq n$ and any particular choice of base point x_0. Note that X is 0-connected if and only if X is path-connected. By Exercise 2.2, we have

LEMMA 4.14. *A contractible space is n-connected for every $n \geq 0$.*

EXERCISE 1.4. Let λ and μ be paths in X from x to y. Suppose that X is simply connected. Then $\lambda \simeq \mu$.

LEMMA 4.15. *Suppose that $X = U \cup V$ with U, V open and simply connected and $U \cap V$ non-empty and path connected. Then X is simply connected.*

PROOF. Let f be any path in X. Then $f^{-1}(U)$ is an open set of I and so $f^{-1}(U)$ is a disjoint union of open intervals. Let

$$f^{-1}(U) = \bigcup_{\alpha}(a_\alpha, b_\alpha)$$

be a disjoint union of open intervals (a_α, b_α). Since $f^{-1}(V)$ is open in I,

$$f^{-1}(V) = \bigcup_{\beta}(c_\beta, d_\beta).$$

Since

$$I = \bigcup_{\alpha,\beta}(a_\alpha, b_\alpha) \cup (c_\beta, d_\beta)$$

and I is compact, there exists a finite subcover

$$I = \bigcup_{i=1}^{m}(a_i, b_i) \cup \bigcup_{j=1}^{n}(c_j, d_j).$$

It follows that there are finite numbers

$$t_1 = 0 < t_2 < \cdots < t_q = 1$$

such that $[t_s, t_{s+1}]$ is either contained in (a_i, b_i) for some i or in (c_j, d_j) for some j. Let

$$f_s(t) = f(t_s + t(t_{s+1} - t_s)).$$

Then f_s is a path that starts with $f(t_s)$ and ends with $f(t_{s+1})$. If $[t_s, t_{s+1}] \subseteq (a_i, b_i)$ for some i, then $f_s(I) = f([t_s, t_{s+1}]) \subseteq U$, that is f_s is a path in U. Otherwise, $[t_s, t_{s+1}] \subseteq (c_j, d_j)$ for some j and f_s is a path in V. It follows that

$$f = f_1 * f_2 * \cdots * f_q,$$

where f_s is either in U or V and so

$$[f] = [f_1][f_2]\cdots[f_q].$$

We show by induction that

If f is a loop with $f(0) = f(1) \in U \cap V$ such that $[f] = [f_1][f_2]\cdots[f_q]$ with f_j is either a path in U or a path in V, then $[f] = 0$.

The assertion will follow from this statement.

If $q = 1$, then $[f] = [f_1]$ and so f_1 must be a loop. If f_1 is a loop in U, then $[f_1] = 0$ because U is simply connected and so $[f] = 0$. Otherwise f_1 is a loop in V and $[f] = [f_1] = 0$ because V is simply connected. Assume that the statement holds for $< q$. Let $[f] = [f_1]\cdots[f_q]$. We may assume that f_1 is a path in U without loss of generality. Let $i \geq 1$ be the largest integer such that f_j is a path in U for $j \leq i$. Then $f_1 * f_2 * \cdots * f_i$ is a path in U and f_{i+1} is a path in V. It follows that

$$f_i(1) = f_{i+1}(0) \in U \cap V.$$

Since $U \cap V$ is path connected, there is a path λ in $U \cap V$ from $f(0)$ to $f_i(1)$. Since U is simply connected, and $f_1 * \cdots * f_i$ and λ are paths in U from $f(0)$ to $f_i(1)$, we have

$$[f_1]\cdots[f_i] = [\lambda]$$

and so

$$[f] = [\lambda][f_{i+1}][f_{i+2}]\cdots[f_q] = [\lambda * f_{i+1}][f_{i+2}]\cdots[f_q] = 0$$

by induction, where $\lambda * f_{i+1}$ is a path in V. By induction. □

COROLLARY 4.16. *S^n is simply connected for $n \geq 2$.*

EXERCISE 1.5. Let X be a space. The *unreduced suspension* $\Sigma^u X$ is the quotient space of $I \times X$ obtained by identifying $0 \times X$ to a point and $1 \times X$ to a (different) point. Suppose that X is path-connected. Show that $\Sigma^u X$ is simply connected.

Note: $\Sigma X = \Sigma^u X / I \times *$. If $I \times * \to \Sigma^u X$ is a cofibration (this is true if $* \to X$ is a cofibration), then $\Sigma^u X \simeq \Sigma X$. Thus if $* \to X$ is a cofibration and X is path-connected, then ΣX is simply connected.

2. The Seifert-Van Kampen Theorem

In this section, we provide a useful theorem for calculations of fundamental groups.

2.1. Free Groups and Free Products of groups. Let X be a set. The *free group* $F(X)$ generated by X is a group that satisfies the following universal property:

1) $X \subseteq F(X)$ is a subset.
2) Let G be any group and let $f\colon X \to G$ be any function. There exists a unique homomorphism of groups $\tilde{f}\colon F(X) \to G$ such that $\tilde{f}|_X$.

It is known that for any X $F(X)$ exists and unique up to isomorphism. There is an explicit construction of the free group $F(X)$ in terms of words:

$$w = x_1^{\epsilon_1}\cdots x_k^{\epsilon_k},$$

where $x_j \in X$ and $\epsilon_j = \pm 1$. For instance, if $X = \{x_1, \cdots, x_n\}$, the words on X are given by

$$x_{i_1}^{\epsilon_1}\cdots x_{i_k}^{\epsilon_k}$$

for $k \leq 0$, $\epsilon_j = \pm 1$ and $1 \leq i_j \leq n$. A word w is called *reduced* if for each $1 \leq j \leq k$ $x_j \neq x_{j+1}$ or $x_j = x_{j+1}$ with $\epsilon_j \neq -\epsilon_{j+1}$. As a set $F(X)$ is given by all reduced words and the multiplication on $F(X)$ is given by the formal product of words, where we use the rule:

$$x_i^{-1}x_i = x_i x_i^{-1} = 1$$

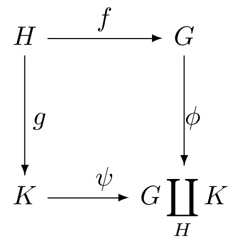
for each i. For example,

$$(x_1 x_2^{-1} x_3) \cdot (x_3^{-1} x_1) = x_1 x_2^{-1} x_1.$$

Clearly $F(X)$ is NOT a commutative group if X has more than one element because x_1x_2 and x_2x_1 are different words in $F(X)$ for $x_1, x_2 \in X$.

DEFINITION 4.17. Let $f\colon H \to G$ and $g\colon H \to K$ be homomorphisms of groups. The *push-out* $G \coprod_H K$ is a group that satisfies the following universal properties:

1) There are homomorphisms of groups $\phi\colon G \to G \coprod_H K$ and $\psi\colon K \to G \coprod_H K$ such that $\phi \circ f = \psi \circ g$, that is the diagram



,

commutes;

2) Let Γ be any group and let $\phi'\colon G \to \Gamma$ and $\psi'\colon K \to \Gamma$ be homomorphisms with $\phi' \circ f = \psi' \circ g$. Then there is a unique homomorphism $\theta\colon G \coprod_H K \to \Gamma$ such that $\phi' = \theta \circ \phi$ and $\psi' = \theta \circ \psi$.

When H is the trivial group, $G \coprod K = G \coprod_{\{1\}} K$ is called the *free product* of G and K. It is known in group theory that the push-out (so-called free product with amalgamation in group theory) always exists. The universal property show that $G \coprod_H K$ must be unique up to isomorphism if it exists. The combinatorial construction $G \coprod_H K$ can be given as follows:

First we construct the free product $G \coprod K$ can be given by the words

$$w = \alpha_1 \cdots \alpha_k,$$

where $\alpha_j \in G$ or K for each j. w is reduced if each $\alpha_j \neq 1$ and α_j and α_{j+1} are not lie in the same group. The product of two reduced words is the reduced words obtained from the formal product of them. For instance, let $\alpha_1, \alpha_2 \in G$ and $\beta_1 \in K$. Then

$$(\alpha_1\beta_1\alpha_2)(\alpha_2^{-1}\beta_1^{-1}\alpha_2) = \alpha_1\alpha_2 \in G.$$

The push-out $G \coprod_H K$ is the quotient group of $G \coprod K$ by the normal subgroup generated by

$$f(h)g(h)^{-1}$$

for $h \in H$. We can check that this construction satisfies the universal property: The homomorphisms $\phi\colon G \to G \coprod_H K$ and $\psi\colon K \to G \coprod_H K$ are canonical map given by $\phi(g)$ is word represented by g and $\psi(k)$ is the word represented by k for $g \in G$ and $k \in K$. By the relation above $\phi \circ f = \psi \circ g$ in the group $G \coprod_H K$ (NOT $G \coprod K$). Assume that $\phi'\colon G \to \Gamma$ and $\psi'\colon K \to \Gamma$ be homomorphisms with $\phi' \circ f = \psi' \circ g$. First there is a unique homomorphism $\theta' \colon G \coprod K \to \Gamma$ such that $\theta'(g) = \phi'(g)$ and $\theta'(k) = \psi'(k)$. Since $\phi' \circ f = \psi' \circ g$, we have that

$$\theta'(f(h)g(h)^{-1}) = 1$$

for $h \in H$ and so θ' induces a unique homomorphism $\theta \colon G \coprod_H K \to \Gamma$ with the desired property.

EXAMPLE 4.18. If K is the trivial group, then $G \coprod K = G$ and so $G \coprod_H K$ is the quotient group of G by the normal subgroup generated by the image of $f \colon H \to G$.

$\mathbb{Z} \coprod \mathbb{Z} = F(x_1, x_2)$ is a free group generated by two generators. In general, the n-fold free product of $\mathbb{Z}$ is a free group of rank n, that is n free generators.

$\mathbb{Z}/m \coprod \mathbb{Z}/n$ is the quotient group of $F(x_1, x_2)$ by the relations:

$$x_1^m = 1, x_2^n = 1.$$

2.2. The Seifert-Van Kampen Theorem.

THEOREM 4.19 (Seifert-Van Kampen Theorem). *Let X be a pointed space. Suppose that $X = U_1 \cup U_2$ such that U_1, U_2 are open and $U_1 \cap U_2$ is non-empty and path connected. Let $x_0 \in U_1 \cap U_2$ be a base-point of X. Then*

$$\pi_1(X, x_0) = \pi_1(U_1, x_0) \coprod_{\pi_1(U_1 \cap U_2, x_0)} \pi_1(U_2, x_0).$$

Sketch of Proof. Let $j^1 \colon U_1 \to X$, $j^2 \colon U_2 \to X$, $i^1 \colon U_1 \cap U_2 \to U_1$ and $i^2 \colon U_1 \cap U_2 \to U_2$ be inclusions. Since $j^1 \circ i^1 = j^2 \circ i^2$, the homomorphisms $j^1_* \colon \pi_1(U_1) \to \pi_1(X)$ and $j^2_* \colon \pi_1(U_2) \to \pi_1(X)$ induces a homomorphism

$$\theta \colon \pi_1(U_1) \coprod_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \to \pi_1(X).$$

Let $\lambda \colon S^1 \to X$ be a loop in X. By the proof of Lemma 4.15, we have

$$[\lambda] = [\lambda_1][\lambda_2] \cdots [\lambda_k],$$

where λ_j is a path either in U_1 or U_2. We may assume that λ_1 is a path in U_1. Let i be the largest number such that $\lambda_1, \cdots, \lambda_i$ are paths in U_1. Then $\lambda_i(1) = \lambda_{i+1}(0) \in U_1 \cap U_2$. Since $U_1 \cap U_2$ is path connected, there is a path μ in $U_1 \cap U_2$ from $\lambda(0)$ to $\lambda_i(1) = \lambda_{i+1}(0)$. Then

$$[\lambda] = [\lambda_1 * \cdots * \lambda_i * \mu^{-1}][\mu\lambda_{i+1}][\lambda_{i+2}] \cdots [\lambda_k].$$

Now $\lambda_1 * \cdots * \lambda_i * \mu^{-1}$ is a loop in U_1 and $\mu * \lambda_{i+1}$ is a path in U_2. By repeating this step (one can do this by induction), finally one can written down $[\lambda]$ as a product of elements from $\pi_1(U_1)$ or $\pi_1(U_2)$ and so θ is an epimorphism.

It is more complicated to show that θ is a monomorphism. So we omit this part of proof. □

A slight generalization of Seifert-Van Kampen Theorem can be found in Hatcher's book.

COROLLARY 4.20. *Let $f \colon X \to Y$ be a pointed map. Let $j \colon Y \to Y \cup_f CX$ be the inclusion. Suppose that X is path-connected. Then $\pi_1(Y \cup_f CX)$ is isomorphic to the cokernel of the group homomorphism*

$$\pi_1(X) \xrightarrow{f_*} \pi_1(Y).$$

Proof. Recall that $Y \cup_f CX = Y \coprod CX/(x,0) \sim f(x)$ and $CX = X \times I/(X \times 1 \cup x_0 \times I)$. Let U be the image of $Y \coprod X \times [0,1/2)$ in $Y \cup_f CX$ and let V be the image of $X \times (1/4,1]$ in $Y \cup_f CX$. Then $U \cap V \cong X \times (1/4,1/2)/(x_0 \times (1/4,1/2) \simeq X$ is path-connected. Note that V is contractible and $U \simeq Y$. By the Seifert-Van Kampen Theorem

$$\pi_1(Y \cup_f CX) = \pi_1(U) \coprod_{\pi_1(U\cap V)} \pi_1(V) = \pi_1(U) \coprod_{\pi_1(U\cap V)} \{1\}$$

is the cokernel of $i_*\colon \pi_1(U \cap V) \to \pi_1(U)$, where $i\colon U \cap V \to U$ is the inclusion. From there is homotopy commutative diagram

$$\begin{array}{ccc} U \cap V & \xrightarrow{\ i\ } & U \\ \uparrow & & \uparrow \\ X \times \{1/3\} & \xrightarrow{\ f\ } & Y, \end{array}$$

there is a commutative diagram

$$\begin{array}{ccc} \pi_1(U \cap V) & \xrightarrow{\ i_*\ } & \pi_1(U) \\ \uparrow \cong & & \uparrow \cong \\ \pi_1(X) & \xrightarrow{\ f_*\ } & \pi_1(Y). \end{array}$$

Thus $\pi_1(Y \cup_f CX)$ is the cokernel of $f_*\colon \pi_1(X) \to \pi_1(Y)$ and hence the result. □

Lemma 4.21. *Let (X,A) be a pair of spaces satisfying the homotopy extension property with respect to any spaces. That is the inclusion $A \to X$ is a cofibration. Then $X \cup CA \simeq X/A$.*

Proof. Let $r\colon X \times I \to X \times 0 \cup A \times I$ be a retraction. Then

$$r \cup \mathrm{id}_{CA\times I}\colon (X \cup CA) \times I \longrightarrow X \times 0 \cup (CA \times I)$$

is a well-defined retraction. Thus $(X \cup CA, CA)$ satisfies the homotopy extension property with respect to any spaces. Since CA is contractible, $X \cup CA \simeq X \cup CA/CA \cong X/A$ and hence the result. □

Corollary 4.22. *Let (X,A) be a pair of spaces satisfying the homotopy extension property with respect to any spaces. Suppose that A is path-connected. Then $\pi_1(X/A)$ is the cokernel of the group homomorphism*

$$\pi_1(A) \longrightarrow \pi_1(X).$$

Proof. By the above lemma, $X \cup CA \simeq X/A$ and hence the result. □

Theorem 4.23. *Let X and Y be path-connected spaces with basepoints x_0 and y_0, respectively. Suppose that $(X \times Y, X \vee Y)$ satisfies the homotopy extension property with respect to any spaces. Then $X \wedge Y$ is simply connected.*

Proof. Let $i\colon X\vee Y\to X\times Y$ be the inclusion. By Corollary 4.22, $\pi_1(X\wedge Y)$ is the cokernel of the group homomorphismthere is a right short exact sequence

$$\pi_1(X\vee Y)\xrightarrow{i_*}\pi_1(X\times Y)=\pi_1(X)\times\pi_1(Y).$$

The commutative diagram

$$\begin{array}{ccc} X\vee Y & \overset{i}{\hookrightarrow} & X\times Y \\ \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle i_1} \\ X & = & X \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} \pi_1(X\vee Y) & \overset{i_*}{\hookrightarrow} & \pi_1(X\times Y)=\pi_1(X)\times\pi_1(Y) \\ \uparrow{\scriptstyle j_{1*}} & & \uparrow{\scriptstyle i_{1*}} \\ \pi_1(X) & = & \pi_1(X). \end{array}$$

In particular, for any $g\in\pi_1(X)$, the element $(g,1)\in\pi_1(X)\times\pi_1(Y)$ is given by

$$(g,1)=i_{1*}(g)=i_*(j_{1*}(g))$$

and so $(g,1)\in\mathrm{Im}(i_*)$. Similarly

$$(1,h)\in\mathrm{Im}(i_*)$$

for any $h\in\pi_1(Y)$. It follows that

$$(g,h)=(g,1)\cdot(1,h)\in\mathrm{Im}(i_*)$$

for any $(g,h)\in\pi_1(X\times Y)=\pi_1(X)\times\pi_1(Y)$. Thus

$$i_*\colon\pi_1(X\vee Y)\longrightarrow\pi_1(X\times Y)$$

is onto and so $\pi_1(X\wedge Y)=\{1\}$. The assertion follows. ∎

Note. If $x_0\to X$ and $y_0\to Y$ are cofibrations, that is x_0 and y_0 are *nondegenerate* basepoints, then the inclusion $X\vee Y\to X\times Y$ is a cofibration and so $(X\times Y, X\vee Y)$ satisfies the homotopy extension property with respect to any spaces.

Exercise 2.1. *Let X and Y be spaces with basepoints x_0 and y_0, respectively. Suppose that there exist small contractible open neighborhoods of x_0 and y_0, respectively. Then $\pi_1(X\vee Y)=\pi_1(X)\coprod\pi_1(Y)$ is the free product.*

2.3. Calculations of the fundamental Group. By Using the Seifert-van Kampen theorem, we can compute the fundamental groups of a lot of spaces.

EXAMPLE 4.24. $\pi_1(S^1 \vee S^1) = F(x_1, x_2)$. *In general,* $\pi_1(\vee^n S^1) = F(x_1, \cdots, x_n)$.

PROOF. Let x be an element in S^1 different from the base point. Let $U = S^1 \vee (S^1 \setminus \{x\})$ and $V = (S^1 \setminus \{x\}) \vee S^1$. Then U and V are open sets in $S^1 \vee S^1$. Since $U \simeq S^1$ and $V \simeq S^1$, we have $\pi_1(U) = \mathbb{Z}$ and $\pi_1(V) = \mathbb{Z}$. Now $U \cap V = (S^1 \setminus \{x\}) \vee (S^1 \setminus \{x\})$ is contractible, $\pi_1(U \cap V) = \{1\}$. By the Seifert-van Kampen theorem, we have

$$\pi_1(S^1 \vee S^1) = \mathbb{Z} \coprod_{\{1\}} \mathbb{Z} = F(x_1, x_2).$$

By induction, one can show that $\pi_1(\vee^n S^1) = F(x_1, \cdots, x_n)$. □

Note: By this example, we know that $\vee^n S^1$ is NOT an H-space if $n > 1$.

EXAMPLE 4.25. $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$ *and* $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ *for* $n \geq 2$.

PROOF. Clearly $\mathbb{R}P^1 \cong S^1$ and so $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$. Now we compute $\pi_1 \mathbb{R}P^2$.

Recall that $\mathbb{R}P^2/\mathbb{R}P^1 \cong S^2$. Let $x \in \mathbb{R}P^2 \setminus \mathbb{R}P^1$ and let $U = \mathbb{R}P^2 \setminus \{x\}$. Then U is homotopy equivalent to $\mathbb{R}P^1$ and so $\pi_1(U) = \mathbb{Z}$. Let V be an open neighborhood of x that is homeomorphic to the open disk B^2 and is disjoint from $\mathbb{R}P^1$. Then $\pi_1 V = 0$. Clearly $U \cap V \simeq S^1$, $\pi_1(U \cap V) = \mathbb{Z}$. Let $j\colon U \cap V \to U$ be the inclusion. Then $j_*\colon \pi_1(U \cap V) \to \pi_1(U)$ is multiple by 2. Thus by the Seifert-van Kampen theorem $\pi_1(\mathbb{R}P^2) = \pi_1(U \cup V) = \mathbb{Z}/2\mathbb{Z}$.

Now we show that $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2$ by induction. Assume that $\pi_1(\mathbb{R}P^{n-1}) = \mathbb{Z}/2$ with $n \geq 3$. Let $x \in \mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$. Let $U = \mathbb{R}p^n \setminus \{x\}$. Then $U \simeq \mathbb{R}P^{n-1}$ and $\pi_1(U) = \mathbb{Z}/2$. Let V be a small neighborhood of x with $V \cong B^n$. Then $\pi_1(V) = 0$. Clearly $U \cap V \simeq S^{n-1}$. Since $n \geq 3$, S^{n-1} is simply connected and so $\pi_1(U \cap V) = 0$. It follows that $\pi_1(\mathbb{R}P^n) = \pi_1(U \cup V) = \mathbb{Z}/2$. □

EXERCISE 2.2. Show that $\mathbb{C}P^n$ is simply connected for each $n \geq 1$.

Note: By looking at fundamental groups, we already know that any $\mathbb{R}P^m$ is NOT homeomorphic to $\mathbb{C}P^n$.

EXERCISE 2.3. Let $T_g = T\#T\# \cdots \#T$ be the g-fold connected sum of the torus T. Show that $\pi_1(T_g)$ is the quotient group of the free group $F(c_1, d_1, c_2, d_2, \cdots, c_g, d_g)$ by the one relation:

$$c_1 d_1 c_1^{-1} d_1^{-1} c_2 d_2 c_2^{-1} d_2^{-1} \cdots c_g d_g c_g^{-1} d_g^{-1} = 1.$$

2.4. Groups and Spaces. Let X be a space. The *unreduced cone* $CX = I \times X/1 \times X$. Clearly the cone CX is contractible for any X. There is a relation between groups and so-called 2-complexes.

LEMMA 4.26. *Let* $\phi\colon F(x_1, \cdots, x_m) \to F(y_1, \cdots, y_n)$ *be a homomorphism. Then there is a (continuous) map* $f\colon \vee^m S^1 \to \vee^n S^1$ *such that*

$$f_* = \phi\colon \pi_1(\bigvee^m S^1) = F(x_1, \cdots, x_m) \to \pi_1(\bigvee^n S^1) = F(y_1, \cdots y_n).$$

PROOF. The homomorphism ϕ is uniquely determined by the elements $\phi(x_1), \cdots, \phi(x_m)$ in $F(y_1, \cdots, y_n)$. Since $\pi_1(\vee^n S^1) = F(y_1, \cdots, y_n)$, there are maps

$$f_1, \cdots, f_m\colon S^1 \to \bigvee^n S^1$$

such that $[f_j]=(f_j)_*([\mathrm{id}])=\phi(x_j)$. Let $f\colon \vee^m S^1\to\vee^n S^1$ be the map induced by $f_1,\cdots,f_m$. The $f_*=\phi$. □

Let G be a group with generators $x_1,\cdots,x_k$ and relations $R_1,\cdots,R_q$, where each R_j is a word in the free group $F(x_1,\cdots,x_k)$. The group G is the quotient group of $F(x_1,\cdots,x_k)$ by the normal subgroup generated by $R_1,\cdots,R_q$. Now we can construct a space $X=X(G)$ such that $\pi_1(X)=G$ as follows:

First we choose the wedges of circles, $X_1=\vee^k S^1$ and $X_2=\vee^q S^1$. Now we define a map $f\colon \vee^q S^1\to\vee^k S^1$ such that f restricted to the j-th copy of S^1 is a representative of the element $R_j\in\pi_1(\vee^k S^1)=F(x_1,\cdots,x_k)$. Define

$$X=X_1\coprod CX_2/\sim,$$

where $\sim$ is the equivalence relation generated by

$$(0,x)\sim f(x)$$

for $x\in\vee^q S^1$. We show that $\pi_1(X)=G$. Let $x=1\times X_2$ be the element in $CX_2=I\times X_2/1\times X_2$, where $X_2=\vee^q S^1$. Let

$$U=X\setminus x=X_1\coprod(CX_2\setminus\{x\})/\sim.$$

Then $U\simeq X_1=\vee^k S^1$ and so $\pi_1(U)=F(x_1,\cdots,x_k)$. Let V be image of $(2/1,1]\times X_2$ in CX_2. Then V is an open neighborhood of x with $\pi_1(V)=0$ (V is contractible). Clearly that $U\cap V\simeq X_2=\vee^q S^1$. Thus $\pi_1(X)$ is the quotient group of $F(x_1,\cdots x_k)$ by the normal subgroup generated by

$$\mathrm{Im}(\pi_1(U\cap V)\to\pi_1(U))=\mathrm{Im}(f_*\colon\pi_1(\vee^q S^1)\to\pi_1(\vee^k S^1)),$$

which is the normal subgroup generated by $R_1,\cdots R_q$. Thus $\pi_1(X)=G$.

Now let $\phi\colon G\to H$ be a homomorphism. Suppose that G has generators $x_1,\cdots x_k$ with relations $R_1,\cdots R_q$ and H has generators $y_1,\cdots y_s$ with relations $S_1,\cdots S_t$. Then there is a homomorphism $\tilde\phi\colon F(x_1,\cdots,x_k)\to F(y_1,\cdots,y_t)$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & H\\ \uparrow & & \uparrow\\ F(x_1,\cdots,x_k) & \xrightarrow{\ \tilde\phi\ } & F(y_1,\cdots,y_s)\end{array}$$

commutes. Thus there is a map $f\colon X_1(G)\to X_1(H)$ such that

$$f_*=\tilde\phi\colon\pi_1(X_1(G))\to\pi_1(X_1(H)).$$

Let $j\colon X_1(H)\to X(H)$ be the inclusion. Then the composite

$$\theta\colon X_2(G)\longrightarrow X_1(G)\xrightarrow{\ f\ }X_1(H)\hookrightarrow X(H)$$

is null homotopic because its restriction to each copy of S^1 induces the trivial element in the fundamental group of $X(H)$. It follows that there is a map $\tilde\theta\colon CX_2(G)\to X(H)$ such that $\tilde\theta|_{X_2(G)}=\theta$. Now the map $j\circ f$ and $\tilde\theta$ defines a map

$$\bar f\colon X(G)=X_1(G)\coprod CX_2(G)/\sim\to X(H).$$

Clearly $\bar f_*=\phi\colon\pi_1(X(G))\to\pi_1(X(H))$. Thus we have the following theorem.

THEOREM 4.27. *For any group G, there is a space $X(G)$ such that $\pi_1(X(G)) = G$. If $\phi\colon G \to H$ is a homomorphism, there is a map $f\colon X(G) \to X(H)$ such that*

$$f_* = \phi\colon \pi_1(X(G) = G \to \pi_1(X(H)) = H.$$

Note: The space $X(G)$ is not unique (even up to homotopy) because a group G can be written down in terms of different generator-relation systems.

EXAMPLE 4.28. If a group G has only one relation (such group are called *one relator groups*, the construction of $X(G)$ is quite simple which can be described as follows:

Let $x_1, \cdots, x_k$ be generators for G and let $R = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}$ be the only relation for G. We may assume that R is an unreduced word such that all $x_1, \cdots, x_k$ occur in R.

Let Y be a t-sided polygonal region with counter-clockwise orientation. The sides in Y are labeled by $x_{i_1}, \cdots, x_{i_t}$. The j-th side is chosen to be in a positive direction [negative direction] if $\epsilon_j = 1$ [if $\epsilon_j = -1$].

Let X be the quotient space of Y by identifying 1) all vertices to be one point and 2) all oriented sides labeled by the same letter.

We can show that $\pi_1(X) = G$. Let x be an inner point in Y. Let $U = X \setminus \{x\}$. Then $U \simeq \vee^k S^1$. Let V be an open ϵ-neighborhood of x in Y (and so in X). Then $\pi_1(V) = 0$. Clearly $U \cap V \simeq S^1$ and so $\pi_1(U \cap V) = \mathbb{Z}$. Let $j\colon U \cap V \to U$ be the inclusion and let $\alpha = [\mathrm{id}_S^1]$ be the generator for $\pi_1(U \cap V)$. Then

$$j_*(\alpha) = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}.$$

Thus $\pi_1(X) = G$.

3. Covering Spaces

DEFINITION 4.29. A map $p\colon \tilde{X} \to X$ is a *covering projection* and $\tilde{X}$ (or $(\tilde{X}, p)$ is a *covering space* of X if

1) p is onto, and
2) for any $x \in X$ there is an open neighbourhood U (called an *elementary neighbourhood*) of x such that

$$p^{-1}(U) = \coprod_{\alpha \in J} U_\alpha$$

is a topological disjoint union of open sets (called *sheets*), each U_α is mapped homeomorphically onto U by p. So $p^{-1}(U) \cong U \times$ (discrete space.)

Roughly speaking covering space just means that 'locally' the pre-image $p^{-1}(U)$ is disjoint union of copies of U.

EXAMPLE 4.30. (1). Any homeomorphism $p\colon \tilde{X} \to X$ is a one-sheeted covering projection.
(2). Let F be a discrete space and $\tilde{X} = X \times F$. Then the coordinate projection $p\colon \tilde{X} \to X$ is a covering projection.
(3). The projection $p\colon S^n \to \mathbb{R}P^n$ is a two-sheeted covering projection.
(4). $p\colon S^1 \to S^1$, $z \mapsto z^n$, is an n-sheeted covering.
(5). The exponential map $\mathrm{e}\colon \mathbb{R} \to S^1$ is a covering with infinite sheets.

EXERCISE 3.1. Let $p\colon \tilde{X} \to X$ and $q\colon \tilde{Y} \to Y$ be covering projections. Show that $p \times q\colon \tilde{X} \times \tilde{Y} \to X \times Y$ is also a covering projection.

Let G be a group and let Y be a G-space. For $g \in G$ and a subset $S \subseteq Y$, let $g \cdot S$ denote the set $\{g \cdot x | x \in S\}$.

DEFINITION 4.31. Let G be a (discrete) group and let Y be a G-space. A G-action on Y is called *properly discontinuous* if

for any $y \in Y$ there exists a neighbourhood W_y such that

$$g_1 \neq g_2 \quad \Rightarrow \quad g_1 \cdot W_y \cap g_2 \cdot W_y = \emptyset$$

(or, equivalently, $g \neq 1 \quad \Rightarrow \quad g \cdot W_y \cap W_y = \emptyset$).

THEOREM 4.32. *Let X be a G-space. If the G-action on X is properly discontinuous, then $X \to X/G$ is a covering.*

PROOF. Let $p\colon X \to X/G$ be the quotient map. By Theorem 2.30, p is an open map. For any $x \in X$, let W be an open neighbourhood satisfying the condition of proper discontinuity. Then $p(U)$ is an open neighbourhood of $p(x)$ and

$$p^{-1}(W) = \coprod_{g \in G} g \cdot W$$

is a disjoint union of open subsets of X. Furthermore $p|_{g \cdot W} \colon g \cdot W \to p(W)$ is a continuous open bijective map and hence a homeomorphism. □

EXERCISE 3.2. Let X be a G-space. Suppose that $X \to X/G$ is a covering. Show that the G-action on X is properly discontinuous.

Now the next question is how can we know a group-action is properly discontinuous. Recall that a group G acts freely on X if $g \cdot x \neq x$ for all $x \in X$ and $g \in G$ with $g \neq 1$.

EXERCISE 3.3. Let X be a G-space. Suppose that the G-action on X is properly discontinuous. Then G acts freely on X.

THEOREM 4.33. *Let G be a finite group and let X be a Hausdorff G-space. Then the G-action on X is properly discontinuous if and only if G acts freely on X.*

PROOF. $\Rightarrow$ is obvious (see Exercise 3.3).

$\Leftarrow$ Let $G = \{g_0 = 1, g_1, \cdots, g_n\}$. Since X is Hausdorff, there exist open neighbourhoods $U_0, \cdots U_n$ of $g_0 \cdot x, \cdots, g_n \cdot x$, respectively such that $U_0 \cap U_j = \emptyset$ for $1 \leq j \leq n$. Let $U = \cap_{j=0}^{n} g_j^{-1} \cdot U_j$. Then U is an open neighbourhood of x with $g_j \cdot U \cap U = \emptyset$ for each $1 \leq j \leq n$ because

$$\begin{aligned} g_j \cdot U = g_j \cdot \bigcap_{i=0}^{n} g_i^{-1} U_i &= \bigcap_{i=0}^{n} g_j (g_i^{-1} \cdot U_i) \\ &= \bigcap_{i=0}^{n} (g_j g_i^{-1}) \cdot U_i \subseteq (g_j g_j^{-1}) \cdot U_j = U_j. \end{aligned}$$

Thus the G-action on X is properly discontinuous. □

Note: If G has infinite elements, a free G-action may or may not be properly discontinuous. In other words, the quotient $X \to X/G$ may or may not be a covering even if G acts freely on X and X is Hausdorff.

Now we have more examples of covering spaces.

EXAMPLE 4.34. 1) Let $\mathbb{Z}$ act on $\mathbb{R}$ by $x \mapsto x+n$. Then this action is discontinuous and so $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ is a covering.

2) Let $\mathbb{Z}^n = \mathbb{Z}^{\oplus n}$ act on $\mathbb{R}^n$ by $(x_1, \cdots, x_n) \mapsto (x_1 + l_1, \cdots, x_n + l_n)$ for $x_j \in \mathbb{R}$ and $l_j \in \mathbb{Z}$. Then this action is discontinuous and so $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = S^1 \times \cdots \times S^1$ is a covering. In particular, when $n = 2$, we have the covering projection : $\mathbb{R}^2 \to T = S^1 \times S^1$.

3) Let p be a prime integer and let $q_1, \cdots q_n$ be integers prime to p. We define a $\mathbb{Z}/p$-action on

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^n \,|\, |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$l \cdot (z_0, \cdots, z_n) = (\mathrm{e}^{2\pi i l/p}\, z_0, \mathrm{e}^{2\pi i l q_1/p}\, z_1, \cdots, \mathrm{e}^{2\pi i l q_n/p}\, z_n).$$

We show that this action is free. Suppose that

$$l \cdot (z_0, \cdots, z_n) = (z_0, \cdots, z_n).$$

Then

$$\mathrm{e}^{2\pi i l q_j/p}\, z_j = z_j$$

for each $0 \le j \le n$, where $q_0 = 1$. Since $(z_0, \cdots z_n) \in S^{2n+1}$, there exists $z_{j_0} \neq 0$ for some j_0. It follows that

$$\mathrm{e}^{2\pi i l q_{j_0}/p} = 1$$

and so $lq_{j_0} \equiv 0 \bmod p$. Since $q_{j_0} \not\equiv 0 \bmod p$ and p is a prime, $l \equiv 0 \bmod p$, that is l is the identity in $\mathbb{Z}/p$. Thus this action is free.

Since S^{2n+1} is Hausdorff, $S^{2n+1} \to S^{2n+1}/(\mathbb{Z}/p)$ is a covering. The quotient $S^{2n+1}/(\mathbb{Z}/p)$, denoted by $L^n(p, q_1, \cdots, q_n)$, is called a *lens space*. Note that $L^n(2) = \mathbb{R}P^{2n+1}$.

4) Let p be any non-zero integer. We define a $\mathbb{Z}/p$-action on

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^n \,|\, |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$$

by

$$l \cdot (z_0, \cdots, z_n) = (\mathrm{e}^{2\pi i l/p}\, z_0, \mathrm{e}^{2\pi i l/p}\, z_1, \cdots, \mathrm{e}^{2\pi i l/p}\, z_n).$$

The argument above show that this action is free. (**Note:** in this case, we do not need to assume that p is a prime.) The quotient $S^{2n+1}/(\mathbb{Z}/p)$ is denoted by $L^n(p)$. Again we have a covering projection $S^{2n+1} \to L^n(p)$.

5) Let M be a manifold and let

$$F(M, n) = \{(x_1, \cdots, x_n) \in M^n \,|\, x_i \neq x_j \text{ for } i \neq j\}$$

be a ordered configuration space. Let the symmetric group Σ_n act on $F(M, n)$ by permuting positions. Then $F(M, n) \to F(M, n)/\Sigma_n$ is a covering. The quotient $F(M, n)/\Sigma_n$, denoted by $B(M, n)$, is called the space of *unordered configurations*.

6) Let G be a (Hausdorff) topological group and let H be a finite subgroup of G. Let G/H be the set of left cosets with quotient topology. Then $G \to G/H$ is a covering. (**Note:** One can directly show that $G \to G/H$ is a covering if H is a discrete subgroup of G (without assuming that H is finite).

4. The Lifting Theorem For Covering Spaces

If $p\colon \tilde{X} \to X$ is a covering and $f\colon Y \to X$ is a map, then a *lifting* of f is a continuous map $\tilde{f}\colon Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$.

The lifting problem is: Given a map $f\colon Y \to X$.

i) When does there exist a lifting of f?
ii) Must such a lifting be unique?

The 'uniqueness' can be answered as follows.

LEMMA 4.35. *Let $p\colon \tilde{X} \to X$ be a covering and let $\tilde{f}, \bar{f}\colon Y \to \tilde{X}$ be two lifting of $f\colon Y \to X$. Suppose that Y is connected and $\tilde{f}(y_0) = \bar{f}(y_0)$ for some $y_0 \in Y$. Then $\tilde{f} = \bar{f}$.*

PROOF. Let $Y' = \{y \in Y | \tilde{f}(y) = \bar{f}(y)\}$. Then $y_0 \in Z$. It suffices to show that Z is open and closed. (**Note:** A space Y is connected if and only if Y and $\emptyset$ are only open and closed subsets of Y (or, equivalently, Y is not disjoint union of two open subsets). A path-connected space is connected, but a connected space may not be path connected in general.)

First we show that Y' is an open subset of Y. Let $y \in Y'$ and let U be an elementary neighbourhood of $f(y)$ in X. There is a (unique) sheet U_α of $p^{-1}(U)$ such that $\tilde{f}(y) = \bar{f}(y) \in U_\alpha$. Then $\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha)$ is an open neighbourhood of y. Since $p|_{U_\alpha}\colon U_\alpha \to U$ is a homeomorphism,

$$\tilde{f}|_{\tilde{f}^{-1}(U_\alpha)\cap\bar{f}^{-1}(U_\alpha)} = \bar{f}|_{\tilde{f}^{-1}(U_\alpha)\cap\bar{f}^{-1}(U_\alpha)}.$$

Thus

$$\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\alpha) \subseteq Y'$$

and so Y' is open.

Now we show that $Y \setminus Y'$ is open. Let $y \in Y \setminus Y'$ and let U be an elementary neighbourhood of $f(y)$ in X. Since $\tilde{f}(y) \neq \bar{f}(y)$, there are two different sheets U_α and U_β of $p^{-1}(U)$ such that $\tilde{f}(y) \in U_\alpha$ and $\bar{f}(y) \in U_\beta$. ($\alpha \neq \beta$ because p restricted to each sheet is a homeomorphism.) Now $\tilde{f}^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\beta)$ is an open neighbourhood of y. Since $U_\alpha \cap U_\beta = \emptyset$, $\tilde{f}(z) \neq \bar{f}(z)$ for any $z \in f^{-1}(U_\alpha) \cap \bar{f}^{-1}(U_\beta)$ and so

$$\tilde{f}^{-1}(U_\alpha \cap \bar{f}^{-1}(U_\beta) \subseteq Y \setminus Y'.$$

Thus $Y \setminus Y'$ is open and hence the result. □

COROLLARY 4.36. *Suppose that $\tilde{X}$ is connected and $\phi\colon \tilde{X} \to \tilde{X}$ is a map such that $p \circ \phi = p$. If $\phi(x_1) = x_1$ for some $x_i \in \tilde{X}$, then ϕ is the identity map.*

PROOF. Both ϕ and the identity map $\mathrm{id}_{\tilde{X}}$ are liftings of the map $p\colon \tilde{X} \to X$. Since $\phi(x_1) = \mathrm{id}_{\tilde{X}}(x_1)$, the assertion follows from Lemma 4.35. □

Let X be a pointed space with a base-point x_0 and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$.

THEOREM 4.37 (Path-lifting Theorem). *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering. Then*

i) *Every path $\lambda\colon (I, 0) \to (X, x_0)$ has a unique lifting $\tilde{\lambda}\colon (I, 0) \to (\tilde{X}, \tilde{x}_0)$.*
ii) *Every map $F\colon (I \times I, (0, 0)) \to (X, x_0)$ has a unique lifting $\tilde{F}\colon (I \times I, (0, 0)) \to (\tilde{X}, \tilde{x}_0)$.*

PROOF. We already prove the uniqueness of a lifting. So we only need to prove the existence.

i) There exist $0 = t_0 < t_1 < \cdots t_m = 1$ such that $\lambda([t_i, t_{i+1}])$ is contained in some elementary neighborhood of each i. We show that there is a lifting $\tilde{\lambda}_i\colon [0, t_i] \to \tilde{X}$ of $\lambda|_{[0,t_i]}$ by induction on i.

When $i = 0$, $\tilde{\lambda}_0 \colon [0,0] \to \tilde{X}$ is given by $\tilde{\lambda}(0) = \tilde{x}_0$. Suppose that there is a lifting $\tilde{\lambda}_i \colon [0,t_i] \to \tilde{X}$. Since $\lambda([t_i,t_{i+1}])$ lies in an elementary neighbourhood. There is a unique lifting $\mu \colon [t_i,t_{i+1}] \to \tilde{X}$ of $\lambda|_{[t_i,t_{i+1}]}$ such that $\mu(t_i) = \tilde{\lambda}_i(t_i)$ (The map μ is obtained by composing $\lambda|_{[t_i,t_{i+1}]}$ with the inverse homeomorphism to p-restricted-to-the-sheet-containing-$\tilde{\lambda}_i(t_i)$. Let

$$\tilde{\lambda}_{i+1} = \tilde{\lambda}_i \cup \mu \colon [0,t_{i+1}] \to \tilde{X}.$$

Then $\tilde{\lambda}_{i+1}$ is a lifting of $\lambda|_{[0,t_{i+1}]}$. This gives a construction of $\tilde{\lambda}$ by induction.

ii) The proof essentially follows from the same idea, that is there are sequence $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that F maps each small rectangle $R_{i,j} = [s_i,s_{i+1}] \times [t_j,t_{j+1}]$ into an elementary neighbourhood and then defined $\tilde{F}$ inductively over the rectangles

$$R_{0,0}, R_{0,1}, \cdots, R_{0,n}, R_{1,0}, \cdots.$$

□

Corollary 4.38 (Monodromy Lemma). *Let $\tilde{\lambda}_0, \tilde{\lambda}_1 \colon (I,0) \to (\tilde{X},\tilde{x}_0)$ be paths with $p\circ\tilde{\lambda}_0 \simeq p\circ\tilde{\lambda}_1$. Then $\tilde{\lambda}_0 \simeq \tilde{\lambda}_1$. In particular, $\tilde{\lambda}_0(1) = \tilde{\lambda}_1(1)$.*

Proof. Let $\lambda_0 = p\circ\tilde{\lambda}_0$ and $\lambda_1 = p\circ\lambda_1$. Let $F\colon I\times I \to X$ be a homotopy relative to $\{0,1\}$ from λ_0 to λ_1. Then there is a unique lifting $\tilde{F}\colon I\times I \to \tilde{X}$ of F with $\tilde{F}(0,0) = \tilde{\lambda}_0(0) = \tilde{\lambda}_1(0)$. Then

1) $\tilde{F}(t,0) = \tilde{\lambda}_0(t)$ for any t because both of them are lifting of λ_0 with $\tilde{F}(0,0) = \tilde{\lambda}_0(0)$. And $\tilde{F}(1,0) = \tilde{\lambda}_0(1)$.
2) $\tilde{F}(0,s) = \epsilon_{\tilde{\lambda}_0(0)}$ because both of them are liftings of $F(0,s) = \epsilon_{\lambda(0)}$ with $\tilde{F}(0,0) = \lambda_0(0)$. And $\tilde{F}(0,1) = \tilde{\lambda}_0(0) = \tilde{\lambda}_1(0)$.
3) $\tilde{F}(t,1) = \tilde{\lambda}_1(t)$ because $\tilde{F}(0,1) = \tilde{\lambda}_1(0)$ and both of them are liftings of λ_1. In particular, $\tilde{F}(1,1) = \tilde{\lambda}_1(1)$.
4) $\tilde{F}(1,s) = \epsilon_{\tilde{\lambda}_0(1)}$ because $\tilde{F}(1,0) = \tilde{\lambda}_0(1)$ and both of them are liftings of $\epsilon_{\lambda_0(1)}$.

This show that $\tilde{F}$ is a path homotopy from $\tilde{\lambda}_0$ to $\tilde{\lambda}_1$. □

If in Corollary 4.38 we consider only loops, then we immediately have

Theorem 4.39. *If $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ is a covering, then $p_*\colon \pi_1(\tilde{X},\tilde{x}_0) \to \pi_1(X,x_0)$ is a monomorphism.* □

Let $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a covering projection. The function $\psi\colon \pi_1(X,x_0) \to p^{-1}(x_0)$ is defined by $[\alpha] \mapsto \tilde{\alpha}(1)$, where $\tilde{\alpha}\colon (I,0,1) \to (\tilde{X},\tilde{x}_0,\tilde{\alpha}(1))$ is the unique lifting of α as in Theorem 4.37. The function ψ is well-defined by the Monodromy Lemma (Corollary 4.38).

Exercise 4.1. Suppose that $\tilde{X}$ is path-connected. Show that the function $\psi\colon \pi_1(X,x_0) \to p^{-1}(x_0)$ is onto.

Hint: Let $y \in p^{-1}(x_0)$. There is a path β from $\tilde{x}_0$ to y. Let $\alpha = p\circ\beta$. Then $\beta = \tilde{\alpha}$ by the uniqueness of the lifting and so $\psi([\alpha]) = \tilde{\alpha}(1) = \beta(1) = y$.

Theorem 4.40. *If $\tilde{X}$ is simply connected, then ψ is a bijection.*

PROOF. By Exercise 4.1, it suffices to show that ψ is one-to-one.

Suppose that $[\alpha], [\beta] \in \pi_1(X, x_0)$ with $\psi([\alpha]) = \psi([\beta]) = y \in p^{-1}(x_0)$, that is $\tilde{\alpha}(1) = \tilde{\beta}(1) = y$, where $\tilde{\alpha}$ and $\tilde{\beta}$ are the liftings of $[\alpha]$ and β, respectively. Since $\tilde{X}$ is simply connected, $[\tilde{\alpha} * \tilde{\beta}^{-1}] = 1 \in \pi_1(\tilde{X}, \tilde{x}_0)$. Thus

$$[\alpha][\beta]^{-1} = [(p \circ \tilde{\alpha}) * (p \circ \tilde{\beta}^{-1}] = [p \circ (\tilde{\alpha} * \tilde{\beta}^{-1})] = p_*([\tilde{\alpha} * \tilde{\beta}^{-1}]) = p_*(1) = 1.$$

Hence $[\alpha] = [\beta] \in \pi_1(X, x_0)$. □

Now suppose that the quotient $p\colon \tilde{X} \to \tilde{X}/G$, $\tilde{x} \mapsto [\tilde{x}]$, is a covering space arising from a properly discontinuous group action. Here we can do much better.

Since $p^{-1}([\tilde{x}_0]) = G \cdot \tilde{x} = \{g \cdot \tilde{x}_0 | g \in G\}$, we can identify $p^{-1}([\tilde{x}_0])$ with G by $g \cdot \tilde{x}_0 \leftrightarrow g$. (Recall: $g \cdot \tilde{x}_0 = g' \cdot \tilde{x}_0 \Rightarrow g = g'$ by the properly discontinuous property.)

THEOREM 4.41. *If $\tilde{X}$ is path-connected, then $\psi\colon \pi_1(\tilde{X}/G, [\tilde{x}_0]) \to G$ is a group epimorphism with kernel $p_*\pi_1(\tilde{X}, \tilde{x}_0)$.*

PROOF. (i) By Exercise 4.1, the function ψ is onto.

(ii) To see that it's a homomorphism, recall that the lifting $\tilde{\alpha}\colon (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, \tilde{\alpha}(1))$ of a loop α representing $[\alpha] \in \pi_1(\tilde{X}/G, [\tilde{x}_0])$ has $\alpha(1) = g_\alpha \cdot \tilde{x}_0$ for some unique $g_\alpha \in G$ (independent of choice of $\alpha \in [\alpha]$.)

Given $[\alpha], [\beta] \in \pi_1(\tilde{X}/G, [\tilde{x}])$, with α, β lifting to $\tilde{\alpha}\colon (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, g_\alpha \cdot \tilde{x}_0)$, $\tilde{\beta}\colon (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, g_\beta \cdot \tilde{x}_0)$, note that in general $\tilde{\alpha} * \tilde{\beta}$ is not defined (since $g_\alpha \cdot \tilde{x}_0 \neq \tilde{x}_0$). However the map $g_\alpha \cdot\colon \tilde{X} \to \tilde{X}$ composes with $\tilde{\beta}$ to give

$$g_\alpha \cdot \tilde{\beta}\colon (I, 0, 1) \to (\tilde{X}, g_\alpha \cdot \tilde{x}_0, g_\alpha \cdot (g_\beta \cdot \tilde{x}_0))$$

which lifts β (Note $g_\alpha \cdot \tilde{\beta}$ is from $g_\alpha \cdot \tilde{x}_0$ to $g_\alpha \cdot (g_\beta \cdot \tilde{x}_0)$). Thus

$$\tilde{\alpha} * (g_\alpha \cdot \tilde{\beta})\colon (I, 0, 1) \to (\tilde{X}, \tilde{x}_0, g_\alpha g_\beta \cdot \tilde{x}_0)$$

is well-defined and lifts $\alpha * \beta$. Since this lifting of $\alpha * \beta$ has final point $g_\alpha g_\beta \cdot \tilde{x}_0$, we have $\psi([\alpha * \beta]) = g_\alpha g_\beta$ and hence

$$\psi([\alpha][\beta]) = \psi([\alpha * \beta]) = g_\alpha g_\beta = \psi([\alpha])\psi([\beta]).$$

(iii) If $\psi([\alpha]) = e \in G$, then $\tilde{\alpha}(1) = e \cdot \tilde{x}_0 = \tilde{x}_0$, making $\tilde{\alpha}$ a loop. Hence

$$[\alpha] = [p \circ \tilde{\alpha}] = p_*([\tilde{\alpha}]) \in p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Conversely, for any $\tilde{\alpha}\colon (I, \partial I) \to (\tilde{X}, \tilde{x}_0)$, $p \circ \tilde{\alpha}$ has lifting $\tilde{\alpha}$ with $\tilde{\alpha}(1) = e \cdot \tilde{x}_0$, and so $\psi(p_*([\tilde{\alpha}])) = \psi([p \circ \tilde{\alpha}]) = e \in G$. □

COROLLARY 4.42. *Suppose that $\tilde{X}$ is path-connected space on which the group G acts properly discontinuously. Then*

$$\psi\colon \pi_1(\tilde{X}/G, \tilde{x}_0) \longrightarrow G$$

is an isomorphism if and only if $\tilde{X}$ is simply-connected.

EXAMPLE 4.43. 1) Since S^n is simply connected for $n \geq 2$, we have

$$\pi_1(\mathbb{R}P^n) = \pi_1(S^n/\mathbb{Z}/2) = \mathbb{Z}/2$$

for $n \geq 2$.

2) $\pi_1(L^n(p)) = \pi_1(S^{2n+1}/\mathbb{Z}/p) = \mathbb{Z}/p$ $(n \geq 1)$.

3) $\pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$.

A space X is called to be *locally path-connected* if for each point $x \in X$ and **any** neighborhood U of x there exists a path-connected open neighborhood V of x with $V \subseteq U$. (**Note.** In Spanier's book [**21**], the definition of locally path-connected is as follows: A space X is said to be *locally path-connected* if, for each $x \in X$ and any neighborhood U of x, there is an open neighborhood V of x such that $x \in V \subseteq U$ and any two points in V can be connected by a path in U. Thus our definition of locally path-connected is stronger than Spanier's definition. Our definition follows from Hatcher's book [**7**, pp.62]. This more restrictive definition simplifies the proofs. But keep in mind that the following statements hold for locally path-connected in the sense of Spanier's book.)

THEOREM 4.44 (Lifting Theorem). *Let* $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *is a covering space. Let* $f\colon (Y, y_0) \to (X, x_0)$ *be a map. Suppose that* Y *is path-connected and locally path-connected. Then* $f\colon (Y, y_0) \to (X, x_0)$ *admits a unique lifting* $\tilde{f}\colon (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ *if and only if*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

PROOF. $\Rightarrow$ is obvious.

$\Leftarrow$ By Lemma 4.35, if f admits a lifting, then the lifting is unique. Thus it suffices to prove the existence of the lifting. The construction of $\tilde{f}$ is as follows:

For each $y \in Y$, since Y is path-connected, there is a path $\lambda\colon (I, 0, 1) \to (Y, y_0, y)$. So lift $f \circ \lambda\colon (I, 0) \to (X, x_0)$ uniquely (by Theorem 4.37) to $\widetilde{f \circ \lambda}\colon (I, 0) \to (\tilde{X}, \tilde{x}_0)$. Let

$$\tilde{f}(y) = \widetilde{f \circ \lambda}(1).$$

Then $p \circ \tilde{f} = f$.

We must prove that

i) $\tilde{f}(y)$ is independent of choice of $\lambda\colon (I, 0, 1) \to (Y, y_0, y)$, that is $\tilde{f}$ is well-defined as a function, and
ii) $\tilde{f}$ is continuous.

To see (i), let λ and λ' be two paths in Y from y_0 to y. Then the path product $\lambda * \lambda'^{-1}$ is a loop in Y from y_0 to y_0. By the assumption

$$[(f \circ \lambda) * f(\lambda'^{-1})] \in f_*(\pi_1(Y, y_0)) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0).$$

Thus the loop $(f \circ \lambda) * f(\lambda'^{-1})$ admits a lifting in $(\tilde{X}, \tilde{x}_0)$ as a **loop**. By uniqueness of lifting, the first half lifting of this loop is given by $\widetilde{f \circ \lambda}$ and second half lifting is given $\widetilde{f \circ \lambda'}$. In particular, $\widetilde{f \circ \lambda}(1) = \widetilde{f \circ \lambda'}(1)$ because they form a loop.

For showing (ii), let W be an open neighborhood of $\tilde{f}(y)$. Choose a small open neighborhood U of $f(y)$ such that $p^{-1}(U)$ is disjoint union of open sets in $\tilde{X}$ with one piece $\tilde{f}(y) \in \tilde{U} \cong U$ and $\tilde{U} \subseteq W$. By the assumption of locally path-connected, there exists a path open neighborhood V of y with $V \subseteq f^{-1}(U)$. Fix a path λ from y_0 to y. For any $y' \in V$, there is a path η from y to y'. Then the path product $\lambda * \eta$ is a path from y_0 to y'. Since

$$p\colon \tilde{U} \to U$$

is a homeomorphism, $p|_{\tilde{U}}^{-1}(f \circ \eta)$ is a path in $\tilde{U}$ from $\tilde{f}(y)$ from a point in $\tilde{U}$. Now the path product $\widetilde{f \circ \lambda} * p|_{\tilde{U}}^{-1}(f \circ \eta)$ is a lifting of $f(\lambda * \eta)$. By the uniqueness of lifting,

$$\widetilde{f(\lambda * \eta)} = \widetilde{f \circ \lambda} * p|_{\tilde{U}}^{-1}(f \circ \eta).$$

In particular

$$\tilde{f}(y') = \widetilde{f(\lambda * \eta)}(1) \in \tilde{U}.$$

It follows that

$$V \subseteq \tilde{f}^{-1}(\tilde{U}) \subseteq \tilde{f}^{-1}(W)$$

and so f is continuous. This finishes the proof. □

COROLLARY 4.45. *Any maps from a simply-connected locally path-connected (Y, y_0) lifts (uniquely).*

COROLLARY 4.46. *Any map from $(S^n, (1, 0, \cdots, 0))$ lifts uniquely ($n \geq 2$).*

COROLLARY 4.47. *For $n \geq 2$, $p_* \colon \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism.*

PROOF. By Corollary 4.46, p_* is onto. By Corollary 4.45, p_* is one-to-one because $S^n \times I$ is simply connected for $n \geq 2$. □

THEOREM 4.48 (Borsuk-Ulam). *There exists no map $f \colon S^2 \to S^1$ such that $f(-x) = -f(x)$ for any x.*

PROOF. Let $q \colon S^2 \to \mathbb{R}P^2$ be the covering projection, and suppose that for all $x \in S^2$

$$f(-x) = -f(x).$$

Then we can define $g \colon \mathbb{R}P^2 \to S^1$ by $g(\pm x) = (f(x))^2$, making $g \circ q = p \circ f$, where $p \colon S^1 \to S^1$ is defined by $z \mapsto z^2$.

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^1 \\ \downarrow q & & \downarrow p \\ \mathbb{R}P^2 & \xrightarrow{g} & S^1 \end{array}$$

Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, $g_*\pi_1(\mathbb{R}P^2)$ is a torsion subgroup of $\pi_1(S^1) = \mathbb{Z}$ and hence $g_*\pi_1(\mathbb{R}P^2) = 0$. Thus, by Theorem 4.44, there is a lifting $\tilde{g} \colon \mathbb{R}P^2 \to S^1$ such that $g = p \circ \tilde{g}$. (Note the map p is a covering.) Since $\tilde{g} \circ q$ and f are two liftings of $g \circ q$, we have

$$\tilde{g} \circ q = f.$$

It follows that

$$f(x) = \tilde{g} \circ q(x) = \tilde{g} \circ q(-x) = f(-x) = -f(x),$$

a contradiction. □

COROLLARY 4.49. *If $g \colon S^2 \to \mathbb{R}^2$ is an antipode-preserving map, that is $g(-x) = -g(x)$, then some $x \in S^2$ has $g(x) = 0$.*

PROOF. Otherwise $f \colon S^2 \to S^1 \quad x \to \frac{g(x)}{\|g(x\|}$ contradicts Theorem 4.48. □

COROLLARY 4.50. *If $h \colon S^2 \to \mathbb{R}^2$, then some $x \in S^2$ has $h(x) = h(-x)$; so h is not injective.*

PROOF. If this were not the case, then $g \colon S^2 \to \mathbb{R}^2 \quad x \mapsto h(x) - h(-x)$ would contradicts Corollary 4.49. □

COROLLARY 4.51. *No subspace of $\mathbb{R}^2$ is homeomorphic to S^2.*

EXAMPLE 4.52. Regard the Earth as S^2 and the functions

$P\colon S^2 \to \mathbb{R}$, $x \mapsto$ barometric pressure at x,
$T\colon S^2 \to \mathbb{R}$, $x \mapsto$ temperature at x

as continuous. Then Corollary 4.50 says that

$$h\colon S^2 \to \mathbb{R}^2 \quad h(x) = (P(x), T(x))$$

has $h(-x) = h(x)$ for some $x \in S^2$, in other words, there are always two antipodal places on Earth with the same temperature and pressure.

5. Universal Covering

Let X be a path-connected space. A covering $p\colon \tilde{X} \to X$ is called *universal* if $\tilde{X}$ is simply connected.

PROPOSITION 4.53. *Let X be path-connected and locally path-connected. Then the universal covering over X is unique provided that it exists.*

PROOF. Suppose that $p\colon \tilde{X} \to X$ and $p'\colon \tilde{X}' \to X$ be two universal coverings over X. By the definition, both $\tilde{X}$ and $\tilde{X}'$ are simply connected. In particular, both $\tilde{X}$ and $\tilde{X}'$ are path-connected. Since X is locally path-connected, so are $\tilde{X}$ and $\tilde{X}'$. Let x_0 be a basepoint of X. Choose basepoints $\tilde{x}_0 \in p^{-1}(x_0)$ and $\tilde{x}'_0 \in p'^{-1}(x_0)$. By the lifting theorem, there exist liftings

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{f} & \tilde{X}' & \xrightarrow{g} & \tilde{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ X & = & X & = & X \end{array}$$

with $f(\tilde{x}_0) = \tilde{x}'_0$ and $g(\tilde{x}'_0) = \tilde{x}_0$ because $\pi_1(\tilde{X})$ and $\pi_1(\tilde{X}')$ are trivial. By the uniqueness of the lifting, $g \circ f = \mathrm{id}_{\tilde{X}}$ and $f \circ g = \mathrm{id}_{\tilde{X}'}$ and hence the result. □

5.1. Existence of Universal Covering Space.

A space X is called *semi-locally simply-connected* if for each point $x \in X$ there exists a neighborhood U of x such that $\pi_1(U, x) \to \pi_1(X, x)$ is trivial.

THEOREM 4.54. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then there exists the universal covering $\tilde{X} \to X$.*

PROOF. The proof is given by constructing a universal covering over X. Let x_0 be a basepoint of X.

Construction: Define

$$\tilde{X} = \{[\lambda] \mid \lambda(0) = x_0\},$$

where $[\lambda]$ is the homotopy class relative to the ending points, that is, with respect to the homotopies that fix the endpoints. Define $p\colon \tilde{X} \to X$ by $p([\lambda]) = \lambda(1)$.

Topology on $\tilde{X}$. Let

$$\mathcal{U} = \{U \subseteq X \mid U \text{ path-connected open } \pi_1(U) \to \pi_1(X) \text{ is trivial } \}.$$

By the assumption of semi-locally simply connected, for any $x \in X$, there exists a neighborhood U of x such that $\pi_1(U) \to \pi_1(X)$ is trivial. By the assumption of locally path-connected, there exists a path-connected open neighborhood V of x such that $V \subseteq U$ with $\pi_1(V) \to \pi_1(X)$ is trivial as it is the composite $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$. Thus $\mathcal{U}$ is a basis for the topology on X. (**Note.** We use the assumptions that X locally path-connected and semi-locally simply connected.)

For $U \in \mathcal{U}$ and a path λ from x_0 to a point in U, define

$$U_{[\lambda]} = \{[\lambda * \eta] \mid \eta \text{ path in } U \text{ with } \eta(0) = \lambda(1)\}.$$

1) $U_{[\lambda]}$ depends only on the path homotopy class of λ, that is, if $\lambda \simeq \lambda' \operatorname{rel} 0, 1$, then $U_{[\lambda]} = U_{[\lambda']}$.
2) $p\colon U_{[\lambda]} \to U$ is onto because U is path-connected.
3) $p\colon U_{[\lambda]} \to U$ is one-to-one. Let η and η' be two paths in U such that $\eta(0) = \eta'(0) = \lambda(1)$ and $\eta(1) = \eta'(1)$. Then $\eta * \eta'^{-1}$ form a loop in U. Since $\pi_1(U) \to \pi_1(X)$ is trivial, $[\eta] * [\eta'^{-1}]$ is trivial in X and so the loop
$$[(\lambda * \eta) * (\lambda * \eta')^{-1}] = 1.$$
It follows that the path homotopy class $[\lambda * \eta] = [\lambda * \eta']$.
4) If $[\lambda'] \in U_{[\lambda]}$, then $U_{[\lambda']} = U_{[\lambda]}$. For seeing this, let $\lambda' = \lambda * \eta$. For any $[\lambda * \eta'] \in U_{[\lambda]}$, let μ be a path in U from $\eta(1)$ to $\eta'(1)$. Then $\eta * \mu * \eta'^{-1}$ is a loop in U from $\lambda(1)$ to $\lambda(1)$. By using the assumption that $\pi_1(U) \to \pi_1(X)$ is trivial, $[\eta * \mu * \eta'^{-1}]$ is trivial in $\pi_1(X, \lambda(1))$. Thus
$$[(\lambda * \eta * \mu) * (\lambda * \eta')^{-1}] = 1$$
in $\pi_1(X, x_0)$ and so
$$[\lambda * \eta'] = [(\lambda * \eta) * \mu],$$
that is $[\lambda * \eta'] \in U_{[\lambda']}$. Or $U_{[\lambda]} \subseteq U_{[\lambda']}$. Similarly, $U_{[\lambda']} \subseteq U_{[\lambda]}$. Thus $U_{[\lambda']} = U_{[\lambda]}$.

Now we show that

$$\tilde{\mathcal{U}} = \{U_{[\lambda]} \mid U \in \mathcal{U}, \lambda \text{ path from } x_0 \text{ a point in } U\}$$

forms a basis for a topology on $\tilde{X}$. Let $U_{[\lambda]}$, $V_{[\lambda']} \in \tilde{\mathcal{U}}$ with $[\lambda''] \in U_{[\lambda]} \cap V_{[\lambda']}$. Then

$$U_{[\lambda'']} = U_{[\lambda]} \quad V_{[\lambda'']} = V_{[\lambda']}$$

by assertion (4) above. Let W be in $\mathcal{U}$ with $\lambda''(1) \in W \subseteq U \cap V$. Then

$$[\lambda''] \in W_{[\lambda'']} \subseteq U_{[\lambda'']} \cap V_{[\lambda'']} = U_{[\lambda]} \cap V_{[\lambda']}$$

and so $\tilde{\mathcal{U}}$ forms a basis for a topology on $\tilde{X}$.

$p\colon U_{[\lambda]} \to U$ **is a homeomorphism:** Recall that the open subsets of $U_{[\lambda]}$ is given by $U_{[\lambda]} \cap W$ for open sets W in $\tilde{X}$.

First we show that p is continuous. Let $[\lambda * \eta]$ be any element in $U_{[\lambda]}$ and let V be an open subset of U with $x = \lambda * \eta(1) \in V$. There exists $V' \in \mathcal{U}$ such that $x \in V' \subseteq V$.

$$p^{-1}(V') = \{[\lambda * \eta'] \mid \eta'(0) = \lambda(1)\ \eta'(1) \in V'\}.$$

Note that

$$[\lambda * \eta] \in V'_{[\lambda * \eta]} = \{[\lambda * \eta * \eta''] \mid \eta''(0) = x,\ \eta'(1) \in V'\} \subseteq p^{-1}(V') \subseteq p^{-1}(V).$$

Thus p is continuous.

Next we show that $p|_{U_{[\lambda]}}^{-1}$ is continuous. Let x be any point in U. Fix a path η in U from $\lambda(1)$ to x. For any open subset W in $\tilde{X}$ with $p|_{U_{[\lambda]}}^{-1}(x) = [\lambda * \eta] \in W \cap U_{[\lambda]}$,since $\tilde{\mathcal{U}}$ is basis, there is $V_{[\lambda']}$ such that

$$[\lambda * \eta] \in V_{[\lambda']} \subseteq W \cap U_{[\lambda]}.$$

Then

$$V = p(V_{[\lambda']}) = (p|_{U_{[\lambda]}}^{-1})^{-1}(V_{[\lambda']})$$

is open. Thus $p|_{U_{[\lambda]}}^{-1}$ is continuous.

$p\colon \tilde{X} \to X$ **is a covering:** For $U \in \mathcal{U}$,

$$p^{-1}(U) = \bigcup \{U_{[\lambda]} \mid \lambda(1) \in U\}.$$

Assume that $U_{[\lambda]} \cap U_{[\lambda']} \neq \emptyset$. Let $[\lambda''] \in U_{[\lambda]} \cap U_{[\lambda']}$. By assertion (4) above, $U_{[\lambda'']} = U_{[\lambda]} = U_{[\lambda']}$. Thus $p^{-1}(U)$ is a disjoint union of $U_{[\lambda]}$ and so p is a covering map.

The space $\tilde{X}$ is simply connected. First we show that $\tilde{X}$ is path-connected. Given two points $[\lambda], [\lambda'] \in \tilde{X}$. Since X is path connected, there is a path η in X from $\lambda(1)$ to $\lambda'(1)$. Let η_t be part of the path η from $\eta(0)$ to $\eta(t)$. Then $[\lambda * \eta_t]$ gives a path in $\tilde{X}$ from $[\lambda]$ to $[\lambda']$. Thus $\tilde{X}$ is path-connected.

Finally we show that $\pi_1(\tilde{X}, [x_0])$ is trivial. Since p is a covering map, $p_*\colon \pi_1(\tilde{X}, [x_0]) \to \pi_1(X, x_0)$ is a monomorphism. Let ω be a loop in $\tilde{X}$, which means that $\omega(t)$ is the path homotopy class of a from x_0 to $\omega(t)(1)$ with $\omega(0) = \omega(1) = [x_0]$ the constant path. Consider the homotopy $F_s := \omega(t)(s)$, namely the path $\omega(t)$ evaluating at s. Then $F_s(0) = \omega(0)(s) = x_0$, $F_s(1) = \omega(1)(s) = x_0$, $F_0(t) = \omega(t)(0) = x_0$ and $F_1(t) = \omega(t)(1) = (p \circ \omega)(t)$. Thus F_s defines a null homotopy for $p \circ \omega$. Thus $[p \circ \omega] = 1$ and so $\pi_1(\tilde{X})$ is trivial. □

Note. In Spanier's book [**21**], the construction of the universal covering space is given as the quotient space of the path space $PX = \{\lambda\colon I \to X \mid \lambda(0)\}$ with compact-open topology. In his book, the arguments use the knowledge of compact-open topology.

Corollary 4.55. *Let X be path-connected, locally path-connected, and semi-locally simply connected. Then for every subgroup $H \subseteq \pi_1(X, x_0)$ there exists a covering space $p\colon \tilde{X}_H \to X$ such that $p_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$ for a suitably chosen basepoint $\tilde{x}_0 \in \tilde{X}_H$.*

Proof. In the universal covering space $\tilde{X}$, define the equivalence relation

$$[\lambda] \sim [\lambda']$$

if $\lambda(1) = \lambda'(1)$ and $[\lambda * \lambda'^{-1}] \in H$. Define $\tilde{X}_H = \tilde{X}/\sim$. Then the resulting covering $\tilde{X}_H \to X$ is the desired coving. See Hatcher's book [**7**] for details. □

An application to combinatorial group theory is to give a geometric proof of the following theorem:

Theorem 4.56. *Any subgroup of a free group is free.*

Proof. Let F be a free group. Then we can choose X a connected 1-dimensional cell complex such that $\pi_1(X) = F$. Let H be a subgroup of F. Then there is a covering $p\colon \tilde{X}_H \to X$ such that $p_*(\pi_1(\tilde{X}_H)) = H$. Since any covering over X is still a 1-dimensional cell-complex, $\tilde{X}_H$ is homotopy to a wedge of circles. It follows that $H \cong \pi_1(\tilde{X}_H)$ is a free group. □

EXERCISE 5.1. Let $p\colon \tilde{X} \to X$ be a covering and let B be a subspace of X. Let $\tilde{B} = p^{-1}(B)$ with projection

$$p' = p|_{\tilde{B}} \colon \tilde{B} \to B$$

be the induced covering. Suppose that

(1). $\tilde{X}$, X and B are path-connected;
(2). $\pi_1(B) \to \pi_{(}X)$ is onto.

Show that $\tilde{B}$ is path-connected.
[Hint: Try a proof by the following steps:
Step 1. By using the Changing-base theorem, show that $\pi_1(B, b) \to \pi_1(X, b)$ is onto for any $b \in B$.
Step 2. Let $x, y \in \tilde{B}$. Show that there is a path λ in $\tilde{B}$ such that $\lambda(0) = x$ and $p'(\lambda(1)) = p'(y)$. (Thus it suffices to show that there is a path in $\tilde{B}$ joint any two points in the fibre.)
Step 3. Let $x, y \in \tilde{B}$ such that $p'(x) = p'(y)$. Let $b = p'(x)$. Since $\tilde{X}$ is path-connected, there is a path λ in $\tilde{X}$ from x to y. Then $p \circ \lambda$ is a loop in X from b to b. By using the statement in Step 1, there is a loop ω in B from b to b such that $\omega \simeq p \circ \lambda$. Let λ' be a path lifting of ω with $\lambda'(0) = x$. By using Monodromy Theorem, $\lambda' \simeq \lambda \operatorname{rel} 0, 1$. In particular, λ' is a path from x to y. Since λ' is a lifting of a loop ω in B, λ' is a path in $\tilde{B}$ joint x and y.]

PROPOSITION 4.57. *Let X be path-connected and let Y be a simply connected. Suppose that*

(1). *There exist small contractible open neighborhoods of x_0 and y_0, respectively.*
(2). *$p\colon \tilde{X} \to X$ is the universal covering over X.*

Then

$$\widetilde{X \vee Y} = \{(x, y) \in \tilde{X} \times Y \mid (p(x), y) \in X \vee Y\}$$

with $p' = (p \times \mathrm{id}_Y)|_{\widetilde{X \vee Y}} \colon \widetilde{X \vee Y} \to X \vee Y$ is the universal covering over $X \vee Y$.

PROOF. Since $p\colon \tilde{X} \to X$ is a covering, so is $p \times \mathrm{id}_Y \colon \tilde{X} \times Y \to X \times Y$. Thus

$$p' = (p \times \mathrm{id}_Y)|_{\widetilde{X \vee Y}} \colon \widetilde{X \vee Y} \to X \vee Y$$

is a covering because it is induced from $p \times \mathrm{id}_Y$. By the above exercise, $\widetilde{X \vee Y}$ is path-connected. From the commutative diagram

$$\begin{array}{ccc}
\pi_1(\widetilde{X \vee Y}) & \longrightarrow & \pi_1(\tilde{X} \times Y) = \pi_1(\tilde{X}) \times \pi_1(Y) = \{1\} \\
\big\downarrow & & \big\downarrow \\
\pi_1(X \vee Y) = \pi_1(X) \coprod \{1\} = \pi_1(X) & \xrightarrow{\cong} & \pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y) = \pi_1(X),
\end{array}$$

$\pi_1(\widetilde{X \vee Y}) = \{1\}$ and hence the result. □

5.2. An Application to co-H-spaces.

THEOREM 4.58. *Let X be a path-connected co-H-space. Suppose that there exists small contractible open neighborhood of x_0. Then $\pi_1(X)$ is free.*

PROOF. Let $\mu'\colon X \to X \vee X$ be the comultiplication. From the homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mu'} & X \vee X \\ \Vert & & \big\downarrow i \\ X & \xrightarrow{\Delta} & X \times X, \end{array}$$

there is a commutative diagram of groups

$$\begin{array}{ccc} \pi_1(X) & \xrightarrow{\mu'_*} & \pi_1(X \vee X) = \pi_1(X) \coprod \pi_1(X) \\ \Vert & & \big\downarrow i_* \\ \pi_1(X) & \xrightarrow{\Delta} & \pi_1(X \times X) = \pi_1(X) \times \pi_1(X). \end{array}$$

Thus μ'_* is a monomorphism with

$$\mathrm{Im}(\mu'_*) \subseteq i_*^{-1}(\Delta(\pi_1(X))).$$

By the following lemma, the subgroup $i_*^{-1}(\Delta(\pi_1(X)))$ of $\pi_1(X) \coprod \pi_1(X)$ is free. Since any subgroup of a free group is also free, $\pi_1(X) \cong \mathrm{Im}(\mu'_*) \subseteq i_*^{-1}(\Delta(\pi_1(X)))$ is a free group. $\square$

Let G be a group. Write $G^{(1)}$ and $G^{(2)}$ for copies of G. Consider the free product $G \coprod G = G^{(1)} \coprod G^{(2)}$. For $g \in G$, write $g^{(i)}$ for the element $g \in G^{(i)}$. Let

$$\tilde{\Delta}(G) = \{g^{(1)}g^{(2)} \in G \coprod G \mid g \in G\}$$

be the subset of $G \coprod G$. Let

$$\Delta(G) = \{(g, g) \in G \times G \mid g \in G\}$$

be the diagonal subgroup of $G \times G$. Let X be a pointed set with basepoint $*$. Let $F[X]$ be the free group generated by X with the single relation $* = 1$, that is, $F[X]$ is the free group generated by $X \smallsetminus \{*\}$.

LEMMA 4.59. *Let G be a group and let $q\colon G \coprod G \to G \times G$ be the canonical quotient homomorphism. Then $q^{-1}(\Delta(G)) \cong F[\tilde{\Delta}(G)]$ is the subgroup of $G \coprod G$ freely generated by $\tilde{\Delta}(G) \smallsetminus \{1\}$.*

PROOF. Let Γ be the subgroup generated by $\tilde{\Delta}(G)$. Clearly $\Gamma \subseteq q^{-1}(\Delta(G))$. We show that

$$q^{-1}(\Delta(G)) \subseteq \Gamma.$$

For words w and w' in $G \coprod G$, denote $w \sim w'$ if $w \equiv w' \mod \Gamma$. Note that

$$g^{(1)} \sim (g^{(2)})^{-1}$$

for any $g \in G$. Now let

$$g_1^{(1)}h_1^{(2)}g_2^{(1)}h_2^{(2)} \cdots g_n^{(1)}h_n^{(2)}$$

be a word in $q^{-1}(\Delta(G)) \subseteq G \coprod G$, that is,

$$g_1g_2 \cdots g_n = h_1h_2 \cdots h_n.$$

Then

$$\begin{aligned} g_1^{(1)}h_1^{(2)}g_2^{(1)}h_2^{(2)}\cdots g_n^{(1)}h_n^{(2)} &\sim (g_1^{(2)})^{-1}h_1^{(2)}g_2^{(1)}h_2^{(2)}\cdots g_n^{(1)}h_n^{(2)} \\ &\sim (h_1^{(1)})^{-1}g_1^{(1)}g_2^{(1)}h_2^{(2)}\cdots g_n^{(1)}h_n^{(2)} \\ &= (h_1^{(1)})^{-1}(g_1g_2)^{(1)}h_2^{(2)}\cdots g_n^{(1)}h_n^{(2)} \\ &\sim ((g_1g_2)^{(2)})^{-1}h_1^{(2)}h_2^{(2)}\cdots g_n^{(1)}h_n^{(2)} \\ &= ((g_1g_2)^{-1}(h_1h_2))^{(2)}g_3^{(1)}h_3^{(2)}\cdots g_n^{(1)}h_n^{(2)} \\ &\cdots \\ &\sim 1. \end{aligned}$$

Thus

$$g_1^{(1)}h_1^{(2)}g_2^{(1)}h_2^{(2)}\cdots g_n^{(1)}h_n^{(2)} \in \Gamma$$

and so $q^{-1}(\Delta(G)) = \Gamma$.

Next we show that Γ is freely generated by $\tilde{\Delta}(G) \setminus \{1\}$. Let

$$\phi\colon F[G] \longrightarrow \Gamma$$

be the group homomorphism such that $\phi(g) = g^{(1)}g^{(2)}$ for $g \in G$ and let

$$w = g_1^{\epsilon_1}g_2^{\epsilon_2}\cdots g_t^{\epsilon_t}$$

be a reduced word in $F[G]$ for the letters $g_i \in G \setminus \{1\}$. Then

$$\phi(w) = (g_1^{(1)}g_1^{(2)})^{\epsilon_1}(g_2^{(1)}g^{(2)})^{\epsilon_2}\cdots(g_t^{(1)}g_t^{(2)})^{\epsilon_t}.$$

The assertion will follow proved that we prove by induction on t that

$\phi(w)$ ends with $g_t^{(2)}$ if $\epsilon_t = 1$, and ends with $(g_t^{-1})^{(1)}$ if $\epsilon_t = -1$

because if so, then $\phi(w) \neq 1$. Clearly this statement holds for $t = 1$. Suppose that the statement holds for less t.

Case I $\epsilon_t = 1$. Then

$$\begin{aligned} \phi(w) &= w'(g_{t-1}^{(1)}g_{t-1}^{(2)})^{\epsilon_{t-1}}g_t^{(1)}g_t^{(2)} \\ &= \begin{cases} w'g_{t-1}^{(1)}g_{t-1}^{(2)}g_t^{(1)}g_t^{(2)} & \text{ends } g_t^{(2)} \quad \text{if} \quad \epsilon_{t-1} = 1 \\ w'(g_{t-1}^{-1})^{(2)}(g_{t-1}^{-1})^{(1)}g_t^{(1)}g_t^{(2)} & \\ = w'(g_{t-1}^{-1})^{(2)}(g_{t-1}^{-1}g_t)^{(1)}g_t^{(2)} & \text{ends } g_t^{(2)} \quad \text{if} \quad \epsilon_{t-1} = -1, \end{cases} \end{aligned}$$

where $g_{t-1}^{-1}g_t \neq 1$ because w is a reduced word.

Case II $\epsilon_t = -1$. Then

$$\begin{aligned} \phi(w) &= w'(g_{t-1}^{(1)}g_{t-1}^{(2)})^{\epsilon_{t-1}}(g_t^{-1})^{(2)}(g_t^{-1})^{(1)} \\ &= \begin{cases} w'g_{t-1}^{(1)}g_{t-1}^{(2)}(g_t^{-1})^{(2)}(g_t^{-1})^{(1)} & \\ = w'g_{t-1}^{(1)}(g_{t-1}g_t^{-1})^{(2)}(g_t^{-1})^{(1)} & \text{ends } (g_t^{-1})^{(2)} \quad \text{if} \quad \epsilon_{t-1} = 1 \\ w'(g_{t-1}^{-1})^{(2)}(g_{t-1}^{-1})^{(1)}(g_t^{-1})^{(2)}(g_t^{-1})^{(1)} & \text{ends } (g_t^{-1})^{(1)} \quad \text{if} \quad \epsilon_{t-1} = -1, \end{cases} \end{aligned}$$

where $g_{t-1}g_t^{-1} \neq 1$ because w is a reduced word. The induction is finished and hence the proof. □

REMARK 4.60. Theorem 4.58 has an important consequence on co-H-spaces. Let X be a path-connected co-H cell-complex. Then there is homotopy decomposition

$$X \simeq \bigvee_{\alpha} S^1 \vee X',$$

a wedge of circles and a simply connected co-H space. (This result was first given by Eilenberg-Ganea in 1957 with some generalizations given by other people later.)

CHAPTER 5

Homology

In this chapter, we will first start with the Eilenberg-Steenrod axioms for homology and cohomology. By using the axioms, we will be able to do some computations of homology. After going through some basic properties of Δ-groups and simplicial groups, we will then consider singular homology and simplicial homology.

1. Eilenberg-Steenrod Axioms

1.1. Eilenberg-Steenrod Axioms. The axioms for homology theories are given in the book of Eilenberg and Steenrod [**?**].

Let (X, A) be a pair of spaces, that is A is a subspace of X. A map $f\colon (X, A) \to (Y, B)$ means a map $f\colon X \to Y$ such that $f(A) \subseteq B$. Let $f, g\colon (X, A) \to (Y, B)$. We call f is homotopic to g, denoted by $f \simeq g$, if there is a homotopy $F\colon X \times I \to Y$ such that $F_0 = f, F_1 = g$ and $F(A \times I) \subseteq B$. In other words, the homotopy is a map $F\colon (X, A) \times I = (X \times I, A \times I) \to (Y, B)$.

An *unreduced homology theory* h_* means:

(1). a sequence of abelian groups $\{h_n(X, A)\}_{n\in\mathbb{Z}}$ for each pair of spaces (X, A), called *n-relative homology group of X modulo A* and simply denoted by $h_n(X)$ if $A = \emptyset$,
(2). a sequence of group homomorphisms $f_*\colon h_n(X, A) \to h_n(Y, B)$ for $n \in \mathbb{Z}$ and any map $f\colon (X, A) \to (Y, B)$, and
(3). a group homomorphism

$$\partial(n, X, A)\colon h_n(X, A) \longrightarrow h_{n-1}(A)$$

for any $n \in Z$ and any pair of spaces (X, A), called *boundary operator*, with the following six axioms:

Axiom 1. *If $f = \mathrm{id}\colon (X, A) \to (X, A)$, then $f_* = \mathrm{id}\colon h_*(X, A) \to h_*(X, A)$.*

Axiom 2. $(g \circ f)_* = g_* \circ f_*$.

Explicitly, if $f\colon (X, A) \to (Y, B)$ and $g\colon (Y, B) \to (Z, C)$, then

$$(g \circ f)_* = g_* \circ f_*\colon h_*(X, A) \to h_*(Z, C).$$

Note. The above two axioms just means that h_* is a *functor* from the category of pairs of spaces to the category of sequences of abelian groups

Axiom 3. $\partial \circ f_* = (f|_A)_* \circ \partial$

Explicitly the diagram

$$\begin{array}{ccc} h_q(X,A) & \xrightarrow{f_*} & h_q(Y,B) \\ \Big\downarrow \partial & & \Big\downarrow \partial \\ h_{q-1}(A) & \xrightarrow{(f|_A)_*} & h_{q-1}(B) \end{array}$$

commutes.

Note. This axiom means that ∂ is a natural transformation from $h_n \to h_{n-1} \circ R$, where R is a functor sending (X, A) to $(A, \emptyset)$.

Axiom 4 (Exactness Axiom). *For any pair of spaces (X, A), there is a long exact sequence*

$$\cdots \longrightarrow h_{n+1}(X,A) \xrightarrow{\partial_{n+1}} h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{j_*} h_n(X,A) \xrightarrow{\partial_n} \cdots,$$

where $i\colon (A, \emptyset) \to (X, \emptyset)$ and $j\colon (X, \emptyset) \to (X, A)$ are the inclusion maps.

Axiom 5 (Homotopy Axiom). *If $f \simeq g\colon (X, A) \to (Y, B)$, then $f_* = g_*\colon h_*(X, A) \to h_*(Y, B)$.*

Axiom 6 (Excision Axiom). *If U is an open subset of X whose closure $\bar{U}$ is contained in the interior of A (that is $\bar{U} \subseteq V \subseteq A$ for some open set V). Then the inclusion $j\colon (X \smallsetminus U, A \smallsetminus U) \to (X, A)$ induces an isomorphism*

$$j_*\colon h_n(X \smallsetminus U, A \smallsetminus U) \to h_n(X, A)$$

for each $n \in \mathbb{Z}$.

An *ordinary homology theory* means a homology theory h_* that satisfies the following additional axiom:

Axiom 7 (Dimension Axiom). *Let P be a space consisting of a single point. Then $h_q(P) = 0$ for $q \neq 0$.*

In this case, $h_*(X, A)$ is called the homology of (X, A) with coefficients in $G = h_0(P)$, denoted by $H_*(X, A; G)$. Write $H_*(X, A)$ for $H_*(X, A; \mathbb{Z})$ the integral homology.

Axiom 5 can be replaced by the following axiom:

> **Axiom 5′.** Let $j_0, j_1\colon (X, A) \to (X, A) \times I$ be defined by $j_0(x) = (x, 0)$ and $j_1(x) = (x, 1)$. Then $j_{0*} = j_{1*}$.

Proof. Suppose that Axiom 5 holds. Since $j_0 \simeq j_1$, $j_{0*} = j_{1*}$.

Conversely assume that Axiom 5′ holds. Let $f \simeq g\colon (X, A) \to (Y, B)$ under a homotopy $F\colon (X, A) \times I \to (Y, B)$. Then $f = F \circ j_0$ and $g = F \circ j_1$. Thus

$$f_* = F_* \circ j_{0*} = F_* \circ j_{1*} = g_*.$$

□

Note. What we proved is Axioms 2 and Axiom 5 $\Longleftrightarrow$ Axiom 2 and Axiom 5′.

The Excision Axiom can be reformulated as follows

Axiom 6′. Let X_1 and X_2 be subspaces of X such that X_1 is closed and $X = \mathrm{Int}(X_1) \cup \mathrm{Int}(X_2)$. Then the inclusion $i\colon (X_1, X_1 \cap X_2) \to (X, X_2)$ induces an isomorphism

$$i_*\colon h_q(X_1, X_1 \cap X_2) \xrightarrow{\cong} h_q(X, X_2)$$

for each q

Proof. The equivalence of this axiom with the Excision Axiom is easily seen by setting $A = X_2$ and $U = X \smallsetminus U$. □

Exercise 1.1. Prove the following statements:

1) If $(X, A) \simeq (Y, B)$, then $h_*(X, A) \cong h_*(Y, B)$.
2) If A is a deformation retract of X, then $H_n(X, A) = 0$ for all $n \in \mathbb{Z}$. In particular, $H_n(X, X) = 0$ for all $n \in \mathbb{Z}$.

Proposition 5.1. *Let (X, A) be a pair of spaces such that the inclusion $A \to X$ is a cofibration. Then the quotient map $p\colon (X, A) \to (X/A, \{*\})$ induces an isomorphism*

$$p_*\colon h_n(X, A) \to h_n(X/A, \{*\})$$

for all $n \in \mathbb{Z}$.

Proof. By Lemma 4.21, $(X \cup CA, \{*\}) \simeq (X/A, \{*\})$. Thus

$$h_*(X/A, \{*\}) \cong H_*(X \cup CA, \{*\}).$$

By applying the long exact sequence to the sequence

$$(CA, *) \longrightarrow (X \cup CA, *) \longrightarrow (X \cup CA, CA),$$

we obtain

$$h_*(X \cup CA, *) \cong h_*(X \cup CA, CA)$$

because $h_*(CA, *) = 0$. Now we apply Axiom 6′. Let X_1 be the image of $X \coprod A \times [0, 1/2]$ in $X \cup CA$ and let $X_2 = CA$. Then X_1 is closed and $\mathrm{Int}(X_1) \cup \mathrm{Int}(X_2) = X$. Then by Axiom 6′

$$h_*(X \cup CA, CA) \cong H_*(X_1, X_1 \cap X_2).$$

Since $(X_1, X_1 \cap X_2) \simeq (X, A)$,

$$h_*(X_1, X_1 \cap X_2) \cong h_*(X, A)$$

and hence the result. □

For a pointed space X with basepoint x_0, the *reduced homology* $\bar{h}_*$ is defined by setting $\bar{h}_*(X) = h_*(X, \{x_0\})$. Note that $h_*(X) = \bar{h}_*(X) \oplus h_*(P)$.

Corollary 5.2. *Let (X, A) be a pair of spaces such that the inclusion $A \to X$ is a cofibration. Then there is a long exact sequence*

$$\cdots \longrightarrow \bar{h}_{n+1}(X/A) \xrightarrow{\partial_{n+1}} \bar{h}_n(A) \xrightarrow{i_*} \bar{h}_n(X) \xrightarrow{p_*} \bar{h}_n(X/A) \xrightarrow{\partial_n} \cdots,$$

where $i\colon A \to X$ is the inclusion and $p\colon X \to X/A$ is the pinch map. □

Corollary 5.3 (Suspension Isomorphism Theorem). *Let X be a pointed space. Then there is a natural isomorphism*

$$\sigma\colon \bar{h}_n(X) \cong \bar{h}_{n+1}(\Sigma X)$$

for all $n \in \mathbb{Z}$.

PROOF. Recall that $\Sigma X = CX/X$. Since $CX \simeq *$, $\bar{h}_*(CX) \cong \bar{h}_*(\{*\}) = 0$. From the long exact sequence

$$\cdots \longrightarrow \bar{h}_{n+1}(CX/X) \xrightarrow{\partial_{n+1}} \bar{h}_n(X) \xrightarrow{i_*} \bar{h}_n(CX) \xrightarrow{p_*} \bar{h}_n(CX/X) \xrightarrow{\partial_n} \cdots,$$

we have

$$h_{n+1}(\Sigma X) \cong h_n(X)$$

for all $n \in \mathbb{Z}$, which is the assertion. $\square$

Note. From the Barratt-Puppe sequence, the boundary map

$$\partial_{n+1} \colon \bar{h}_{n+1}(X/A) \longrightarrow \bar{h}_n(A)$$

in Corollary 5.2 is given by

$$\bar{h}_{n+1}(X/A) = \bar{h}_{n+1}((X \cup CA)/CA) \underset{\cong}{\xleftarrow{\text{pinch}_*}} \bar{h}_{n+1}(X \cup CA)$$

$$\longrightarrow \bar{h}_{n+1}((X \cup CA)/X) = \bar{h}_{n+1}(\Sigma A) \xrightarrow{\nu_* = -1} \bar{h}_{n+1}(\Sigma A) \underset{\cong}{\xleftarrow{\sigma}} \bar{h}_n(A).$$

The following theorem is useful for computing homology of CW-complexes.

THEOREM 5.4 (Mayer-Vietoris Sequence). *Let $X = A \cup B$ be a CW-complex with A and B subcomplexes. Then there is a long exact sequence*

$$\cdots \xrightarrow{\Delta} h_n(A \cap B) \xrightarrow{\alpha = (i_{1*}, i_{2,*})} h_n(A) \oplus h_n(B) \xrightarrow{\beta = j_{1*} - j_{2*}} h_n(X) \xrightarrow{\Delta} h_{n-1}(A \cap B) \longrightarrow \cdots,$$

where $i_1 \colon A \cap B \to A$, $i_2 \colon A \cap B \to B$, $j_1 \colon A \to X$ and $j_2 \colon B \to X$ are the inclusions and Δ is given by the composite

$$h_n(X) \longrightarrow h_n(X, A) \underset{\cong}{\xleftarrow{j_{2*}}} h_n(B, A \cap B) \xrightarrow{\partial_n} h_{n-1}(A \cap B).$$

PROOF. Since $B/(A \cap B) = X/A$, there is a commutative diagram of long exact sequence

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & h_{n+1}(B, A \cap B) & \xrightarrow{\partial'_{n+1}} & h_n(A \cap B) & \xrightarrow{i_{2*}} & h_n(B) & \xrightarrow{p_{2*}} & h_n(B, A \cap B) & \longrightarrow & \cdots \\
 & & \downarrow \cong & & \downarrow i_{1*} & & \downarrow j_{2*} & & \downarrow \cong & & \\
\cdots & \longrightarrow & h_{n+1}(X, A) & \xrightarrow{\partial_{n+1}} & h_n(A) & \xrightarrow{j_{1*}} & h_n(X) & \xrightarrow{q_{1*}} & h_n(X, A) & \longrightarrow & \cdots .
\end{array}$$

We only prove the following sequence

$$h_n(A \cap B) \xrightarrow{\alpha = (i_{1*}, i_{2,*})} h_n(A) \oplus h_n(B) \xrightarrow{\beta = j_{1*} - j_{2*}} h_n(X)$$

is exact.

From the commutative diagram

$$\begin{array}{ccc}
A \cap B & \xhookrightarrow{i_1} & A \\
\downarrow i_2 & & \downarrow j_1 \\
B & \xhookrightarrow{j_2} & X,
\end{array}$$

$j_1 \circ i_1 = j_2 \circ i_2$. Let $x \in h_n(A \cap B)$. Then $\alpha \circ \beta(x) = j_{1*} \circ i_{1*}(x) - j_{2*} \circ i_{2*}(x) = 0$.

Let $(a, b) \in h_n(A) \oplus h_n(B)$ such that $j_{1*}(a) = j_{2*}(b)$. Then

$$0 = q_{1*} \circ j_{1*}(a) = q_{1*} \circ j_{2*}(b) = p_{2*}(b).$$

Thus there exists $c \in h_n(A \cap B)$ such that $i_{2*}(x) = b$. Now

$$\begin{array}{rcl} j_{1*}(a - i_{1*}(x)) & = & j_{1*}(a) - j_{1*} \circ i_{1*}(x) \\ & = & j_{1*}(a) - j_{2*} \circ i_{2*}(x) \\ & = & j_{1*}(a) - j_{2*}(b) \\ & = & 0. \end{array}$$

There exists $d \in h_{n+1}(X, A)$ such that

$$\partial_{n+1}(d) = a - i_{1*}(x).$$

Let $x' = \partial'_{n+1}(d) + x$. Then

$$i_{1*}(x') = i_{1*}(\partial'_{n+1}(d) + x) = \partial_{n+1}(d) + i_{1*}(x) = a,$$
$$i_{2*}(x') = i_{2*}(\partial'_{n+1}(d) + x) = 0 + i_{2*}(x) = b.$$

This proves that $h_n(A \cap B) \xrightarrow{\alpha=(i_{1*}, i_{2,*})} h_n(A) \oplus h_n(B) \xrightarrow{\beta=j_{1*}-j_{2*}} h_n(X)$ is exact. □

EXERCISE 1.2. Finish the proof for the above theorem.

COROLLARY 5.5. *Let h_* be a homology theory and let X and Y be CW-complexes. Then*

$$\bar{h}_*(X \vee Y) \cong \bar{h}_*(X) \oplus \bar{h}_*(Y).$$

Thus for a finite wedge of CW-complexes, we have

$$\bar{h}_*(\bigvee_{i=1}^{n} X_i) = \bigoplus_{i=1}^{n} \bar{h}_*(X_i).$$

For infinite wedge, we need to give an axiom which is true for many homology theory such as ordinary homology:

Wedge Axiom: $\bar{h}_*(\bigvee_\alpha X_\alpha) = \bigoplus_\alpha \bar{h}_*(X_\alpha)$ for any wedge of CW-complexes.

THEOREM 5.6. *Let $T\colon \bar{h}_* \to \bar{h}'_*$ be a natural transformation of homology theories, that is, there are commutative diagrams*

$$\begin{array}{ccccccc} \bar{h}_*(X) & \xrightarrow{f_*} & \bar{h}_*(Y) & \qquad & \bar{h}_*(X) & \xrightarrow[\cong]{\sigma} & \bar{h}_{*+1}(\Sigma X) \\ \downarrow T(X) & & \downarrow T(Y) & & \downarrow T(X) & & \downarrow T(Y) \\ \bar{h}'_*(X) & \xrightarrow{f_*} & \bar{h}_*(Y) & & \bar{h}'_*(X) & \xrightarrow[\cong]{\sigma} & \bar{h}'_{*+1}(\Sigma X) \end{array}$$

for any $f\colon X \to Y$. Suppose that

$$T(S^0)\colon \bar{h}_q(S^0) \to \bar{h}'_q(S^0)$$

is an isomorphism for each q. Then for every finite complex X

$$T(X)\colon \bar{h}_q(X) \to \bar{h}'_q(X)$$

is an isomorphism for each q.

Proof. The commutative diagram

$$\begin{array}{ccc}
\bar{h}_*(S^0) & \xrightarrow[\cong]{\sigma^n} & \bar{h}_{*+1}(S^n) \\
\Big\downarrow{\scriptstyle T(S^0)} & & \Big\downarrow{\scriptstyle T(Y)} \\
\bar{h}'_*(S^0) & \xrightarrow[\cong]{\sigma} & \bar{h}'_{*+1}(S^n),
\end{array}$$

shows that

$$T(S^n)\colon \bar{h}_*(S^n) \to \bar{h}'_*(S^n)$$

is an isomorphism.

Now we prove that

$$T(X)\colon \bar{h}_*(X) \to \bar{h}'_*(X)$$

is an isomorphism for any finite CW-complex by induction on the number of cells in X. The assertion is true when X is 1-cell complex, that is X is a sphere. Suppose that the assertion holds for all finite CW-complexes with the number of cells less n. Let $X = Y \cup e^q$. Then $X/Y = S^q$ and so there is commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \bar{h}_{k+1}(S^q) & \longrightarrow & \bar{h}_k(Y) & \longrightarrow & \bar{h}_k(X) & \longrightarrow & \bar{h}_k(S^q) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle T(S^q)} & & \Big\downarrow{\scriptstyle T(Y)} & & \Big\downarrow{\scriptstyle T(X)} & & \Big\downarrow{\scriptstyle T(S^q)} & & \\
\cdots & \longrightarrow & \bar{h}'_{k+1}(S^q) & \longrightarrow & \bar{h}'_k(Y) & \longrightarrow & \bar{h}'_k(X) & \longrightarrow & \bar{h}'_k(S^q) & \longrightarrow & \cdots .
\end{array}$$

By induction, $T(Y)\colon \bar{h}_*(Y) \to \bar{h}'(Y)$ is an isomorphism. From the 5-lemma given below,

$$T(X)\colon \bar{h}_k(X) \to \bar{h}'_k(X)$$

is an isomorphism. The induction is finished and hence the result. □

Note. If in addition $\bar{h}_*$ and $\bar{h}'_*$ satisfies the wedge axiom, then $T(X)\colon \bar{h}_*(X) \to \bar{h}'_*(X)$ is an isomorphism for any CW-complexes X (including infinite CW-complexes), see Switzer's book [**25**, Theorem 7.55].

Lemma 5.7 (The Five-Lemma). *Let*

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
{\scriptstyle\cong}\Big\downarrow{\scriptstyle\alpha} & & {\scriptstyle\cong}\Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\gamma} & & {\scriptstyle\cong}\Big\downarrow{\scriptstyle\delta} & & {\scriptstyle\cong}\Big\downarrow{\scriptstyle\epsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

be a commutative diagram of groups such that the rows are exact. Suppose that α, β, δ and ϵ are isomorphisms. Then γ is an isomorphism also.

PROOF. Let $x \in C$ such that $\gamma(c) = 1$. Then

$$\delta(k(x)) = k'(\gamma(x)) = 1.$$

Since δ is a monomorphism, $k(x) = 1$ and so there exists $x' \in B$ such that $j(x') = x$. Since

$$j'(\beta(x')) = \gamma(j(x')) = \gamma(x) = 1,$$

there exists $y' \in A'$ such that $i'(y') = \beta(x')$. Since $\alpha\colon A \to A'$ is onto, there exists x'' such that $y' = \alpha(x'')$. Now

$$\beta(i(x'')) = i'(\alpha(x'')) = i'(y') = \beta(x').$$

Thus $i(x'') = x'$ because β is a monomorphism. It follows that

$$x = j(x') = j(i(x'')) = 1$$

and so γ is a monomorphism.

Let $y \in C'$. Then there exists $\tilde{x} \in D$ such that $\delta(\tilde{x}) = k'(y)$ because δ is onto. Since

$$\epsilon(l(\tilde{x})) = l'(\delta(\tilde{x})) = l'(k'(y)) = 1$$

and ϵ is a monomorphism, $l(\tilde{x}) = 1$ and so there exists $x' \in C$ such that $k(x') = \tilde{x}$. Since

$$k'(\gamma(x')) = \delta(k(x')) = \delta(\tilde{x}) = k'(y),$$

there exists an element $y' \in B'$ such that $j'(y') = y\gamma(x')^{-1}$. Since β is onto, there exists $x'' \in B$ such that $\beta(x'') = y'$. Now

$$\gamma(j(x'')x') = \gamma(j(x''))\gamma(x') = j'(\beta(x''))\gamma(x') = j'(y')\gamma(x') = y\gamma(x')^{-1}\gamma(x') = y$$

and so γ is onto. This finishes the proof. □

1.2. Eilenberg-Steenrod Axioms for Cohomology. An *unreduced cohomology theory* h^* means:

(1). a sequence of abelian groups $\{h^n(X,A)\}_{n\in\mathbb{Z}}$ for each pair of spaces (X,A), called *n-relative cohomology group of X modulo A* and simply denoted by $h^n(X)$ if $A = \emptyset$,

(2). a sequence of group homomorphisms $f^*\colon h_n(X,A) \leftarrow h_n(Y,B)$ for $n \in \mathbb{Z}$ and any map $f\colon (X,A) \to (Y,B)$, and

(3). a group homomorphism

$$\delta(n,X,A)\colon h^n(X,A) \longleftarrow h^{n-1}(A)$$

for any $n \in Z$ and any pair of spaces (X,A), called *boundary operator*, with the following six axioms:

AXIOM 1. *If $f = \mathrm{id}\colon (X,A) \to (X,A)$, then $f^* = \mathrm{id}\colon h^*(X,A) \to h^*(X,A)$.*

AXIOM 2. $(g \circ f)^* = f^* \circ g_*$.

AXIOM 3. $f^* \circ \delta = \delta(f|_A)^*$

Explicitly the diagram

$$\begin{array}{ccc} h^q(X,A) & \xleftarrow{f_*} & h^q(Y,B) \\ \uparrow \delta & & \uparrow \delta \\ h^{q-1}(A) & \xrightarrow{(f|_A)^*} & h^{q-1}(B) \end{array}$$

commutes.

Axiom 4 (Exactness Axiom). *For any pair of spaces* (X,A)*, there is a long exact sequence*

$$\cdots \longleftarrow h^{n+1}(X,A) \xleftarrow{\delta_{n+1}} h^n(A) \xleftarrow{i^*} h^n(X) \xrightarrow{j^*} h^n(X,A) \xleftarrow{\delta_n} \cdots,$$

where $i\colon (A,\emptyset)\to(X,\emptyset)$ *and* $j\colon (X,\emptyset)\to(X,A)$ *are the inclusion maps.*

Axiom 5 (Homotopy Axiom). *If* $f\simeq g\colon (X,A)\to(Y,B)$*, then* $f^*=g^*\colon h^*(X,A)\to h^*(Y,B)$.

Axiom 6 (Excision Axiom). *If* U *is an open subset of* X *whose closure* $\bar U$ *is contained in the interior of* A *(that is* $\bar U\subseteq V\subseteq A$ *for some open set* V*). Then the inclusion* $j\colon (X\smallsetminus U, A\smallsetminus U)\to(X,A)$ *induces an isomorphism*

$$j^*\colon h^n(X\smallsetminus U, A\smallsetminus U)\leftarrow h^n(X,A)$$

for each $n\in\mathbb{Z}$.

An *ordinary cohomology theory* means a cohomology theory h^* that satisfies the following additional axiom:

Axiom 7 (Dimension Axiom). *Let* P *be a space consisting of a single point. Then* $h^q(P)=0$ *for* $q\neq 0$.

In this case, $h^*(X,A)$ is called the cohomology of (X,A) with coefficients in $G=H^0(P)$, denoted by $H^*(X,A;G)$. Write $H^*(X,A)$ for $H^*(X,A;\mathbb{Z})$ the integral homology.

2. Computations and Applications

2.1. Degree. Since $S^n=\Sigma^n S^0$,

$$H_n(S^n)=\bar H_n(S^n)\cong \bar H_0(S^0)=H_0(P)=\mathbb{Z}.$$

Any map $f\colon S^n\to S^n$ induces a homomorphism

$$f_*\colon H_n(S^n)=\mathbb{Z}\longrightarrow H_n(S^n)=\mathbb{Z}.$$

Thus there exists a unique integer $\deg(f)$ such that

$$f_*(\alpha)=\deg(f)\alpha$$

for $\alpha\in H_n(S^n)$.

Proposition 5.8. *Some basic properties of degree are as follows:*

(1). $\deg(\mathrm{id})=1$.
(2). *If* f *is not surjective, then* $\deg(f)=0$.
(3). $f\simeq g \Leftrightarrow \deg(f)=\deg(g)$
(4). $\deg(f\circ g)=\deg(f)\deg(g)$.
(5). *If* f *is a reflection fixing the points in a subspace* S^{n-1} *pointwise, then* $\deg(f)=-1$.
(6). *The antipodal map* $S^n\to S^n$, $x\mapsto -x$ *has degree of* $(-1)^{n+1}$.
(7). *Moreover if* $f\colon S^n\to S^n$ *has no fixed points, then* $\deg(f)=(-1)^{n+1}$.
(8). *The map* $f\colon S^{m+n}=S^m\wedge S^n\to S^{m+n}=S^n\wedge S^m$, $x\wedge y\mapsto y\wedge x$, *has degree of* $(-1)^{mn}$

Assertion (3) is equivalent to that $\pi_n(S^n)\cong H_n(S^n)=\mathbb{Z}$. At moment we are only able to prove $f\simeq g\Rightarrow \deg(f)=\deg(g)$.

PROOF. (1) is obvious.

(2) Since f is not surjective, f null homotopic and so $\deg(f)$.

(3) $f \simeq g \Rightarrow \deg(f) = \deg(g)$ follows from the Homotopy Axiom.

(4) Follows from $(g \circ f)_* = g_* \circ f_*$.

(5) Consider $S^n = S^{n-1} \wedge S^1 = \Sigma^{n-1} S^1$. Let $\nu \colon S^1 \to S^1$, $t \mapsto 1-t$, the inverse of the loop. Then $f = \simeq \Sigma^{n-1}\nu'$. By the Suspension Isomorphism Theorem, there is a commutative diagram

$$\begin{array}{c}
H_1(S^1) = \mathbb{Z} \cong H_n(S^n) = H_n(\Sigma^{n-1}S^1) \\
\Big\downarrow \nu_* \qquad\qquad f_* = \Sigma^{n-1}\nu_* \\
H_1(S^1) = \mathbb{Z} \cong H_n(S^n) = H_n(\Sigma^{n-1}S^1).
\end{array}$$

Thus $\deg(f) = \deg(\nu')$. Since the path product $\nu' * \mathrm{id}_{S^1}$ is null homotopic (because $[\nu'][\mathrm{id}_{S^1}] = 1$), the composite

$$S^1 \xrightarrow{\text{pinch}} S^1 \vee S^1 \xrightarrow{\nu' \vee \mathrm{id}_{S^1}} S^1 \vee S^1 \xrightarrow{\text{fold}} S^1$$

is null homotopic. By applying H_1 to the above composite,

$$0 = \nu'_* + \mathrm{id}_{S^1 *}$$

and so $\deg(\nu') = -1$.

(6) The antipodal map is the composite of $n+1$ reflections, each changing the sign of one of the coordinates in $\mathbb{R}^{n+1}$.

(7) If $f(x) \neq x$ for any x, then

$$F(x,t) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

gives a homotopy from f to the antipodal map. Thus $\deg(f) = (-1)^{n+1}$.

(8) Consider $S^m = I^m/\partial(I^m)$ is the quotient space of I^m by pinching out its boundary to a point. Then there is a commutative diagram

$$\begin{array}{ccc}
S^{m+n} = S^m \wedge S^n & \xrightarrow{\quad f \quad} & S^{m+n} = S^m \wedge S^n \\
\Big\uparrow & & \Big\uparrow \\
I^{m+n}/\partial(I^{m+n}) = I^m/\partial(I^m) \wedge I^n/\partial(I^n) & \xrightarrow{g} & I^{m+n}/\partial(I^{m+n} = I^m/\partial(I^m) \wedge I^n/\partial(I^n),
\end{array}$$

where

$$g(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = (x_{n+1}, \ldots, x_{n+m}, x_1, \ldots, x_n).$$

Note $\tau_i \colon I^{m+n}/\partial(I^{m+n}) \to I^{m+n}/\partial(I^{m+n})$ given by

$$\tau(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$$

has degree of -1 because the map $I^2/\partial(I^2) \to I^2/\partial(I^2)$, $(x,y) \mapsto (y,x)$, is the reflection along the circle obtained from the diagonal line in I^2. Then the map g has degree of $(-1)^{mn}$ and so does f. $\square$

THEOREM 5.9. *There exists a continuous nowhere zero vector field over S^n if and only if n is odd.*

PROOF. Assume that n is odd. Let $n = 2k - 1$. Define

$$\nu(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}).$$

Then $\nu(x) \perp x$ for any $x \in S^n$ with $|\nu(x)| = 1$. Thus ν defines a nowhere vector field over S^n.

Conversely suppose that there exists a nowhere zero vector field ν over S^n, that is, $\nu(x) \neq 0$ for any $x \in S^n$ and $\nu(x) \perp x$. Define $\tilde{\nu}(x) = \nu(x)/|\nu(x)|$. Then $\tilde{\nu}\colon S^n \to S^n$ with $\tilde{\nu}(x) \neq x$ for any $x \in S^n$ because $\tilde{\nu}(x) \perp x$. It follows that $\deg(\tilde{\nu}) = (-1)^{n+1}$. Now define the homotopy by

$$F(x,t) = \cos(\frac{\pi}{2}t)x + \sin(\frac{\pi}{2}t)\tilde{\nu}(x).$$

Then $F_0 = \mathrm{id}$ and $F_1 = \tilde{\nu}$. Thus $\tilde{\nu} \simeq \mathrm{id}$ and so

$$(-1)^{n+1} = \deg(\tilde{\nu}) = \deg(\mathrm{id}) = 1.$$

It follows that n is odd and hence the result. □

PROPOSITION 5.10. *If n is even, then $\mathbb{Z}/2\mathbb{Z}$ is the only nontrivial group that can act freely on S^n.*

PROOF. Let G be a group that acts freely on S^n. Then there is a group homomorphism:

$$\phi\colon G \longrightarrow \mathrm{Aut}(H_n(S^n)) = \mathrm{Aut}(\mathbb{Z}) = \{\pm 1\} \quad g \mapsto (g_*\colon H_n(S^n) \to H_n(S^n)).$$

Suppose that $G \neq \mathbb{Z}/2\mathbb{Z}$. There exists $g \in \mathrm{Ker}(\phi)$ with $g \neq 1$. Since G acts freely on S^n, g has no fixed point because G. Hence $\deg(g) = -1$. This contradicts to that $g \in \mathrm{Ker}(\phi)$. The proof is finished. □

2.2. Cellular Homology of CW-complexes. Let X be a CW-complex. Then there is a skeleton filtration $\{\mathrm{sk}_n(X)\}_{n\geq 0}$, where $\mathrm{sk}_n(X)$ is the subspace of X consisting of the cells up to dimension n. If X is a finite dimensional CW-complex, then this is a finite filtration, that is, $X = \mathrm{sk}_k(X)$ for some large k.

PROPOSITION 5.11. *Let $H_*(-;G)$ be the ordinary homology with coefficients in G. Let X be a finite dimensional CW-complex. Then*

(1). $H_j(\mathrm{sk}_n(X);G) = 0$ *for* $j > n$.
(2). *the inclusion* $\mathrm{sk}_n(X) \to X$ *induces an isomorphism*

$$H_j(\mathrm{sk}_n(X);G) \cong H_j(X;G)$$

for $j \leq n-1$.
(3). $H_n(\mathrm{sk}_n(X);G) \to H_{n+1}(X;G)$ *is onto.*

PROOF. Note that $\mathrm{sk}_{n-1}(X) \hookrightarrow \mathrm{sk}_n(X)$ is a cofibration with $\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X) = \bigvee_{\alpha\in J_n} S^n$ a wedge of n-spheres. The assertions follow by applying long exact sequence of the homology to the sequence $\mathrm{sk}_{n-1}(X) \to \mathrm{sk}_n(X) \to \bigvee_\alpha S^n$ and by using the fact that

$$H_j(\bigvee_\alpha S^n; G) = \begin{cases} G & \text{if} \quad j = 0 \\ \bigoplus_\alpha G & \text{if} \quad j = n \\ 0 & \text{otherwise} \end{cases}$$

□

From the above proposition, $H_n(X;G) \cong H_n(\mathrm{sk}_{n+1}(X);G)$. From the commutative diagram

$$\begin{array}{ccccc}
\mathrm{sk}_{n-1}(X) & = & \mathrm{sk}_{n-1}(X) & & \\
\downarrow & & \downarrow & & \\
\mathrm{sk}_n(X) & \hookrightarrow & \mathrm{sk}_{n+1}(X) & \twoheadrightarrow & \mathrm{sk}_{n+1}(X)/\mathrm{sk}_n(X) = \bigvee_{\alpha\in J_{n+1}} S^{n+1} \\
\downarrow & & \downarrow & & \| \\
\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X) = \bigvee_{\alpha\in J_n} S^n & \hookrightarrow & \mathrm{sk}_{n+1}(X)/\mathrm{sk}_{n-1}(X) & \twoheadrightarrow & \mathrm{sk}_{n+1}(X)/\mathrm{sk}_n(X) = \bigvee_{\alpha\in J_{n+1}} S^{n+1},
\end{array}$$

there is a commutative diagram of exact sequences of abelian groups

$$\begin{array}{ccccc}
H_{n+1}(\bigvee_{\alpha\in J_{n+1}} S^{n+1};G) & \xrightarrow{\partial_{n,n+1}} & H_n(\mathrm{sk}_n(X);G) & \twoheadrightarrow & H_n(\mathrm{sk}_{n+1}(X);G) \\
\| & & \downarrow p_{n*} & & \downarrow \\
H_{n+1}(\bigvee_{\alpha\in J_{n+1}} S^{n+1};G) & \xrightarrow{\partial_{n+1}} & H_n(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X);G) & \twoheadrightarrow & H_n(\mathrm{sk}_{n+1}(X)/\mathrm{sk}_{n-1}(X);G) \\
 & & \downarrow \partial_{n-1,n} & & \downarrow \\
 & & H_{n-1}(\mathrm{sk}_{n-1}(X);G) & = & H_{n-1}(\mathrm{sk}_{n-1}(X);G).
\end{array} \tag{3}$$

Note that

$$p_{n-1*}\colon H_{n-1}(\mathrm{sk}_{n-1}(X);G) \to H_{n-1}(\mathrm{sk}_{n-1}(X)/\mathrm{sk}_{n-2}(X);G)$$

is a monomorphism. The kernel of

$$\partial_{n-1,n}\colon H_n(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X);G) \to H_{n-1}(\mathrm{sk}_{n-1}(X);G)$$

is the same as the kernel of

$$\partial_n = p_{n-1*} \circ \partial_{n-1,n}\colon H_n(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X);G) \to H_{n-1}(\mathrm{sk}_{n-1}(X)/\mathrm{sk}_{n-2}(X);G).$$

From Diagram (3), we have $\partial_n \circ \partial_{n+1} = 0$ and

$$H_n(X;G) = H_n(\mathrm{sk}_{n+1}(X);G) \cong \mathrm{Ker}(\partial_n)/\operatorname{Im}(\partial_{n+1}).$$

This proves the following theorem when X is a finite dimensional CW-complex.

THEOREM 5.12. *Let X be a CW-complex. Define the groups*

$$C_n^{\mathrm{CW}}(X;G) = H_n(\mathrm{sk}_n(X),\mathrm{sk}_{n-1}(X);G) = \bar{H}_n(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X);G) = \bigoplus_{\alpha\in J_n} G$$

with $\partial_n\colon C_n^{\mathrm{CW}}(X;G)\to C_{n-1}^{\mathrm{CW}}(X;G)$ *as the composite*

$$H_n(\mathrm{sk}_n(X),\mathrm{sk}_{n-1}(X);G)\longrightarrow H_{n-1}(\mathrm{sk}_{n-1}(X);G)\longrightarrow H_{n-1}(\mathrm{sk}_{n-1}(X),\mathrm{sk}_{n-2};G).$$

Then $C_*^{\mathrm{CW}}(X;G)$ *is a chain complex with*

$$H_*(X;G)\cong H_*(C_*^{\mathrm{CW}}(X;G)).$$

□

Remark. The proof for the case when X is an infinite dimensional CW-complex requires the following lemma. A proof of this lemma can be found in Switzer's book [**25**, pp.118-121], which requires a construction for a sequence of spaces so-called *infinite telescope*. Also see Hatcher's book [**7**] for the proof for the case when X is an infinite dimensional CW-complex and the construction of telescope.

LEMMA 5.13. *Let* X *be a* CW*-complex and let*

$$X_0\subseteq X_1\subseteq X_2\subseteq\cdots\subseteq X_n\subseteq\cdots$$

be a sequence of subcomplexes of X *such that* $X=\bigcup_n X_n$. *Let* h_* *be a homology theory satisfying the Wedge Axiom. Then* $h_j(X)$ *is given by the direct limit of the sequence of groups*

$$h_j(X_0)\to h_j(X_1)\to\cdots\to h_j(X_n)\to h_j(X_{n+1})\to\cdots.$$

for each $j\in\mathbb{Z}$. □

REMARK 5.14. Let $f_n\colon\bigvee_{\alpha\in J_n}S^{n-1}\to\mathrm{sk}_{n-1}(X)$ be the attaching map. Then

$$\mathrm{sk}_n(X)=\mathrm{sk}_{n-1}(X)\cup_{f_{n-1}}C\left(\bigvee_{\alpha\in J_{n-1}}S^{n-1}\right)$$

and so there is a long exact sequence

$$\cdots\longrightarrow\bar H_*(\bigvee_{\alpha\in J_n}S^{n-1};G)\xrightarrow{f_{n*}}\bar H_*(\mathrm{sk}_{n-1}(X);G)\longrightarrow H_*(\mathrm{sk}_n(X);G)$$

$$\longrightarrow\bar H_{*-1}(\bigvee_{\alpha\in J_{n-1}}S^{n-1})=\bar H_*(\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X);G)\xrightarrow{f_{n*}=\partial_{n,*}}\bar H_{*-1}(\mathrm{sk}_{n-1}(X);G)\longrightarrow\cdots.$$

with $\partial_{n,*}=-f_*$ as in the Note to Corollary 5.2. Thus $-\partial_n$ is given by the composite

$$H_n(\bigvee_{\alpha\in J_n}S^n;G)\cong H_{n-1}(\bigvee_{\alpha\in J_n}S^{n-1};G)\xrightarrow{f_{n*}}H_{n-1}(\mathrm{sk}_{n-1}(X);G)\longrightarrow H_{n-1}(\mathrm{sk}_{n-1}(X)/\mathrm{sk}_{n-2}(X);G).$$

The cellular homology will not be affected if we replace ∂_n by the above composite.

EXAMPLE 5.15 (Computation of $H_*(\mathbb{R}\mathrm{P}^m)$). Let $p\colon S^n\to\mathbb{R}\mathrm{P}^n$ be the quotient map, that is p is a 2-sheeted covering map. Observe that

$$\mathbb{R}\mathrm{P}^{n+1}=\mathbb{R}\mathrm{P}^n\cup_p CS^n.$$

There is a commutative diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\;p\;} & \mathbb{R}\mathrm{P}^n \\ \downarrow{\scriptstyle \text{pinch}} & & \\ S^n \vee S^n & & \downarrow \\ \downarrow{\scriptstyle \mathrm{id}_{S^n} \vee \text{antipodal}} & & \\ S^n \vee S^n & \xrightarrow{\text{fold}} & S^n = \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-1} \end{array}$$

because one can think that S^n is union of the upper hemisphere and lower hemisphere and the lower hemisphere is identified via the antipodal if we consider the upper hemisphere is identified as the identity. It follows that

$$\partial_{n+1} \colon \mathbb{Z} = H_{n+1}(\mathbb{R}\mathrm{P}^{n+1};\mathbb{Z}) \cong H_n(S^n;\mathbb{Z}) \to H_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \to H_n(\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{n-1};\mathbb{Z}) = \mathbb{Z}$$

has degree of $(1+(-1)^{n+1})$ and so the cellular chain complex for $\mathbb{R}\mathrm{P}^m$ is as follows

$$\mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \xrightarrow{\;2\;} \cdots \xrightarrow{\;0\;} \mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \qquad \text{if } m \text{ even},$$

$$\mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \xrightarrow{\;0\;} \cdots \xrightarrow{\;0\;} \mathbb{Z} \xrightarrow{\;2\;} \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \qquad \text{if } m \text{ odd}.$$

Hence

$$H_k(\mathbb{R}\mathrm{P}^m) = \begin{cases} \mathbb{Z} & \text{if} \quad k = 0 \\ \mathbb{Z} & \text{if} \quad k = m \text{ odd} \\ \mathbb{Z}/2 & \text{if} \quad 0 < k < n \text{ and } k \text{ odd} \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Research Project. Study the skeleton filtration of CW-complexes for a general homology theory. You will then get Atiyah-Hizebruch-Whitehead spectral sequence: for any homology theory h_* and CW-complex X there is a spectral sequence $\{E^r_{p,q}, d^r\}$ with $E^2_{pq} \cong H_p(X; h_q(\text{point}))$ and converging to $h_*(X)$. See Switzer's book [**25**, Chapter 15].

2.3. Cellular Structure on Products of CW-complexes.

For simplicity, in this subsection we only consider finite CW-complexes and the attaching maps are pointed maps.

First let's refresh the meaning of attaching a cell. Assume that we have $\mathrm{sk}_{n-1}(X)$ already and we want to attach an n-cell to $\mathrm{sk}_{n-1}(X)$. This means that there is map $f \colon S^{n-1} \to \mathrm{sk}_{n-1}(X)$ for obtaining the new space

$$\mathrm{sk}_{n-1}(X) \cup_f CS^{n-1},$$

which is a subspace of $\mathrm{sk}_n(X)$. (The construction of $\mathrm{sk}_n(X)$ is obtained by attaching cell-by-cell.) Thus there is a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{sk}_{n-1}(X) & \hookrightarrow & \mathrm{sk}_n(X) & \xrightarrow{\text{pinch}} & \mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X) = \bigvee_{\alpha\in J_n} S^n \\
\uparrow f & & \uparrow \tilde{f} & & \uparrow \bar{f} \\
S^{n-1} & \hookrightarrow & CS^{n-1} & \xrightarrow[\text{pinch}]{} & CS^{n-1}/S^{n-1} = S^n,
\end{array}
\tag{4}
$$

where $CS^{n-1} = (I \times S^{n-1})/(\{1\} \times S^{n-1} \cup I \times \{*\})$.

Now let X and Y be CW-complexes. The product skeleton filtration on $X \times Y$ is given by

$$\mathrm{sk}_n(X \times Y) = \bigcup_{i+j\leq n} \mathrm{sk}_i(X) \times \mathrm{sk}_j(Y).$$

PROPOSITION 5.16. *Let X and Y be finite CW-complexes. Then*

$$\mathrm{sk}_n(X \times Y)/\mathrm{sk}_{n-1}(X \times Y) \cong \bigvee_{i+j=n} (\mathrm{sk}_i(X)/\mathrm{sk}_{i-1}(X)) \wedge (\mathrm{sk}_j(Y)/\mathrm{sk}_{j-1}(Y)).$$

PROOF. The composite

$$\mathrm{sk}_i(X) \times \mathrm{sk}_j(Y) \longrightarrow \mathrm{sk}_n(X \times Y) \longrightarrow \mathrm{sk}_n(X \times Y)/\mathrm{sk}_{n-1}(X \times Y)$$

factors through the quotient $(\mathrm{sk}_i(X)/\mathrm{sk}_{i-1}(X)) \wedge (\mathrm{sk}_j(Y)/\mathrm{sk}_{j-1}(Y))$ and so it induces a map

$$\bigvee_{i+j=n} (\mathrm{sk}_i(X)/\mathrm{sk}_{i-1}(X)) \wedge (\mathrm{sk}_j(Y)/\mathrm{sk}_{j-1}(Y)) \cong \mathrm{sk}_n(X \times Y)/\mathrm{sk}_{n-1}(X \times Y),$$

which is one-to-one and onto. Since both sides are compact as they are finite wedges of spheres, the above map is a homeomorphism. □

Let $\mathrm{sk}_i(X)/\mathrm{sk}_{i-1}(X) = \bigvee_{\alpha\in I_i} S^i$ and let $\mathrm{sk}_j(Y)/\mathrm{sk}_{j-1}(Y) = \bigvee_{\beta\in J_j} S^j$. For $\alpha \in I_i$ and $\beta \in J_j$ with $i + j = n$ and $i, j > 0$, consider the attaching maps

$$f_\alpha \colon S^{i-1} \to \mathrm{sk}_{i-1}(X) \quad f_\beta \colon S^{j-1} \to \mathrm{sk}_{j-1}(Y).$$

Then there is a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{sk}_{n-1}(X \times Y) & \hookrightarrow & \mathrm{sk}_n(X \times Y) & \twoheadrightarrow & \mathrm{sk}_n(X \times Y)/\mathrm{sk}_{n-1}(X \times Y) \\
\uparrow {\scriptstyle (\tilde{f}_\alpha \times f_\beta) \cup (f_\alpha \times \tilde{f}_\beta)} & & \uparrow & & \uparrow \\
 & & \mathrm{sk}_i(X) \times \mathrm{sk}_j(Y) & \twoheadrightarrow & (\mathrm{sk}_i(X)/\mathrm{sk}_{i-1}(X)) \wedge (\mathrm{sk}_j(Y)/\mathrm{sk}_{j-1}(Y)) \\
 & & \uparrow {\scriptstyle \tilde{f}_\alpha \times \tilde{f}_\beta} & & \uparrow {\scriptstyle \bar{f}_\alpha \wedge \bar{f}_\beta} \\
A & \hookrightarrow & CS^{i-1} \times CS^{j-1} & \twoheadrightarrow & S^i \wedge S^j,
\end{array}
$$

where

$$A = (CS^{i-1} \times S^{j-1}) \cup (S^{i-1} \cup CS^{j-1}) \cong S^{i+j-1}.$$

Define

$$f_{\alpha,\beta} = (\tilde{f}_\alpha \times f_\beta) \cup (f_\alpha \times \tilde{f}_\beta)\colon S^{i+j-1} = A \longrightarrow (\mathrm{sk}_i(X) \times \mathrm{sk}_{j-1}(Y)) \cup (\mathrm{sk}_{i-1}(X) \times \mathrm{sk}_i(Y)) \longrightarrow \mathrm{sk}_{n-1}(X \times Y).$$

Then $f_{\alpha,\beta}$ is the attaching map for the n-cell of $X \times Y$ corresponding to $(\alpha, \beta) \in I_i \times J_i$. Consider the composite

$$S^{i+j-1} \xrightarrow{f_{\alpha,\beta}} \mathrm{sk}_{n-1}(X \times Y) \longrightarrow \mathrm{sk}_{n-1}(X \times Y)/\mathrm{sk}_{n-2}(X \times Y).$$

There is a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{(\tilde{f}_\alpha \times f_\beta) \cup (f_\alpha \times \tilde{f}_\beta)} & (\mathrm{sk}_i(X) \times \mathrm{sk}_{j-1}(Y)) \times (\mathrm{sk}_{i-1}(X) \times \mathrm{sk}_j(Y) & \longrightarrow & \mathrm{sk}_{n-1}(X \times Y) \\
\downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{(\bar{f}_\alpha \wedge f_\beta) \vee (f_\alpha \wedge \bar{f}_\beta)} & ((\mathrm{sk}_i(X)/\mathrm{sk}_{i-1}(X)) \wedge \mathrm{sk}_j(Y)) \vee (\mathrm{sk}_{i-1}(X) \wedge (\mathrm{sk}_j(Y)/\mathrm{sk}_{j-1}(Y))) & \longrightarrow & \mathrm{sk}_{n-1}/\mathrm{sk}_{n-2},
\end{array}$$

where

$$B = (CS^{i-1}/S^{i-1} \wedge S^{j-1}) \vee (S^{i-1} \wedge CS^{j-1}/S^{j-1}),$$

because

$$(\mathrm{sk}_i(X) \times *) \cup (\mathrm{sk}_{i-1}(X) \times \mathrm{sk}_{j-1}(Y)) \cup (* \times \mathrm{sk}_j(Y)) \subseteq \mathrm{sk}_{n-2}(X \times Y).$$

Define

$$\phi\colon I \times S^{i-1} \times S^{j-1} \longrightarrow A = (CS^{i-1} \times S^{j-1}) \cup (S^{i-1} \times CS^{j-1})$$

by

$$\phi(t, x, y) = \begin{cases} (x, [1-2t, y]) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ ([2t-1, x], y) & \text{if} \quad \frac{1}{2} \le t \le 1. \end{cases}$$

Since $\phi(0, x, y) = (x, ast)$ and $\phi(1, x, y) = (*, y)$, the map ϕ factors the join $S^{i-1} \# S^{j-1} = S^{i+j-1}$ and it induces a map

$$\bar{\phi}\colon S^{i-1} \# S^{j-1} \longrightarrow A = (CS^{i-1} \times S^{j-1}) \cup (S^{i-1} \times CS^{j-1}),$$

which is a homotopy equivalence. Now the composite

$$S^{i-1} \# S^{j-1} \longrightarrow A = (CS^{i-1} \times S^{j-1}) \cup (S^{i-1} \times CS^{j-1}) \xrightarrow{\text{pinch}} (CS^{i-1}/S^{i-1}) \wedge S^{j-1}$$

has degree of 1 because from the construction of ϕ this map is given by pinching $[0, 1/2] \times S^{i-1} \times S^{j-1}$. Then composite

$$S^{i-1} \# S^{j-1} \longrightarrow A = (CS^{i-1} \times S^{j-1}) \cup (S^{i-1} \times CS^{j-1}) \xrightarrow{\text{pinch}} S^{i-1} \wedge (CS^{j-1}/S^{j-1})$$

has degree of $(-1)^i$ because from the construction of ϕ, by pinching $[1/2, 1] \times S^{i-1} \times S^{j-1}$, this map is given by switching x and t and taking the inverse on t. Together with the above commutative diagram, we prove the following theorem:

THEOREM 5.17. *Let X and Y be finite complexes. Then the integral cellular chain complex for $X \times Y$ is given by*

$$C_n^{\mathrm{CW}}(X \times Y) = \bigoplus_{i+j=n} C_i^{\mathrm{CW}}(X) \otimes C_j^{\mathrm{CW}}(Y)$$

with

$$\partial_n^{X\times Y}(a \otimes b) = \partial_i^X(a) \otimes b + (-1)^i a \otimes \partial_j^Y(b)$$

for $a \in C_i^{\mathrm{CW}}(X)$ and $b \in C_j^{\mathrm{CW}}(Y)$. Moreover for any abelian group G

$$C_n^{\mathrm{CW}}(X \times Y; G) = \bigoplus_{i+j=n} C_i^{\mathrm{CW}}(X;\mathbb{Z}) \otimes C_j^{\mathrm{CW}}(Y;\mathbb{Z}) \otimes G$$

with the differential induced from above. □

Note. The skeleton filtration on $X \times Y$ determines $C_n^{\mathrm{CW}}(X \times Y)$. The construction of the map ϕ together with above commutative diagram determine the differentials in the cellular chain complex for $X \times Y$, where one can see why there is a sign $(-1)^i$ by the construction of ϕ. The sign $(-1)^i$ can be understood in a way by thinking that a has degree of i as it is in C_i and the differential ∂ has degree of -1 and the sign $(-1)^{|a||\partial|} = (-1)^i$ occurs when interchange a and ∂.

3. Δ-sets, Simplicial Sets and Homology

3.1. Δ-sets. A Δ-*set* means a sequence of sets $X = \{X_n\}_{n\geq 0}$ with *faces* $d_i\colon X_n \to X_{n-1}$, $0 \leq i \leq n$, such that

$$d_i d_j = d_j d_{i+1}$$

for $i \geq j$, which is called the Δ-*identity*.
Note. One can use coordinate projections for catching Δ-identity:

$$d_i\colon (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Let $\mathcal{O}^+$ be the category whose objects are finite ordered sets and whose morphisms are functions $f\colon X \to Y$ such that $f(x) < f(y)$ if $x < y$. Note that the objects in $\mathcal{O}^+$ are given by $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and the morphisms in $\mathcal{O}^+$ are generated by $d^i\colon [n-1] \longrightarrow [n]$ with

$$d^i(j) = \begin{cases} j & \text{if} \quad j < i \\ j+1 & \text{if} \quad j \geq i \end{cases}$$

for $0 \leq i \leq n$, that is d^i is the ordered embedding missing i. We may write the function d^i in matrix form:

$$d^i = \begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n-1 \\ 0 & 1 & \cdots & i-1 & i+1 & i+2 & \cdots & n. \end{pmatrix}.$$

The morphisms d^i satisfy the following identity:

$$d^j d^i = d^{i+1} d^j$$

for $i \geq j$.
Note. For seeing that morphisms in $\mathcal{O}^+$ are generated by d^i, observe that any morphism in $\mathcal{O}^+$ means an ordered embedding, which can be written as the compositions of d^i's.

Let $\mathcal{S}$ denote the category of sets.

PROPOSITION 5.18. *Δ-sets are one-to-one correspondent to cofunctors from $\mathcal{O}^+$ to $\mathcal{S}$.*

Proof. Let $F\colon \mathcal{O}^+ \to \mathcal{S}$ be a cofunctor. Define $X_n = F([n])$ and
$$d_i = F(d^i)\colon X_n = F([n]) \to X_{n-1} = F([n-1]).$$
Then X is a Δ-set.

Conversely suppose that X is a Δ-set. Define the $F\colon \mathcal{O}^+ \to \mathcal{S}$ by setting $F([n]) = X_n$ and $F(d^i) = d_i$. Then F is a cofunctor. □

A Δ-*map* $f\colon X \to Y$ means a sequence of functions $f\colon X_n \to Y_n$ for each $n \geq 0$ such that $f \circ d_i = d_i \circ f$, that is the diagram
$$\begin{array}{ccc} X_n & \xrightarrow{f} & Y_n \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ X_{n-1} & \longrightarrow & Y_{n-1} \end{array}$$
commutes.

Example 5.19 (n-simplex). The n-simplex $\Delta^+[n]$, as a Δ-set, is as follows:
$$\Delta^+[n]_k = \{(i_0, i_1, \ldots, i_k) \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n\}$$
for $k \leq n$ and $\Delta^+[n]_k = \emptyset$ for $k > n$. The face $d_j\colon \Delta^+[n]_k \to \Delta^+[n]_{k-1}$ is given by
$$d_j(i_0, i_1, \ldots, d_k) = (i_0, i_1, \ldots, \hat{i}_j, \ldots, i_k),$$
that is deleting i_j. Let $\sigma_n = (0, 1, \ldots, n)$. Then
$$(i_0, i_1, \ldots, i_k) = d_{j_1} d_{j_2} \cdots d_{j_{n-k}} \sigma_n,$$
where $j_1 < j_2 < \ldots < j_{n-k}$ with $\{j_1, \ldots, j_k\} = \{0, 1, \ldots, n\} \smallsetminus \{i_0, i_1, \ldots, i_k\}$. In other words, any elements in $\Delta^{[}n]$ can be written an iterated face of σ_n.

Proposition 5.20. *Let X be a Δ-set and let $x \in X_n$ be an element. Then there exists a unique Δ-map*
$$f_x\colon \Delta^+[n] \to X$$
such that $f_x(\sigma_n) = x$.

Proof. From the assumption $f_x(\sigma_n) = x$, we have
$$f_x(i_0, i_1, \ldots, i_k) = f_x(d_{j_1} d_{j_2} \cdots d_{j_{n-k}} \sigma_n) = d_{j_1} d_{j_2} \cdots d_{j_{n-k}} f_x(\sigma_n) = d_{j_1} d_{j_2} \cdots d_{j_{n-k}} x.$$
This defines a Δ-map f_x such that $f_x(\sigma_n) = x$. □

3.2. Homology of Δ-sets. A Δ-set $G = \{G_n\}_{n\geq 0}$ is called a Δ-*group* if each G_n is a group, and each face d_i is a group homomorphism. In other words, a Δ-group means a cofunctor from $\mathcal{O}^+$ to the category of groups. More abstractly, for any category $\mathcal{C}$, a Δ-object over $\mathcal{C}$ means a cofunctor from $\mathcal{O}^+$ to $\mathcal{C}$. In other words, a Δ-object over $\mathcal{C}$ means a sequence of objects over $\mathcal{C}$, $X = \{X_n\}_{n\geq 0}$ with faces $d_i\colon X_n \to X_{n-1}$ as morphisms in $\mathcal{C}$.

Recall that a chain complex of groups means a sequence $C = \{C_n\}$ of groups with differential $\partial_n\colon C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1}$ is trivial, that is $\mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$ and so the homology is defined by
$$H_n(C) = \mathrm{Ker}(\partial_n)/\,\mathrm{Im}(\partial_{n+1}),$$

which is a coset in general. A chain complex C is called *normal* if $\mathrm{Im}(\partial_{n+1})$ is a normal subgroup of $\mathrm{Ker}(\partial_n)$ for each n. In this case $H_n(C)$ is a group for each n.

PROPOSITION 5.21. *Let G be a Δ-abelian group. Define*

$$\partial_n = \sum_{i=0}^{n}(-1)^i d_i \colon G_n \to G_{n-1}.$$

Then $\partial_{n-1}\circ\partial_n = 0$, that is, G is a chain complex under ∂_*.

PROOF.

$$\begin{aligned}
\partial_{n-1}\circ\partial_n &= \sum_{i=0}^{n-1}(-1)^i d_i \sum_{j=0}^{n}(-1)^j d_j \\
&= \sum_{0\le i<j\le n}(-1)^{i+j} d_i d_j + \sum_{0\le j\le i\le n-1}(-1)^{i+j} d_i d_j \\
&= \sum_{0\le i<j\le n}(-1)^{i+j} d_i d_j + \sum_{0\le j<i+1\le n}(-1)^{i+j} d_j d_{i+1} \\
&= \sum_{0\le i<j\le n}(-1)^{i+j} d_i d_j + \sum_{0\le j<i\le n}(-1)^{i+j-1} d_j d_i \\
&= 0.
\end{aligned}$$

□

Let X be a Δ-set. The homology $H_*(X;G)$ of X with coefficients in an abelian group G is defined by

$$H_*(X;G) = H_*(\mathbb{Z}(X)\otimes G, \partial_*),$$

where $\mathbb{Z}(X) = \{\mathbb{Z}(X_n)\}_{n\ge 0}$ and $\mathbb{Z}(X_n)$ is the free abelian group generated by X_n.

PROPOSITION 5.22. *Let*

$$1 \longrightarrow C' \xrightarrow{\ i\ } C \xrightarrow{\ p\ } C'' \longrightarrow 1$$

be any short exact sequence of chain complexes of (possibly non-commutative) groups. Then there is a long exact sequence

$$\cdots \longrightarrow H_{k+1}(C'') \xrightarrow{\partial_{k+1}} H_k(C') \xrightarrow{\ i_*\ } H_k(C) \xrightarrow{\ p_*\ } H_k(C'') \longrightarrow \cdots.$$

Moreover if C' and C'' are normal chain complexes, then ∂_{k+1} is a group homomorphism for each k.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc}
C'_{k+2} & \overset{i}{\hookrightarrow} & C_{k+2} & \overset{p}{\twoheadrightarrow} & C''_{k+2} \\
\downarrow \partial' & & \downarrow \partial & & \downarrow \partial'' \\
C'_{k+1} & \overset{i}{\hookrightarrow} & C_{k+1} & \overset{p}{\twoheadrightarrow} & C''_{k+1} \\
\downarrow \partial' & & \downarrow \partial & & \downarrow \partial'' \\
C'_{k} & \overset{i}{\hookrightarrow} & C_{k} & \overset{p}{\twoheadrightarrow} & C''_{k} \\
\downarrow \partial' & & \downarrow \partial & & \downarrow \partial'' \\
C'_{k-1} & \overset{i}{\hookrightarrow} & C_{k-1} & \overset{p}{\twoheadrightarrow} & C''_{k-1}.
\end{array}$$

Let $x \in C''_{k+1}$ with $\partial''(x) = 1$. There exists $\tilde{x} \in C_{k+1}$ such that $p(\tilde{x}) = x$. Since

$$p(\partial(\tilde{x})) = \partial''(p(\tilde{x})) = \partial''(x) = 1,$$

there exists $\bar{x} \in C'_k$ such that $i(\bar{x}) = \partial(\tilde{x})$. Now

$$i(\partial'(\bar{x})) = \partial(i(\bar{x})) = \partial \circ \partial(\tilde{x}) = 1.$$

Thus $\bar{x}$ is a circle in C' and so $\{\bar{x}\}$ defines an element in $H_k(C')$.

Let $\hat{x}$ be another element in C_{k+1} such that $p(\hat{x}) = x$. Then

$$p(\tilde{x}\hat{x}^{-1}) = 1$$

and so there exists an element $z \in C'_{k+1}$ such that $i(z) = \tilde{x}^{-1}\hat{x}$. Now

$$i(\bar{x}\partial'(z)) = \partial(\tilde{x})(\partial(\tilde{x}))^{-1}\partial(\hat{x}) = \partial(\hat{x}).$$

Thus $\{\bar{x}\} \in H_k(C')$ is independent on the choice of the pre-image of x in C_{k+1}.

Suppose that $x' = x\partial''(y)$ with $\partial''(x) = 1$ for some $y \in C''_{k+2}$. There exists $\tilde{y} \in C_{k+2}$ such that $p(\tilde{y}) = y$. Then

$$x' = p(\tilde{x}\partial(\tilde{y}))$$

with

$$\bar{x}' = \partial(\tilde{x}\partial(\tilde{y})) = \partial(\tilde{x}) = \bar{x}.$$

This shows that

$$\partial_{k+1} \colon H_{k+1}(C'') \to H_k(C') \quad \{x\} \mapsto \{\bar{x}\}$$

is well-defined. Assume that C' and C'' are normal chain complexes. For $x, x' \in C''_{k+1}$ with $\partial''(x) = \partial''(x') = 1$. Then $p(\tilde{x}\tilde{x}') = xx'$ and so

$$\partial_{k+1}(\{x\}\{x'\}) = \partial_{k+1}(\{x\})\partial_{k+1}(\{x'\})$$

provided that C' and C'' are normal.

The composite $i_* \circ \partial_{k+1}$ is trivial because $i(\bar{x}) = \partial(\tilde{x})$. Let $y \in C'_k$ with $\partial'(y) = 1$ and $i_*(y)$ is trivial in $H_k(C)$. Then there exists $\tilde{y} \in C_{k+1}$ such that

$$i(y) = \partial(\tilde{y}).$$

By the construction of ∂_{k+1}, $\partial_{k+1}(p(\tilde{y})) = y$. This shows that

$$H_{k+1}(C'') \xrightarrow{\partial_{k+1}} H_k(C') \xrightarrow{i_*} H_k(C)$$

is exact.

Now we show that

$$H_{k+1}(C) \xrightarrow{p_*} H_{k+1}(C'') \xrightarrow{\partial_{k+1}} H_k(C')$$

is exact. Let $y \in C_{k+1}$ such that $\partial(y) = 1$. Then by the construction of ∂_{k+1}, $\partial_{k+1}(p(y)) = 1$. Thus the composite $\partial_{k+1} \circ p_*$ is trivial. Suppose that $x \in C''_{k+1}$ with $\partial''(x) = 1$ and $\bar{x} = \partial_{k+1}(x)$ is trivial in $H_k(C')$. There exists an element $z \in C'_{k+1}$ such that

$$\partial'(z) = \bar{x}.$$

Let $\hat{x} = i(z)^{-1}\tilde{x}$. Then

$$p(\hat{x}) = p(i(z)^{-1}\tilde{x}) = p(\tilde{x}) = x$$

with

$$\partial(\hat{x}) = \partial(i(z)^{-1}\tilde{x}) = i(\partial'(z)^{-1}\bar{x}) = 1.$$

Thus $\hat{x}$ defines an elements in $H_{k+1}(C)$ with $p_*(\{\hat{x}\}) = \{x\}$.

Finally we show that

$$H_k(C') \xrightarrow{i_*} H_k(C) \xrightarrow{p_*} H_k(C'')$$

is exact. Since $p \circ i$ is trivial, so is $p_* \circ i_*$. Let $y \in C_k$ with $\partial(y) = 1$ and $p_*(y)$ is trivial in $H_k(C'')$. There exists an element $z \in C''_{k+1}$ such that

$$p(y) = \partial''(z).$$

Let $\tilde{z} \in C_{k+1}$ such that $p(\tilde{z}) = z$. Then

$$\begin{aligned} p(y\partial(\tilde{z}^{-1})) &= \partial''(z)p(\partial(\tilde{z}^{-1})) \\ &= \partial''(z)\partial''(p(\tilde{z})^{-1}) \\ &= \partial''(z)\partial''(z^{-1}) \\ &= 1. \end{aligned}$$

Thus there exists $w \in C'_k$ such that $i(w) = y\partial(\tilde{z}^{-1})$ with

$$i(\partial'(w)) = \partial(i(w)) = \partial(y\partial(\tilde{z}^{-1})) = 1.$$

and so $\partial'(w) = 1$. Hence $i_*(\{w\}) = \{y\}$. The proof is finished now. □

Let X' be a Δ-subset of X. The relative homology $H_*(X, X'; G)$ is defined by

$$H_*(X, X'; G) = H_*((\mathbb{Z}(X)/\mathbb{Z}(X') \otimes G, \partial_*).$$

COROLLARY 5.23. *Let X' be a Δ-subset of X. Then there is a long exact sequence*

$$\cdots \longrightarrow H_{k+1}(X, X'; G) \xrightarrow{\partial_{k+1}} H_k(X'; G) \xrightarrow{i_*} H_k(X; G) \xrightarrow{p_*} H_k(X, X'; G) \longrightarrow \cdots$$

for abelian group G. □

3.3. Simplicial Sets. A *simplicial set* means a Δ-set X with *degeneracies* $s_i\colon X_n \to X_{n+1}$ such that

$$s_j s_i = s_{i+1} s_j$$

for $j \leq i$ and

$$d_j s_i = \begin{cases} s_{i-1} d_j & j < i \\ \mathrm{id} & j = i, i+1 \\ s_i d_{j-1} & j > i+1. \end{cases}$$

The three identities for $d_i d_j$, $s_j s_i$ and $d_i s_j$ are called the *simplicial identities.*

Note. One can use *deleting-doubling* for catching simplicial identities:

$$d_i\colon (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

$$s_i\colon (x_0, \ldots, x_n) \longrightarrow (x_0, \ldots, x_{i-1}, x_i, x_i, x_{i+1}, \ldots, x_n).$$

Let $\mathcal{O}$ be the category whose objects are finite ordered sets and whose morphisms are functions $f\colon X \to Y$ such that $f(x) \leq f(y)$ if $x < y$. The objects in $\mathcal{O}$ are given by $[n] = \{0, \ldots, n\}$ for $n \geq 0$, which are the same as the objects in $\mathcal{O}^+$. The morphisms in $\mathcal{O}$ are generated by d^i, which is defined in $\mathcal{O}^+$, and the following morphism

$$s^i\colon [n+1] \to [n] \quad s^i = \begin{pmatrix} 0 & 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n+1 \\ 0 & 1 & \cdots & i-1 & i & i & i+1 & \cdots & n \end{pmatrix}$$

for $0 \leq i \leq n$, that is, s^i hits i twice.

Exercise 3.1. Show that simplicial sets are one-to-one correspondent to cofunctors from $\mathcal{O}$ to $\mathcal{S}$.

A *simplicial map* $f\colon X \to Y$ means a sequence of functions $f\colon X_n \to Y_n$ for each $n \geq 0$ such that $f \circ d_i = d_i \circ f$ and $f \circ s_i = s_i \circ f$, that is the diagram

$$\begin{array}{ccccc} X_{n+1} & \xleftarrow{s_i} & X_n & \xrightarrow{d_i} & X_{n-1} \\ \downarrow f & & \downarrow f & & \downarrow f \\ Y_{n+1} & \xleftarrow{s_i} & Y_n & \xrightarrow{d_i} & Y_{n-1} \end{array}$$

commutes.

Example 5.24 (n-simplex). The n-simplex $\Delta[n]$, as a simplicial set, is as follows:

$$\Delta[n]_k = \{(i_0, i_1, \ldots, i_k) \mid 0 \leq i_0 \leq i_1 \leq \cdots \leq i_k \leq n\}$$

for $k \leq n$. The face $d_j\colon \Delta[n]_k \to \Delta[n]_{k-1}$ is given by

$$d_j(i_0, i_1, \ldots, d_k) = (i_0, i_1, \ldots, \hat{i}_j, \ldots, i_k),$$

that is deleting i_j. The degeneracy $s_j\colon \Delta[n]_k \to \Delta[n]_{k+1}$ is defined by

$$s_j(i_0, i_1, \ldots, d_k) = (i_0, i_1, \ldots, i_j, i_j, \ldots, i_k),$$

that is doubling i_j. Let $\sigma_n = (0, 1, \ldots, n) \in \Delta[n]_n$. Then any elements in $\Delta[n]$ can be written as iterated compositions of faces and degeneracies of σ_n.

Proposition 5.25. *Let X be a simplicial set and let $x \in X_n$ be an element. Then there exists a unique simplicial map*

$$f_x \colon \Delta[n] \to X$$

such that $f_x(\sigma_n) = x$.

Proof. We leave the proof as an exercise to the reader. □

Let X be a simplicial set and $A = \{A_n\}_{n\geq 0}$ with $A_n \subseteq X_n$. The *simplicial subset of X generated by A* is defined by

$$\langle A\rangle = \{A \subseteq Y \subseteq X \mid Y \text{ is a simplicial set}\},$$

namely $\langle A\rangle$ consists of elements in X that can be written as iterated compositions of faces and degeneracies of the elements in A.

Example 5.26 (n-sphere). The simplicial n-sphere S^n is defined by

$$S^n = \Delta[n]/\partial(\Delta[n]),$$

where $\partial(\Delta[n])$ is the simplicial subset of $\Delta[n]$ generated by $\Delta[n]_k$ for $k < n$. Let's write explicitly for the elements in the simplicial circle S^1.

Now that

$$\begin{aligned}\Delta[1]_k &= \{(i_0, \ldots, i_k) \mid 0 \leq i_0 \leq \cdots \leq i_k \leq 1\}\\ &= \{(\overbrace{0, \ldots, 0}^{i}, 1, \ldots, 1 \mid 0 \leq i \leq k+1\}\end{aligned}$$

has $k+2$ elements. Now

$$\partial(\Delta[1])_k = \{(0, \ldots, 0), (1, \ldots, 1)\}.$$

By definition, $S^1 = \Delta[1]/\partial(\Delta[1])$. Thus

$$\begin{aligned}S^1_k &= \{*, (i_0, \ldots, i_k) \mid 0 \leq i_0 \leq \cdots \leq i_k \leq 1\}\\ &= \{(\overbrace{0, \ldots, 0}^{i}, 1, \ldots, 1 \mid 1 \leq i \leq k\}\end{aligned}$$

has $k+1$ elements including the basepoint $* = (0, \ldots, 0) \sim (1, \ldots, 1)$.

For general simplicial n-sphere S^n, we have $S^n_k = \{*\}$ for $k < n$ and

$$S^n_k = \{*, (i_0, \ldots, i_k) \mid 0 \leq i_0 \leq \cdots \leq i_k \leq n \text{ with } \{i_0, \ldots, i_k\} = \{0, 1, \ldots, n\}\}$$

for $k \geq n$. □

Let X be any simplicial set. Define

$$\mathrm{sk}_n(X) = \langle X_k \mid k \leq n\rangle.$$

Then we obtain the *skeleton filtration* of X

$$\mathrm{sk}_0(X) \subseteq \mathrm{sk}_1(X) \subseteq \cdots$$

with $X = \bigcup_n \mathrm{sk}_n(X)$ and $\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X) = \bigvee_{\alpha\in J_n} S^n$, a wedge of simplicial n-spheres.

Proposition 5.27. *Let G be an abelian group. Then the inclusion $\mathrm{sk}_n(X) \to X$ induces an isomorphism*

$$H_j(\mathrm{sk}_n(X); G) \xrightarrow{\cong} H_j(X; G)$$

for $j < n$ and epimorphism for $j = n$.

Proof. Let $C' = \mathbb{Z}(\mathrm{sk}_n(X)) \otimes G$ and let $C = \mathbb{Z}(\mathrm{sk}_n(X)) \otimes G$. Then C' is a sub chain complex of C with

$$C'_k = C_k$$

for $k \leq n$. Let $C'' = C/C'$. Then $C''_k = 0$ for $k \leq n$. The assertion follows from the long exact sequence from $C' \to C \to C''$. □

3.4. Geometric Realization of Simplicial Sets. The standard *geometric n-simplex* Δ^n is defined by

$$\Delta^n = \{(t_0, t_1, \ldots, t_n) \mid t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1\}.$$

Define $d^i\colon \Delta^{n-1} \to \Delta^n$ and $s^i\colon \Delta^{n+1} \to \Delta^n$ by setting

$$d^i(t_0, t_1, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}),$$
$$s^i(t_0, t_1, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_{n+1})$$

for $0 \leq i \leq n$.

Let X be a simplicial set. Then its *geometric realization* $|X|$ is a CW-complex defined by

$$|X| = \coprod_{\substack{x \in X_n \\ n \geq 0}} (\Delta^n, x)/\sim = \coprod_{n=0}^{\infty} \Delta^n \times X_n/\sim,$$

where (Δ^n, x) is Δ^n labeled by $x \in X_n$ and $\sim$ is generated by

$$(z, d_i x) \sim (d^i z, x)$$

for any $x \in X_n$ and $z \in \Delta^{n-1}$ labeled by $d_i x$, and

$$(z, s_i x) \sim (s^i z, x)$$

for any $x \in X_n$ and $z \in \Delta^{n+1}$ labeled by $s_i x$. Note that the points in $(\Delta^{n+1}, s_i x)$ and $(\Delta^{n-1}, d_i x)$ are identified with the points in (Δ^n, x).

Exercise 3.2. Prove that $|\Delta[n]| \cong \Delta^n$ and $|S^n| \cong S^n$.

Let $f\colon X \to Y$ be a simplicial map. Then its *geometric realization* $|f|$ is defined by

$$|f|(z, x) = (z, f(x))$$

for any $x \in X_n$ and $z \in \Delta^n$ labeled by x. Clearly $|f|$ is continuous. Thus the geometric realization gives a functor from the category of simplicial sets to the category of CW-complexes.

Let X and Y be simplicial sets. The *Cartesian product* $X \times Y$ is defined by

$$(X \times Y)_n = X_n \times Y_n$$

with $d_i^{X\times Y} = (d_i^X, d_i^Y)$ and $s_i^{X\times Y} = (s_i^X, s_i^Y)$. There is an important theorem due to Milnor [**16**] which can be described as follows:

Theorem 5.28 (Milnor). *Let X and Y be simplicial sets. Then there is a one to one continuous map*

$$\eta\colon |X \times Y| \to |X| \times |Y|$$

which is onto, and which is a homeomorphism if either a) K and L are countable, or b) K or L is locally finite. □

Remark 5.29. We give some remarks:

(1). In Milnor's theorem, the map η is always one-to-one and one. Following from Steenrod's notion, redefine a *new topology* on $|X|\times|Y|$ as a *compact generated topology*, that is, define $A\subseteq|X|\times|Y|$ to be closed if and only if $A\cap C$ is closed for any compact subset C of $|X|\times|Y|$. Under this new topology for $|X|\times|Y|$, the map η is always a homeomorphism.

(2). One can also have the geometric realization $|X|$ of a Δ-set X by setting

$$|X|=\coprod_{n=0}^{\infty}\Delta^n\times X_n/\sim,$$

where (Δ^n,x) is Δ^n labeled by $x\in X_n$ and $\sim$ is generated by

$$(z,d_ix)\sim(d^iz,x)$$

for any $x\in X_n$ and $z\in\Delta^{n-1}$ labeled by d_ix. But it does not have such a good property for Cartesian products in direct way.

(3). **Research Project:** Define the Cartesian product $X\times Y$ for Δ-sets X and Y such that it has the property that $|X\times Y|\cong|X|\times|Y|$.

A simplicial set $G=\{G_n\}_{n\geq 0}$ is called a *simplicial group* if each G_n is a group, and all faces d_i and degeneracies s_i are group homomorphisms.

Corollary 5.30. *Let G be a simplicial group such that each G_n is countable. Then $|G|$ is a topological group. Moreover this property holds for any simplicial group by Steenrod's notion for replacing any topology to be compact generated topology.*

Proof. The multiplication $\mu\colon G\times G\to G$ induces a continuous multiplication

$$\mu\colon |G\times G|=|G|\times|G|\longrightarrow|G|$$

such that $|G|$ is a topological group. □

3.5. Singular Simplicial Sets and Singular Homology. Let Δ^n be the geometric n-simplex. Recall that coface $d^i\colon\Delta^{n-1}\to\Delta^n$ and codegeneracy $s^i\colon\Delta^{n+1}\to\Delta^n$ are defined by

$$d^i(t_0,t_1,\ldots,t_{n-1})=(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}),$$
$$s^i(t_0,t_1,\ldots,t_{n+1})=(t_0,\ldots,t_{i-1},t_i+t_{i+1},\ldots,t_{n+1})$$

for $0\leq i\leq n$.

For any space X, define

$$S_n(X)=\mathrm{Map}(\Delta^n,X)$$

be the set of all continuous maps from Δ^n to X with

$$d_i=d^{i*}\colon S_n(X)=\mathrm{Map}(\Delta^n,X)\longrightarrow S_{n-1}(X)=\mathrm{Map}(\Delta^{n-1},X),$$
$$s_i=s^{i*}\colon S_n(X)=\mathrm{Map}(\Delta^n,X)\longrightarrow S_{n+1}(X)=\mathrm{Map}(\Delta^{n+1},X)$$

for $0\leq i\leq n$. Then $S_*(X)=\{S_n(X)\}_{n\geq 0}$ is a simplicial set, called *singular simplicial set*. In particular, $S_*(X)$ is a Δ-set. This allows us to define singular homology:

Definition 5.31. For a pair of spaces (X,A), the *singular homology* $H_*(X,A;G)$ with coefficients in an abelian group G is defined by

$$H_*(X,A;G)=H_*(S_*(X),S_*(A);G).$$

We suggest you to read Hatcher's book [**7**] for checking that singular homology satisfies the Eilenberg-Steenrod axioms.

3.6. Δ-complexes and Simplicial Homology. We still use the model of geometric n-simplex Δ^n together with $d^i\colon \Delta^{n-1}\to\Delta^n$ given by

$$d^i(t_0,t_1,\ldots,t_{n-1})=(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1})$$

for $0\le i\le n$. Note the boundary

$$\partial\Delta^n=\cup_{i=0}^n d^i(\Delta[n-1])$$

is the union of all faces of Δ^n. Let $\mathrm{Int}(\Delta^n)=\Delta^n\smallsetminus\partial\Delta^n$ be the interior of Δ^n, called *open simplex*.

DEFINITION 5.32. A Δ-*complex structure* on a space X is a collection of maps

$$\mathcal{C}(X)=\{\sigma_\alpha\colon \Delta^n\to X\mid \alpha\in J_n\ n\ge 0\}.$$

such that

(1). $\sigma_\alpha|_{\mathrm{Int}(\Delta^n)}\colon \mathrm{Int}(\Delta^n)\to X$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathrm{Int}(\Delta^n)}$.
(2). For each $\sigma_\alpha\in\mathcal{C}(X)$,

$$\sigma_\alpha\circ d^i\in\mathcal{C}(X).$$

(3). A set $A\subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each $\sigma_\alpha\in\mathcal{C}(X)$.

Define

$$C_n^\Delta(X)=\{\sigma_\alpha\colon \Delta^n\to X\mid \alpha\in J_n\}\subseteq\mathcal{C}(X)$$

with $d_i\colon C_n^\Delta(X)\to C_{n-1}^\Delta(X)$ given by

$$d_i(\sigma_\alpha)=\sigma_\alpha\circ d^i$$

for $0\le i\le n$. Then $C_*^\Delta(X)=\{C_n^\Delta(X)\}_{n\ge0}$ is a Δ-set. The *simplicial homology* of X with coefficients in an abelian group G is defined by

$$H_*^\Delta(X;G)=H_*(C_*^\Delta(X);G).$$

If A is a subcomplex of X, define

$$H_*^\Delta(X,A;G)=H_*(C_*^\Delta(X),C_*^\Delta(A);G).$$

Since each $\sigma_\alpha\colon\Delta^n\to X$ is a map, there is an inclusion of Δ-map

$$C_*^\Delta(X)\longrightarrow S_*(X)$$

and induces a natural transformation

$$H_*^\Delta(X,A;G)\longrightarrow H_*(X,A;G)$$

for any pair of Δ-complexes (X,A).

THEOREM 5.33. *Let X be a Δ-complex and let A be a subcomplex. Then*

$$H_n^\Delta(X,A;G)\longrightarrow H_n(X,A;G)$$

is an isomorphism for each n.

Ideas of Proof. It suffices to show that the assertion holds for the absolute case that $A = \emptyset$ because there is a long exact sequence for homologies for relative cases. For showing that $H^{\Delta}_*(X;G) \cong H_*(X;G)$, one can prove it by induction on the skeleton of X. Observe that

$$\mathrm{sk}_n(X)/\mathrm{sk}_{n-1}(X) \cong \bigvee_{\alpha\in J_n} S^n.$$

By the definition of simplicial homology, one directly gets that

$$H^{\Delta}_k(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X); G) = \begin{cases} \bigoplus_{\alpha\in J_n} G & \text{if} \quad k = n \\ 0 & \text{otherwise.} \end{cases}$$

This proves that

$$H^{\Delta}_*(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X); G) \cong H_*(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X); G),$$

where one needs to show that $H_*(\mathrm{sk}_n(X), \mathrm{sk}_{n-1}(X); G) \cong \bar{H}_*(\bigvee_{\alpha\in J_n} S^n; G)$ is the same as the simplicial homology. Assume that $H^{\Delta}_*(\mathrm{sk}_{n-1}(X); G) \cong H_*(\mathrm{sk}_{n-1}; G)$. Then one concludes that $H^{\Delta}_*(\mathrm{sk}_n(X); G) \cong H_*(\mathrm{sk}_n(X); G)$ by applying the long exact sequence for homologies.

See Hatcher's book for the details of the proof. □

CHAPTER 6

Some Suggested Topics for Your Further Reading/Study

The topics in this chapter will not be covered in class. These topics are only for your further study on algebraic topology after this module in future.

1. Künneth Theorem, Cohomology and Hopf Algebras

1.1. Künneth Theorem. Let R be a commutative ring with identity and let C and D be chain complexes of R-modules. Define the tensor product $C \otimes_R D$ by setting

$$(C \otimes_R D)_n = \bigotimes_{i+j=n} C_i \otimes_R D_j.$$

with differential

$$\partial^{\otimes}(x \otimes y) = \partial(x) \otimes y + (-1)^{|x|} x \otimes \partial(y),$$

where $|x| = i$ for $x \in C_i$.

Lemma 6.1. *The tensor product $C \otimes D$ is a chain complex.*

Proof.

$$\begin{aligned}
\partial^{\otimes} \circ \partial^{\otimes}(x \otimes y) &= \partial^{\otimes}(\partial(x) \otimes y + (-1)^{|x|} x \otimes \partial(y)) \\
&= \partial^{\otimes}(\partial(x) \otimes y) + (-1)^{|x|} \partial^{\otimes}(x \otimes \partial(y)) \\
&= \partial^2(x) \otimes y + (-1)^{|\partial(x)|} \partial(x) \otimes \partial(y) + (-1)^{|x|} \partial(x) \otimes \partial(y) + (-1)^{|x|}(-1)^{|x|} x \otimes \partial^2(y) \\
&= 0
\end{aligned}$$

Thus $C \otimes D$ is a chain complex. □

Recall that for CW-complexes, the integral cellular chain complex

$$C^{\mathrm{CW}}(X \times Y) = C^{\mathrm{CW}}(X) \otimes C^{\mathrm{CW}}(Y),$$

where $R = \mathbb{Z}$.

The Künneth Theorem arising from the problem how to computing $H_*(C \otimes_R D)$. The algebraic statement from homological algebra, see [**8**, Theorem 2.1, p.172], is as follows:

An R-module M is called *flat* if for every short exact sequence

$$A' \hookrightarrow A \twoheadrightarrow A''$$

of R-modules the induced sequence

$$0 \to A' \otimes_R M \to A \otimes_R M \to A'' \otimes_R M \to 0$$

is exact. A chain complex C over R is called flat if each C_n is flat.

Let A be a right R-module and let B be a left R-module. Let

$$K \hookrightarrow P \twoheadrightarrow A$$

be a short exact sequence of R-modules such that P is projective, that is P is summand of a free R-module. Then $\mathrm{Tor}^R(A,B)$ is defined to be the kernel of $K\otimes_R B\to P\otimes_R B$ and so there is an exact sequence

$$0\to \mathrm{Tor}^R(A,B)\longrightarrow K\otimes_R B\longrightarrow P\otimes_R B\longrightarrow A\otimes_R B\longrightarrow 0.$$

(**Note.** One needs to show that $\mathrm{Tor}^R(A,B)$ is independent on the projective presentation $K\hookrightarrow P\twoheadrightarrow A$, that is, if

$$K'\hookrightarrow P'\twoheadrightarrow A$$

is another short exact sequence with P' projective. Then

$$\mathrm{Ker}(K'\otimes_R B\to P'\otimes_R B)\cong \mathrm{Ker}(K\otimes_R B\to P\otimes_R B).$$

THEOREM 6.2. *Let R be a PID (principal ideal domain) and let C and D be chain complexes over R such that one of C and D is flat. Then there is a natural short exact sequence*

$$\bigoplus_{p+q=n} H_p(C)\otimes_R H_q(D)\hookrightarrow H_n(C\otimes D)\twoheadrightarrow \bigoplus_{p+q=n-1}\mathrm{Tor}(H_p(C),H_q(D)).$$

Mover the sequence splits, but not naturally. □.

By using this algebraic theorem, we obtain the Künneth Formula:

THEOREM 6.3 (Künneth Formula). *Let X and Y be CW-complexes and R be a PID. Then there is a natural short exact sequence*

$$\bigoplus_{p+q=n} H_p(X;R)\otimes_R H_q(Y;R)\hookrightarrow H_n(X\times Y;R)\twoheadrightarrow \bigoplus_{p+q=n-1}\mathrm{Tor}(H_p(X;R),H_q(Y;R)).$$

Mover the sequence splits, but not naturally.

PROOF.

$$\begin{aligned} C^{\mathrm{CW}}(X\times Y;R) &= C^{\mathrm{CW}}(X\times Y)\otimes_{\mathbb{Z}} R\\ &= (C^{\mathrm{CW}}(X)\times C^{\mathrm{CW}}(Y))\otimes_{\mathbb{Z}} R\\ &= C^{\mathrm{CW}}(X;R)\otimes_R C^{\mathrm{CW}}(Y;R),\end{aligned}$$

because each $C_n^{\mathrm{CW}}(Z)$ is a free abelian group for any CW-complex Z. The assertion follows from the algebraic Künneth Formula. □

COROLLARY 6.4. *Let X and Y be CW-complexes and let $\mathbb{F}$ be a field. Then*

$$H_*(X\times Y;\mathbb{F})\cong H_*(X;\mathbb{F})\otimes H_*(Y;\mathbb{F}),$$

that is $H_n(X\times Y;\mathbb{F})\cong \bigoplus_{i+j=n} H_i(X;\mathbb{F})\otimes H_j(Y;\mathbb{F})$ *for each n.* □

This result tells that it is much easier to compute the homology with coefficients in a field for Cartesian products. The typical coefficient fields used in algebra topology are $\mathbb{F}=\mathbb{Z}/p\mathbb{Z}$ (mod p homology) and $\mathbb{F}=\mathbb{Q}$ (rational homology).

REMARK 6.5. We give some remarks:

(1). Theorem 6.3 and Corollary 6.4 actually hold for singular homology for any spaces. As we see that the singular homology was defined using singular simplicial sets. So one just show that these statements holds for any simplicial sets by using the method similar to cellular homology. Namely by looking at the skeleton filtration of simplicial sets, we will obtain that the cellular chain complex for the product of simplicial sets is the tensor product, and so we can apply the algebraic Künneth formula.

(2). Künneth Formula also holds for smash product that there is a natural short exact sequence

$$\bigoplus_{p+q=n} \bar{H}_p(X;R)\otimes_R \bar{H}_q(Y;R) \hookrightarrow \bar{H}_n(X\wedge Y;R) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(\bar{H}_p(X;R),\bar{H}_q(Y;R))$$

for an PID R. Mover the sequence splits, but not naturally.

(3). One may ask how to generalize the Künneth formula for general homology theory. My suggestions are to consider the skeleton filtration of the product of CW-complexes (or the product of simplicial sets). Generally speaking, if we have a filtration on a space (or simplicial set) X:

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots$$

with $X=\bigcup\limits_n X_n$, we will obtain a long exact sequence in homology for

$$X_{k-1} \hookrightarrow X_k \longrightarrow (X_k, X_{k-1}),$$

for each k. By putting all of these long exact sequence in a system, one obtains a spectral sequence. So I expect that you will obtain Künneth spectral sequence for the Cartesian product for any homology theory.

The *Universal Coefficient Theorem* can be obtained from Künneth Formula. First let's look at algebraic situation:

Let R be a PID and let C be a flat chain complex over R. Let M be any R-module. We can construct a chain complex D by setting $D_0=M$ and $D_i=0$ for $i\neq 0$. In this case, the tensor product $C\otimes D$ is given by

$$(C\otimes_R D)_n = C_n\otimes_R M.$$

Note that $H_0(D)=M$ and $H_i(D)=0$ for $i\neq 0$. By using Künneth Formula we have the natural short exact sequence

$$H_n(C)\otimes_R M \hookrightarrow H_n(C\otimes_R M) \twoheadrightarrow \mathrm{Tor}^{\mathrm{R}}(H_{n-1}(C),M).$$

This sequence splits, but the splitting is not natural in C.

For any space X there is a natural short exact sequence (with un-natural splitting)

$$H_n(X;R)\otimes_R M \hookrightarrow H_n(X;M) \twoheadrightarrow \mathrm{Tor}^{\mathrm{R}}(H_{n-1}(X,R),M)$$

for any PID R and any R-module M. In particular, for any abelian group G, there is a natural short exact sequence (with un-natural splitting)

$$H_n(X)\otimes G \hookrightarrow H_n(X;G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(X),G)$$

because any abelian group G can be regarded as a $\mathbb{Z}$-module.

1.2. Cohomology. Let X be a Δ-set. Recall that $\mathbb{Z}(X)$ is chain complex, where

$$\partial_n=\sum_{i=0}^n(-1)^i d_i\colon \mathbb{Z}(X_n)\to\mathbb{Z}(X_{n-1}).$$

Let G be any abelian group. Consider

$$\mathrm{Hom}(\mathbb{Z}(X_0),G) \xrightarrow{\partial_0^*} \mathrm{Hom}(\mathbb{Z}(X_1),G) \xrightarrow{\partial_1^*} \mathrm{Hom}(\mathbb{Z}(X_2),G) \longrightarrow \cdots,$$

where

$$\partial_n^*(f)=f\circ\partial_n$$

for $f \in \mathrm{Hom}(\mathbb{Z}(X_n), G)$. Let $\delta_n = \partial_n^*$. Then

$$\delta_{n+1} \circ \delta_n = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0.$$

Thus the sequence of groups $\mathrm{Hom}(Z(X), G) = \{\mathrm{Hom}(\mathbb{Z}(X_n), G)\}$ with differential δ is cochain complex. (Here we use the word cochain because the differential δ has degree of $+1$.)

The *cohomology* of X with coefficients in G is defined by

$$H^n(X; G) = H_n(\mathrm{Hom}(\mathbb{Z}(X), G)) = \mathrm{Ker}(\delta_n)/\,\mathrm{Im}(\delta_{n-1})$$

for each n.

From this, one gets singular cohomology and simplicial cohomology. I suggest you to read Chapter 3 of Hatcher's book for cohomology theory. Below I just highlight few points:

First we need to have a relation between homology and cohomology. This will be so-called universal coefficient theorem for cohomology. It needs a notion of extension groups. Let R be a commutative ring with identity. Let A and let

$$K \hookrightarrow P \twoheadrightarrow A$$

be a short exact sequence of R-modules with P projective, which is called a *projective presentation* of A. Then for any R-module B there is an exact sequence

$$0 \to \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(K, B),$$

where $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(K, B)$ is not onto in general. Define

$$\mathrm{Ext}_R(A, B)$$

to be the cokernel of $\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(K, B)$, called the *extension group*. Again one needs to check that $\mathrm{Ext}_R(A, B)$ is independent on the choice of the projective presentations of A. By using this notion, we have the following theorem, which can be found in [**8**, Theorem 3.3, p.179].

Theorem 6.6 (Universal Coefficient Theorem). *Let R be a PID and let M be an R-module. Then there is a natural short exact sequence*

$$0 \longrightarrow \mathrm{Ext}_R(H_{n-1}(X; R), M) \longrightarrow H^n(X; M) \longrightarrow \mathrm{Hom}_R(H_n(X; R); M) \longrightarrow 0.$$

Moreover this sequence splits; but not natural in X. In particular, for a field $\mathbb{F}$,

$$H^n(X; \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}}(H_n(X; \mathbb{F}); \mathbb{F})$$

which is the dual vector space of $H_n(X; \mathbb{F})$. □

An important structure on cohomology is that it has the product structure. For simplicity, we only discuss for the cases when the (co-)homology is given with coefficients in a field and X is of finite type, that is, each $H_n(X; \mathbb{F})$ is a finite dimensional vector space over $\mathbb{F}$. In this case,

$$\begin{aligned} H^n(X \times X; \mathbb{F}) &\cong \mathrm{Hom}_{\mathbb{F}}(H_n(X \times X; \mathbb{F}); \mathbb{F}) \\ &= \mathrm{Hom}_{\mathbb{F}}(\textstyle\bigoplus_{i+j=n} H_i(X; \mathbb{F}) \otimes H_j(X; \mathbb{F}); \mathbb{F}) \\ &= \bigoplus_{i+j=n} \mathrm{Hom}_{\mathbb{F}}(H_i(X; \mathbb{F}) \otimes H_j(X; \mathbb{F}); \mathbb{F}) \\ &= \bigoplus_{i+j=n} \mathrm{Hom}_{\mathbb{F}}(H_i(X; \mathbb{F}); \mathbb{F}) \otimes \mathrm{Hom}_{\mathbb{F}}(H_j(X; \mathbb{F}); \mathbb{F}) \\ &\cong \bigoplus_{i+j=n} H^i(X; \mathbb{F}) \otimes H^j(X; \mathbb{F}). \end{aligned}$$

In other words, $H^*(X \times X; \mathbb{F}) \cong H^*(X; \mathbb{F}) \otimes H^*(X; \mathbb{F})$. Now the diagonal map

$$\Delta \colon X \to X \times X$$

induces a multiplication

$$\Delta^*\colon H^*(X\times X;\mathbb{F})\cong H^*(X;\mathbb{F})\otimes H^*(X;\mathbb{F})\longrightarrow H^*(X;\mathbb{F})$$

such that $H^*(X;\mathbb{F})$ is a *graded commutative algebra*, that is

$$ab=(-1)^{|a||b|}ba$$

for $a\in H^{|a|}(X;\mathbb{F})$ and $b\in H^{|b|}(X;\mathbb{F})$. This product is called the *cup product*. By keeping this fact in mind, it may help you to read cup product in more general setting. Also there is a cup product for some general cohomology theory. One can see Switzer's book [**25**] for details.

In dual situation, the diagonal map

$$\Delta\colon X\to X\times X$$

induces a comultiplication

$$\Delta_*\colon H_*(X;\mathbb{F})\longrightarrow H_*(X\otimes X;\mathbb{F})\cong H_*(X;\mathbb{F})\otimes H_*(X;\mathbb{F}).$$

In this case, you do not have to assume that X is of finite type because the right side isomorphism directly follows from the Künneth formula. This means that $H_*(X;\mathbb{F})$ is always a graded cocommutative coalgebra. (Think about coalgebras as the dual version of algebras.)

A Connection between Cohomology and Homotopy: Given any group π, there exists a path-connected CW-complex $K(\pi,1)$ which has the property that $\pi_1(K(\pi,1))=\pi$ and $\pi_j(K(\pi,1))=0$ for $j>1$. For any abelian group π, there exists a path-connected CW-complex $K(\pi,n)$ such that $\pi_n(K(\pi,n))=\pi$ and $\pi_j(K(\pi,n))=0$ for $j\neq\pi$. The spaces $K(\pi,n)$ are called *Eilenberg-MacLane Spaces*. Such a space exists and unique up to homotopy for each π and n. (For $n>1$, we need to require that π is abelian.) In Hatcher's book, $K(\pi,1)$ is constructed in chapter 1, as the classifying space of the group π, and $K(\pi,n)$ is constructed in chapter 4 with a proof of uniqueness.

There is another way to construct $K(\pi,n)$ for abelian group π by using simplicial sets. Let S^n be the standard simplicial n-sphere and let π be any abelian group. Consider the simplicial group

$$G=(\mathbb{Z}(S^n)/\mathbb{Z}(*))\otimes\pi,$$

that $G_q=\mathbb{Z}(S^n_q)\otimes\pi$, is a direct sum of copies of π with labeled by elements in S^n_q modulo the relation $*=0$. The faces and degeneracies in G are induced from the faces and degeneracies in S^n. Then the geometric realization $|G|$ of G is $K(\pi,n)$. The proof can be given by the following steps:

(1). It suffices to show that $\pi_i(|G|)=\pi$ for $i=n$ and $\pi_i(|G|)=0$ for $i\neq n$.
(2). There is a combinatorial way to define homotopy group $\pi_n(\Gamma)$ for any simplicial group Γ, called *Moore homotopy group*, with the property that $\pi_n(\Gamma)\cong\pi_n(|\Gamma|)$. (We will give a brief review for Moore homotopy groups later.)
(3). If Γ is an abelian simplicial group, then the Moore homotopy group $\pi_*(\Gamma)$ is isomorphic to the homology by regarding Γ as a chain complex with $\partial_n=\sum_{i=0}^n(-1)^i d_i$.
(4). Now come back to the abelian simplicial group G. The homology of G with the differential given by $\partial_n=\sum_{i=0}^n(-1)^i d_i$ is just the reduced homology of the sphere with coefficients in π. From the above step, we have the Moore homotopy group $\pi_i(G)=\pi$ for $i=n$ and $\pi_i(G)=0$ for $i\neq n$. From the previous step, we conclude that $\pi_i(|G|)=\pi$ for $i=n$ and $\pi_i(|G|)=0$ for $i\neq n$.

THEOREM 6.7. *There is an isomorphism*

$$H^n(X;G)\cong[X,K(G,n)]$$

for each n and any abelian group G.

One can find this theorem in Whitehead's book [**26**, Theorem 7.14, p.250]. There are similar results for general homology theories discussed in Switzer's book [**25**].

Steenrod Operations: These operations play a key role in algebraic topology. One can read Steenrod's book [**22**]. The operations are on mod p cohomology. The ideas can be described as follows:

We want to consider *natural transformations:* $H^*(X;\mathbb{Z}/p) \to H^*(X;\mathbb{Z}/p)$. According to the above theorem,

$$H^n(X;\mathbb{Z}/p) \cong [X, K(\mathbb{Z}/p, n)].$$

Consider the mod p cohomology $H^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$. Let

$$\alpha \in H^q(K(\mathbb{Z}/p,n);\mathbb{Z}/p) \cong [K(\mathbb{Z}/p,n), K(\mathbb{Z}/p,q)]$$

Then α determines a map

$$f_\alpha \colon K(\mathbb{Z}/p,n) \longrightarrow K(\mathbb{Z}/p,q)$$

and so we have a natural transformation

$$f_{\alpha *} \colon H^n(X;\mathbb{Z}/p) = [X, K(\mathbb{Z}/p,n)] \longrightarrow H^q(X;\mathbb{Z}/p) = [X, K(\mathbb{Z}/p,q)].$$

The cohomology $H^*(K(\mathbb{Z}/p,n);\mathbb{Z}/p)$ has been determined by H. Cartan. By using that, Steenrod studied self natural transformations of $H^*(-;\mathbb{Z}/p)$ and produced so-called *Steenrod algebra.*

1.3. Hopf Algebras. Let $H_*(X)$ denote the homology of X with coefficients in a field. As we have seen, the diagonal map $\Delta\colon X \to X \times X$ induces a comultiplication

$$\psi = \Delta_* \colon H_*(X) \longrightarrow H_*(X \times X) = H_*(X) \otimes H_*(X).$$

Thus for any space X, the homology of X with coefficients in a field is a graded cocommutative coalgebra. Let $f\colon X \to Y$ be a map. Then there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ X \times X & \xrightarrow{f \times f} & Y \times Y. \end{array}$$

By taking homology, we have the commutative diagram

$$\begin{array}{ccc} H_*(X) & \xrightarrow{f_*} & H_*(Y) \\ \downarrow{\scriptstyle\Delta_*} & & \downarrow{\scriptstyle\Delta_*} \\ H_*(X) \otimes H_*(X) & \xrightarrow{f_* \otimes f_*} & H_*(Y) \otimes H_*(Y). \end{array}$$

It follows that $f_*\colon H_*(X) \to H_*(Y)$ is a coalgebra map.

Now assume that X is an H-space. Then the multiplication $\mu\colon X \times X \to X$ induces a product

$$\mu_* \colon H_*(X) \otimes H_*(X) = H_*(X \times X) \longrightarrow H_*(X).$$

Moreover μ_* is a morphism of coalgebras. This arises an algebraic notion called *Hopf algebra.*

A *Hopf algebra* A means (1). A is a graded algebra with a multiplication $\mu\colon A\otimes A\to A$, (2). A is a graded coalgebra with a comultiplication $\psi\colon A\to A\otimes A$, and (3). the multiplication $\mu\colon A\otimes A\to A$ is a coalgebra map. (**Note.** The assumption (3) is equivalent to say that the comultiplication ψ is an algebra map.) A is called *connected* if $A_0=\mathbb{F}$.

In algebraic topology, we assume that the comultiplication is graded cocommutative and coassociative if we think that the Hopf algebra comes from the homology of an H-space. But we may not assume that the multiplication is associative if we think that our H-space may not be homotopy associative. Some people [**19**] called such kind of objects as *quasi-Hopf algebra*, namely associativity may not hold for multiplication. If X is a path-connected H-space, then $H_0(X)=\mathbb{F}$ and $H_*(X)$ is connected. Algebraically, any connected quasi-Hopf algebra A is isomorphic to the tensor product of a collection of the following *monogenic Hopf algebras* H with the property that H is generated by single element $x\in H_n$:

(1). If n is odd and $\mathrm{char}(\mathbb{F})\neq 2$, then $H=E(x)$ the exterior algebra generated by x.
(2). If n is even and $\mathrm{char}(\mathbb{F})=0$, $H=P(x)$ the polynomial algebra $P(x)$ generated by x.
(3). If n is even and $\mathrm{char}(\mathbb{F})=0$, either $H\cong P(x)$ or H is the truncated polynomial algebra $P(x)/(x^{p^t})$.
(4). If $\mathrm{char}(\mathbb{F})=2$, either $H\cong P(x)$ or H is the truncated polynomial algebra $P(x)/(x^{2^t})$.

See [**19**] for details.

As an application, we can see many finite complex does not has H-space structure: Let X be a path-connected finite CW-complex such that X is an H-space. Then $H_*(X;\mathbb{Q})$ must be a tensor product of exterior algebra with generators in odd dimensions and polynomial algebra with generators in even dimensions. Since X is finite complex, $H_*(X;\mathbb{Q})$ has to be exterior algebra with generators in odd dimensions. In particular, we can see many complex such as S^{2n}, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$ are not H-spaces.

A *finite H-space* means a finite CW-complex with an H-space structure.

Problem 6.8. *Classify path-connected finite H-spaces.*

This is one of open problems in algebraic topology. There have been a lot of people to study this problem.

2. Hurewicz Theorem and Whitehead Theorem

The statements in this section are some fundamental theorems in algebraic topology. The proofs can be found in Hatcher's book [**7**].

A relation between homotopy groups and homology is as follows: Let X be a pointed space and $[f]\in\pi_n(X)$. Then the map
$$f\colon S^n\to X$$
induces a group homomorphism
$$f_*\colon H_n(S^n)=\mathbb{Z}\longrightarrow H_n(X).$$
Denote by ι the generator for $H_n(S^n)=\mathbb{Z}$. Define
$$h_n\colon \pi_n(X)\to H_n(X)$$
by setting $h_n([f])=f_*(\iota)$. The function H is well defined because if $f\simeq f'$ then $f_*=f'_*$.

Lemma 6.9. *The function $h_n\colon\pi_n(X)\to H_n(X)$ is a group homomorphism for $n\geq 1$.*

PROOF. Let $[f], [g] \in \pi_n(X)$. By definition, the product $[f][g]$ is the homotopy class represented by the composite

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{f \vee g} X \vee X \xrightarrow{\text{fold}} X.$$

By taking homology, we have

$$H_n(S^n) \xrightarrow{\text{pinch}_*} H_n(S^n \vee S^n) = H_n(S^n) \oplus H_n(S^n) \xrightarrow{(f\vee g)_* = f_* \oplus g_*} H_n(X \vee X) = H_n(X) \oplus H_n(X) \xrightarrow{\text{fold}_*} H_n(X).$$

Since $\text{pinch}_*(\iota) = (\iota, \iota)$,

$$h_n([f][g]) = f_*(\iota) + g_*(\iota) = h_n([f]) + h_n([g])$$

and hence the result. □

The group homomorphism $h_n \colon \pi_n(X) \to H_n(X)$ is called *Hurewicz map.*

2.1. The Relation between the Fundamental Group and the First Homology. Now consider the first case $n = 1$. Let G be a group. The abelianization G^{ab} of G is defined to be the quotient group of G modulo the *commutator subgroup.* Observe that G^{ab} has the following universal property:

> Let $\phi \colon G \to A$ be a group homomorphism. If A is abelian, then ϕ factors through the abelianization G^{ab}.

The Hurewicz map $h_1 \colon \pi_1(X) \to H_1(X)$ is a group homomorphism and it incudes a group homomorphism

$$\bar{h}_1 \colon \pi_1(X)^{\text{ab}} \longrightarrow H_1(X).$$

THEOREM 6.10. *Let X be a path-connected space. Then the induced group homomorphism*

$$\bar{h}_1 \colon \pi_1(X)^{\text{ab}} \longrightarrow H_1(X)$$

is an isomorphism. □

In Hatcher's book, the proof of this theorem is given in Section 2.A.

2.2. Whitehead Theorem and Hurewicz Theorem. The Whitehead Theorem is as follows (See [**7**, Theorem 4.5, p.346]):

THEOREM 6.11 (Whitehead Theorem). *If a map $f \colon X \to Y$ between connected CW-complex induces isomorphisms $f_* \colon \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a strong deformation retract of Y.* □

The Hurewicz Theorem is as follows.

THEOREM 6.12 (Hurewicz Theorem). *Let X be a path-connected CW-complex. Suppose that $\pi_i(X) = 0$ for $1 \le i < n$ with $n \ge 2$. Then*

$$h_n \colon \pi_n(X) \to H_n(X)$$

is an isomorphism. □

In Hatcher's book, this theorem is given in a more general form discussed in Section 4.2.

A space X is called *k-connected* if X is path-connected and $\pi_i(X) = 0$ for $i \le k$. Since homology groups are usually more computable than homotopy groups, the following theorem is useful for checking whether a CW-complex is k-connected.

COROLLARY 6.13. *Let X be a connected CW-complex. Suppose that*

(1). $\pi_1(X)$ *is trivial and*
(2). $H_i(X) = 0$ *for* $i < n$.

Then X is $(n-1)$-connected, that is $\pi_i(X) = 0$ for $i < n$, and $\pi_n(X) \cong H_n(X)$.

PROOF. Show by induction that $\pi_i(X) = 0$ for $i < n$. By the assumption, $\pi_1(X) = 0$. Suppose that $\pi_j(X) = 0$ for $j < i$ with $i < n$. Then

$$\pi_i(X) \cong H_i(X) = 0$$

by Hurewicz Theorem. The induction is finished and so $\pi_i(X) = 0$ for all $i < n$. By using Hurewicz Theorem again, $\pi_n(X) \cong H_n(X)$. □

In practice, the following version of Whitehead Theorem is often easier to apply (See [**7**, Corollary 4.33]):

THEOREM 6.14 (Whitehead Theorem). *Let $f\colon X \to Y$ be a map between simply-connected CW-complexes. Suppose that $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for all n. Then f is a homotopy equivalence.* □

A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism for each $n \geq 0$ and all choices of basepoint x_0. A map $f\colon X \to Y$ is called *homology equivalence* if $f_*\colon H_q(X) \cong H_q(Y)$ for all q. Whitehead Theorem states that a homology equivalence between simply connected CW-complexes is a homotopy equivalence. The following Theorem is given in Hatcher's book [**7**, Proposition 4.22, p.357].

THEOREM 6.15. *Let $f\colon X \to Y$ be a map. Then f is a weak homotopy equivalence if and only if*

$$f_*\colon [K, X] \longrightarrow [K, Y]$$

is one-to-one and onto for every CW-complexes K. □

We conclude this section with some examples from Whitehead's book [**26**, pp.183-184] relevant to proceeding important theorems.

EXAMPLE 6.16. *There are path-connected CW-complexes X and Y with $\pi_n(X) \cong \pi_n(Y)$ for all n, but $H_*(X) \not\cong H_*(Y)$. (Therefore there is no map $f\colon X \to Y$ inducing isomorphisms of the homotopy groups.)*

For giving an example, let $X = \mathbb{R}\mathrm{P}^m \times S^n$ and $Y = S^m \times \mathbb{R}\mathrm{P}^n$ for $m > n > 1$ with m even and n odd. Since $S^q \to \mathbb{R}\mathrm{P}^q$ is a 2-sheeted covering,

$$S^m \times S^n \longrightarrow X \quad S^n \times S^m \longrightarrow Y$$

are 2-sheeted covering. Thus

$$\pi_1(X) \cong \pi_1(Y) \cong \mathbb{Z}/2$$

and

$$\pi_j(X) \cong \pi_j(Y) \cong \pi_j(S^m \times S^n)$$

for $j > 1$. By Künneth Formula,

$$H_{n+m}(X) \cong H_m(\mathbb{R}\mathrm{P}^m) \otimes H_n(S^n) = H_m(\mathbb{R}\mathrm{P}^m) = 0$$
$$H_{n+m}(Y) \cong H_m(S^m) \otimes H_n(\mathbb{R}\mathrm{P}^n) = H_n(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}.$$

Thus $H_{n+m}(X) \not\cong H_{n+m}(Y)$ □

EXAMPLE 6.17. *There are simply connected CW-complexes X and Y with $H_n(X) \cong H_n(Y)$ for all n, but $\pi_*(X) \not\cong \pi_*(Y)$. (Therefore there is no map $f\colon X \to Y$ inducing isomorphisms of the homotopy groups.)*

For giving an example, let $X = S^2 \vee S^4$ and $Y = \mathbb{C}\mathrm{P}^2$. Then X and Y do indeed have isomorphic homology with $H_0 = \mathbb{Z}$, $H_2 = \mathbb{Z}$, $H_4 = \mathbb{Z}$ and $H_j = 0$ for $j \neq 0, 2, 4$. Since S^4 is a retract of X, $\pi_4(S^4) = \mathbb{Z}$ is a summand of $\pi_4(X)$ and so $\pi_4(X) \neq 0$. On the other hand, there is a fibre bundle

$$S^1 \longrightarrow S^5 \longrightarrow \mathbb{C}\mathrm{P}^2.$$

(We will briefly go through fibrations and fibre bundles later.) It induces a long exact sequence on homotopy groups

$$\cdots \longrightarrow \pi_4(S^1) \longrightarrow \pi_4(S^5) = 0 \longrightarrow \pi_4(\mathbb{C}\mathrm{P}^2) \longrightarrow \pi_3(S^1) = 0 \longrightarrow \cdots.$$

Thus $\pi_4(Y) = 0$. Hence $\pi_4(X) \not\cong \pi_4(Y)$. □

EXAMPLE 6.18. *The reader may wonder whether a map $f\colon X \to Y$ between path-connected CW-complexes is a homotopy equivalence provided that $f_*\colon \pi_1(X) \to \pi_1(Y)$ and $f_*\colon H_q(X) \to H_q(Y)$ are isomorphisms for all q. The example in Whitehead's book* [**26**, Example 3, p.183] *shows that this is not the case.* □

3. Fibrations and Fibre Sequences

3.1. Exact Sequences. Let $f\colon A \to B$ be a pointed map. Let PB be the path space over B, that is

$$PB = \{\lambda\colon I \to B \mid \lambda(1) = b_0\}$$

with compact open topology. The basepoint in PB is the constant path ω_0 with $\omega_0(t) = b_0$ for $0 \le t \le 1$. Define the *mapping path space*

$$P_f = \{(a, \lambda) \in A \times PB \mid f(a) = \lambda(0)\},$$

that is there is a pull-back diagram

$$\begin{array}{ccc} P_f & \hookrightarrow & PB \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{\ f\ } & B, \end{array}$$

where $q(\lambda) = \lambda(0)$.

LEMMA 6.19. *Let $f\colon A \to B$ be a pointed map. Then there is an exact sequence*

$$[X, P_f] \xrightarrow{f_{1*}} [X, A] \xrightarrow{f_*} [X, B]$$

for any pointed space X.

PROOF. Let $g\colon X \to A$ such that $f \circ g\colon X \to B$ is null homotopic under a pointed homotopy. Then $G\colon X \times I \to B$ such that $G_0 = f \circ g$, $G_1(x) = b_0$ and $G(x_0, t) = b_0$ for $0 \le t \le 1$. Let

$$\tilde{G}\colon X \to \mathrm{Map}(I, B)$$

be the adjoint map of G. Since $\tilde{G}(x)(1) = G(x,1) = b_0$, $\tilde{G}$ maps into the path space $PB \subseteq \mathrm{Map}(I,B)$. Since $\tilde{G}(x_0)(t) = b_0$, $\tilde{G}(x_0) = \omega_0$ and so the map $\tilde{G}\colon X \to PB$ is a pointed map. Now

$$(g, \tilde{G})\colon X \longrightarrow A \times PB$$

maps X into P_f because $\tilde{G}(x)(0) = G(x,0) = f(g(x))$. This defines a map $(g, \tilde{G})\colon X \to P_f$ with

$$f_1(g, \tilde{G}) = g.$$

Thus $f_{1*}([(g, \tilde{G})]) = [g]$ and hence the result. □

Thus we have a tower

$$\cdots \longrightarrow P_{f_2} \xrightarrow{f_3} P_{f_1} \xrightarrow{f_2} P_f \xrightarrow{f_1} A \xrightarrow{f} B$$

with the long exact sequence

$$\cdots \longrightarrow [X, P_{f_2}] \xrightarrow{f_{3*}} [X, P_{f_1}] \xrightarrow{f_{2*}} [X, P_f] \xrightarrow{f_{1*}} [X, A] \xrightarrow{f_*} [X, B]$$

for any pointed space X.

From the construction, $P_f \subseteq A \times PB$ and with $f_1\colon P_f \to A$ by first coordinate projection, that is $f_1(a, \lambda) = a$. Thus

$$P_{f_1} \subseteq P_f \times PA \subseteq A \times PB \times PA.$$

More precisely

$$P_{f_1} = \{(a, \lambda, \mu) \in A \times PB \times PA \mid \lambda(0) = f(a),\ \mu(0) = a\}.$$

Lemma 6.20. *The space P_{f_1} is homeomorphic to the subspace*

$$\{(\lambda, \mu) \in PB \times PA \mid \lambda(0) = f(\mu(0))\}$$

of $PB \times PA$.

Proof. Let

$$T = \{(\lambda, \mu) \in PB \times PA \mid \lambda(0) = f(\mu(0))\}.$$

The coordinate projection

$$A \times PB \times PA \longrightarrow PB \times PA$$

induces a map $\phi\colon P_{f_1} \to T$. Now the map

$$PB \times PA \longrightarrow A \times PB \times PA \quad (\lambda, \mu) \mapsto (\mu(0), \lambda, \mu)$$

induces a map $\psi\colon T \to P_{f_1}$. Clearly $\phi \circ \psi = \mathrm{id}_T$ and $\psi \circ \phi = \mathrm{id}_{P_{f_1}}$. The proof is finished. □

By using the identification of Lemma 6.20, the map f_2 is given by the following commutative diagram:

(5)
$$\begin{array}{ccccc} P_{f_1} & \hookrightarrow & PB \times PA & & (\lambda, \mu) \\ \downarrow{\scriptstyle f_2} & & \downarrow & & \downarrow \\ P_f & \hookrightarrow & A \times PB & & (\mu(0), \lambda). \end{array}$$

Theorem 6.21. *There is a homotopy equivalence: $P_{f_1} \simeq \Omega B$*

Proof. From the previous lemma,

$$P_{f_1} = \{(\lambda, \mu) \in PB \times PA \mid \lambda(0) = f(\mu(0))\}.$$

Let $j\colon \Omega B \to P_{f_1}$ be the map given by

$$j(\omega) = (\omega, \omega_0^A),$$

where ω_0^A is the constant path in A with $\omega_0^A(t) = a_0$ for $0 \le t \le 1$.

Define the map $\theta\colon P_{f_1} \to \Omega B$ by

$$\theta(\lambda,\mu)(t) = \begin{cases} f(\mu(1-2t)) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \lambda(2t-1) & \text{if} \quad \frac{1}{2} \le t \le 1, \end{cases}$$

that is $\theta(\lambda, \mu) = f(\mu)^{-1} * \lambda$ as path product. Then

$$\theta \circ j(\omega)(t) = \theta(\omega, \omega_0^A)(t) = \begin{cases} b_0 & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \lambda(2t-1) & \text{if} \quad \frac{1}{2} \le t \le 1, \end{cases}$$

It follows that $\theta \circ j \simeq \mathrm{id}_{\Omega B}$ by the homotopy

$$G(\omega, s)(t) = \begin{cases} b_0 & \text{if} \quad 0 \le t \le \frac{s}{2} \\ \lambda(\frac{t-\frac{s}{2}}{1-\frac{s}{2}}) & \text{if} \quad \frac{s}{2} \le t \le 1, \end{cases}$$

Define $F_A\colon P_{f_1} \times I \to PA$ and $F_B\colon P_{f_1} \times I \longrightarrow PB$ by

$$F_A(\lambda, \mu; s)(t) = \mu(1 - s(1-t));$$

$$F_B(\lambda, \mu; s)(t) = \begin{cases} f(\mu(1-s-2t)) & \text{if} \quad 0 \le t \le \frac{1-s}{2} \\ \lambda(\frac{2}{1+s}\left(t - \frac{1-s}{2}\right)) & \text{if} \quad \frac{1-s}{2} \le t \le 1. \end{cases}$$

For each $0 \le s \le 1$, we have

$$f(F_A(\lambda, \mu; s)(0)) = f(\mu(1-s)) = F_B(\lambda, \mu; s)(0)$$

and so it defines a pointed homotopy

$$(F_B, F_A)\colon P_{f_1} \times I \longrightarrow P_{f_1} \quad (\lambda, \mu; s) \mapsto (F_B(\lambda, \mu; s), F_A(\lambda, \mu; s)).$$

When $s = 1$, then

$$(F_B, F_A)(\lambda, \mu; 1)(t) = (\lambda(t), \mu(t)$$

and $(F_B, F_A)_1 = \mathrm{id}_{P_{f_1}}$. When $s = 0$,

$$(F_B, F_A)(\lambda, \mu; 0)(t) = j \circ \theta(t)$$

for $0 \le t \le 1$. Thus $j \circ \theta \simeq \mathrm{id}_{P_{f_1}}$. The proof is finished. □

Proposition 6.22. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} P_{f_2} & \xrightarrow{\;f_3\;} & P_{f_1} \\ \uparrow{\scriptstyle j} & & \downarrow{\scriptstyle \theta} \\ \Omega A & \xrightarrow{\;f^{\nu}\;} & \Omega B, \end{array}$$

where θ and j are given in the proof of Theorem 6.21 and $\nu\colon S^1 \to S^1$ is the inverse map.

Proof. From the commutative diagram (5), there is a commutative diagram

$$\begin{array}{ccc} P_{f_2} \subseteq PA \times PP_f & & (\mu, \Lambda) \\ \downarrow f_3 \qquad \downarrow & & \downarrow \\ P_{f_1} \subseteq P_f \times PA & & (\Lambda(0), \mu). \end{array}$$

Let $\mu\Omega A$ be a loop in A. Then $j(\mu) = (\mu, e)$, where e is the constant path in PP_f. Thus

$$\theta \circ f_3 \circ j(\mu) = \theta(e(0), \mu) = \begin{cases} f(\mu(1-2t)) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ b_0 & \text{if} \quad \frac{1}{2} \le t \le 1, \end{cases}$$

Thus $\theta \circ f_3 \circ j \simeq f^\nu$ and hence the result. □

Thus we have the homotopy commutative diagram

$$\begin{array}{ccccccccccc} P_{f_2} & = & P_{f_2} & \xrightarrow{f_3} & P_{f_1} & = & P_{f_1} & = & P_{f_1} & \xrightarrow{f_2} & P_f \\ \downarrow\theta & & \uparrow j & & & & \downarrow\theta & & \uparrow j & & \\ \Omega A & = & \Omega A & \xrightarrow{\Omega f} & \Omega B & \xrightarrow{\mathrm{id}_B^\nu = -1} & \Omega B & = & \Omega B. & & \end{array}$$

Define $\partial\colon \Omega B \to P_f$ to be the composite

$$\Omega B \xrightarrow{\mathrm{id}_B^\nu} \Omega B \xrightarrow{j} P_{f_1} \xrightarrow{f_2} P_f.$$

Then we have the following theorem.

Theorem 6.23. *Let $f\colon A \to B$ be any pointed map. Then there is a natural long exact sequence*

$$\cdots \xrightarrow{\Omega\partial_*} [X, \Omega P_f] \xrightarrow{\Omega f_{1*}} [X, \Omega A] \xrightarrow{\Omega f_*} [X, \Omega B] \xrightarrow{\partial_*} [X, P_f] \xrightarrow{f_{1*}} [X, A] \xrightarrow{f_*} [X, B]$$

for any pointed space X. □

Proposition 6.24. *A commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow h \\ A' & \xrightarrow{f'} & B' \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \Omega A & \xrightarrow{\Omega f} & \Omega B & \xrightarrow{\partial} & P_f & \xrightarrow{f_1} & A & \xrightarrow{f} & B \\ & & \downarrow\Omega g & & \downarrow\Omega h & & \downarrow k & & \downarrow g & & \downarrow h \\ \cdots & \longrightarrow & \Omega A' & \xrightarrow{\Omega f'} & \Omega B' & \xrightarrow{\partial'} & P_{f'} & \xrightarrow{f_1'} & A' & \xrightarrow{f'} & B'. \end{array}$$

Proof. Let $k\colon P_f \to P_{f'}$ be given by $k(a,\lambda) = (g(a), h^{\mathrm{id}}(\lambda)$. □

3.2. Fibrations. A map $p\colon E \to B$ be a map. A *homotopy lifting problem* can be symbolized by a commutative diagram

$$\begin{array}{ccc} X \times 0 & \xrightarrow{\;g\;} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ X \times I & \xrightarrow{\;G\;} & B. \end{array} \tag{6}$$

A map $p\colon E \to B$ is called to have *homotopy lifting property* with respect to X if the above homotopy lifting problem has a solution for every $g\colon X \to E$ and $G\colon I \times X \to B$ such that $p \circ g = G_0$. A map $p\colon E \to B$ is called a *fibration* if it has the homotopy lifting property with respect to all spaces.

Let $p\colon E \to B$ be a fibration. The subspace $p^{-1}(b)$ is called the *fibre* over $b \in B$. If b_0 is the basepoint, then $F = p^{-1}(b_0)$ is called the fibre of p.

Proposition 6.25. *Let (X, A) be a pair of spaces. Suppose that the inclusion $i\colon A \to X$ is a cofibration. Then*

$$p = \mathrm{id}_Z^i\colon \operatorname{Map}(X, Z) \to \operatorname{Map}(A, Z)$$

is a fibration for any space Z.

Proof. By taking adjoint to the homotopy lifting problem

$$\begin{array}{ccc} Y \times 0 & \longrightarrow & \operatorname{Map}(X, Z) \\ \big\downarrow & \nearrow & \big\downarrow p = \mathrm{id}_Z^i \\ Y \times I & \longrightarrow & \operatorname{Map}(A, Z), \end{array}$$

we have the following diagram

$$\begin{array}{ccc} X \times Y \times 0 & \longrightarrow & Z \\ \big\downarrow & \nearrow & \big\uparrow \\ X \times Y \times I & \hookleftarrow & A \times Y \times I \end{array}$$

and the assertion follows from the homotopy extension property of

$$(X \times Y, A \times Y) = (X, A) \times (Y, \emptyset).$$

□

Corollary 6.26. *The Maps*

$$\operatorname{Map}(I, X) \longrightarrow X \times X \quad \lambda \mapsto (\lambda(0), \lambda(1)),$$
$$\operatorname{Map}(I, X) \longrightarrow X \quad \lambda \mapsto \lambda(1)$$

are fibrations. □

A basepoint x_0 of X is called *nondegenerate* if $x_0 \to X$ is a cofibration.

Corollary 6.27. *Let X be a pointed space with nondegenerate basepoint x_0. Then the evaluation map*

$$\mathrm{Map}(X, Y) \longrightarrow Y \quad f \mapsto f(x_0)$$

is a fibration.

Proposition 6.28. *If $p\colon E \to B$ is a fibration, then*

$$p^{\mathrm{id}_Z}\colon \mathrm{Map}(Z, E) \longrightarrow \mathrm{Map}(Z, B)$$

is a fibration for any locally compact space Z.

Proof. By taking adjoint to the homotopy lifting problem

$$\begin{array}{ccc} X \times 0 & \longrightarrow & \mathrm{Map}(Z, E) \\ \big\downarrow & \nearrow & \big\downarrow p^{id} \\ X \times I & \longrightarrow & \mathrm{Map}(Z, B), \end{array}$$

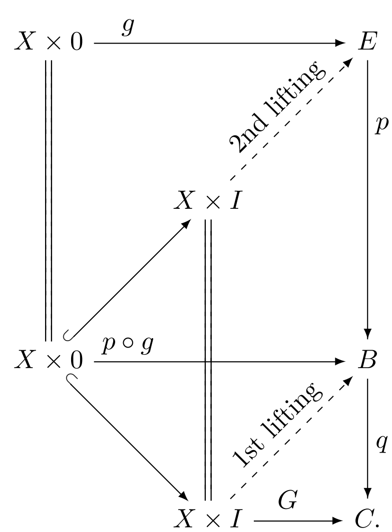
we have the following diagram

$$\begin{array}{ccc} Z \times X \times 0 & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ Z \times X \times I & \longrightarrow & B \end{array}$$

and hence the result. □

Proposition 6.29. *If $p\colon E \to B$ and $q\colon B \to C$ are fibrations, then the composite $q \circ p\colon E \to C$ is a fibration.*

Proof. The assertion follows from the following commutative diagram

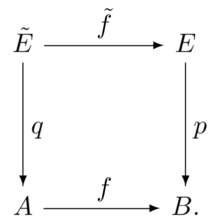


□

PROPOSITION 6.30. *Let $p\colon E \to B$ be a fibration. Let $f\colon A \to B$ be any map. Let*

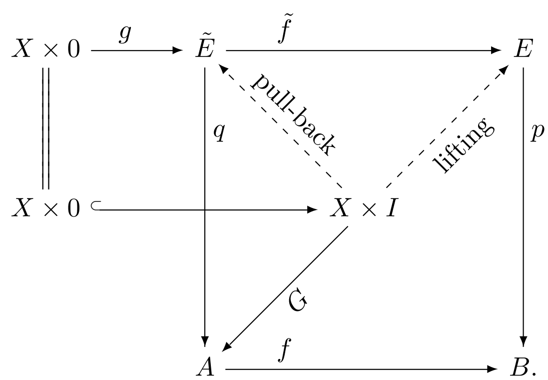
$$\tilde{E} = \{(a, y) \in A \times E \mid f(a) = p(y)\}$$

with $q\colon \tilde{E} \to X$, $q(a, y) = a$, that is there is pull-back



Then $q\colon \tilde{E} \to A$ is a fibration.

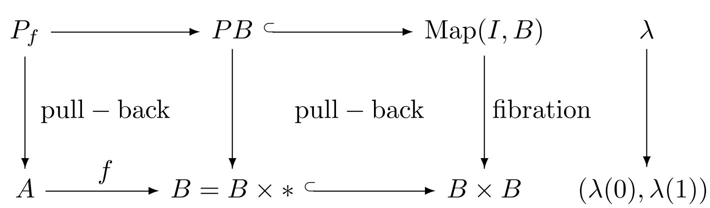
PROOF. The assertion follows from the commutative diagram



□

COROLLARY 6.31. *The map $PB \to B$, $\lambda \mapsto \lambda(0)$ is a fibration. Moreover for any map $f\colon A \to B$ the map $f_1\colon P_f \to A$ is a fibration.*

PROOF. The assertions follow from the following commutative diagram



□

Let $f\colon A \to B$ be a pointed map. Define

$$\tilde{P}_f = \{(x, b, \lambda) \in A \times \mathrm{Map}(I, B) \mid f(x) = \lambda(0)\ \lambda(1) = b\},$$

that is $\tilde{P}_f$ is the pull-back of the diagram

$$\begin{array}{ccccc}
\tilde{P}_f & \longrightarrow & \mathrm{Map}(I, B) & & \lambda \\
\Big\downarrow \tilde{q} & & \Big\downarrow & & \Big\downarrow \\
A \times B & \xrightarrow{f \times \mathrm{id}_B} & B \times B & & (\lambda(0), \lambda(1)).
\end{array}$$

Define $q\colon \tilde{P}_f \to B$ by

$$q(x, b, \lambda) = \lambda(1).$$

Then $q\colon \tilde{P}_f \to B$ is a fibration with fibre P_f because q is given by the composite of fibrations

$$\tilde{P}_f \xrightarrow{\tilde{q}} A \times B \xrightarrow{\text{proj.}} B.$$

Define $j\colon A \to \tilde{P}_f$ by setting

$$j(x) = (x, f(x), \omega_{f(x)}),$$

where $\omega_{f(x)}$ is the constant path with $\omega_{f(x)}(t) = f(x)$ for $0 \leq t \leq 1$. The space A can identified with the subspace of $\tilde{P}_f$ consisting of $\tilde{A} = \{(x, f(x), \omega_{f(x)}\}$ because the inverse map $\tilde{A} \to A$ of j can be given by projecting to the first coordinate. Note that

$$f = q \circ j\colon A \to B$$

with q a fibration.

THEOREM 6.32. *The subspace $\tilde{A}$ is a strong deformation retract of $\tilde{P}_f$. Thus any pointed map $f\colon A \to B$ admits a decomposition*

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\simeq \Big\downarrow j & \nearrow q & \\
\tilde{P}_f & &
\end{array}$$

with j a homotopy equivalence and $q\colon \tilde{P}_f \to B$ a fibration with fibre P_f.

The importance of this theorem is that for any pointed map we can always write it as the composite of a fibration with a homotopy equivalence. This is very useful in practice.

PROOF. Define the homotopy

$$H\colon \tilde{P}_f \times I \longrightarrow P_f$$

by setting

$$H(x, b, \lambda; t) = (x, b, \phi(\lambda, t)),$$

where

$$\phi(\lambda, t)(s) = \lambda(s(1 - t)).$$

Then

$$H(x, b, \lambda; 0) = (x, b, \lambda)$$

for any $(x, b, \lambda) \in \tilde{P}_f$. For any $(x, f(x), \omega_{f(x)})$,

$$H(x, f(x), \omega_{f(x)}) = (x, f(x), \omega_{f(x)})$$

because $\omega_{f(x)}$ is a constant path. When $t = 1$, then

$$\phi(\lambda, 1)(s) = \lambda(0)$$

for $1 \leq s \leq 1$. Thus

$$H(x, b, \lambda; 1) = (x, b, \omega_{\lambda(0)}) = (x, f(x), \omega_{f(x)}) \subseteq \tilde{A}$$

and hence the result. □

A pair of space (X, A) is called a *DR-pair* if there are continuous maps $u\colon X \to I$ and $h\colon X \times I \to X$ such that

(1). $A = u^{-1}(0)$.
(2). $h(x, 0) = x$ for all $x \in X$;
(3). $h(x, t) = x$ for all $x \in A$ and $t \in I$;
(4). $h(X \times 1) \subseteq A$.

Conditions $(2) - (4)$ tell that A is a strong deformation retract of X. Condition (1) tells that $A = u^{-1}(0)$ for a map $u\colon X \to I$. The pair (u, h) is said to *represent* (X, A) as a DR-pair.

A pair of space (X, A) is called an *NDR-pair* if there are continuous maps $u\colon X \to I$ and $h\colon X \times I \to X$ such that

(1). $A = u^{-1}(0)$.
(2). $h(x, 0) = x$ for all $x \in X$;
(3). $h(x, t) = x$ for all $x \in A$ and $t \in I$;
(4). $h(x, 1) \subseteq A$ for all $x \in X$ such that $u(x) < 1$.

The pair (u, h) is said to *represent* (X, A) as an NDR-pair.

The following theorem is from Steenrod's paper [**22**]

LEMMA 6.33. *If X is compactly generated and A is closed in X, then the following statements are equivalent each other:*

(1). *(X, A) is an NDR-pair.*
(2). *$(X \times I, X \times 0 \cup A \times I)$ is a DR-pair.*
(3). *$X \times 0 \cup A \times I$ is a retract of $X \times I$.*
(4). *(X, A) has the homotopy extension property with respect to any spaces, that is the inclusion $A \to X$ is a cofibration.*

PROOF. We only prove (1) $\Longrightarrow$ (2) (without using the assumption that X is compactly generated). We refer to Steenrod's paper for having a complete proof.

(1) $\Longrightarrow$ (2). Assume that (X, A) is an NDR-pair represented by (u, h). Define

$$\tilde{u}\colon X \times I \to I$$

by setting

$$\tilde{u}(x, t) = tu(x).$$

Then

$$\tilde{u}^{-1}(0) = X \times 0 \cup A \times I.$$

Now define

$$H\colon (X \times I) \times I \to X \times I$$

by setting

$$H(x,s,t) = (h(x,t),(1-tu(x))s).$$

Then

$$H(x,s,0) = (h(x,0),s) = (x,s).$$

For $x \in A$, since $u(x) = 0$ and $h(x,t) = x$,

$$H(x,s,t) = (h(x,t),s) = (x,s).$$

Now

$$H(x,s,1) = (h(x,1),(1-u(x))s).$$

If $u(x) < 1$, then $h(x,1) \in A$ and so $H(x,s,1) \in A \times I$. If $u(x) = 1$, then

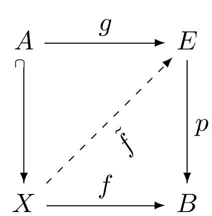
$$H(x,s,1) = (h(x,1),0) \in X \times 0.$$

It follows that

$$H(X \times I \times 1) \subseteq X \times 0 \cup A \times I.$$

This shows that $(X \times I, X \times 0 \cup A \times I)$ is a DR-pair represented by $(\tilde{u}, H)$. □

LEMMA 6.34. *Let $p\colon E \to B$ be a fibration and let (X,A) be a pair of spaces. Suppose that (X,A) is DR-pair. Then every lifting problem*



has a solution.

PROOF. Let $u\colon X \to I$ and $h\colon X \times I \to X$ satisfy the conditions for (X,A) to be a DR-pair. Let $r\colon X \to A$ be the map given by $r(x) = h(x,1)$. Define a new homotopy $\Phi\colon X \times I \to X$ by setting

$$\Phi(x,t) = \begin{cases} h(x,1) & \text{if} \quad u(x) = 0, \\ h\left(x, 1 - \frac{t}{u(x)}\right) & \text{if} \quad 0 \le t < u(x), \\ h(x,0) = x & \text{if} \quad t \ge u(x) > 0. \end{cases}$$

We need to show that Φ is continuous. Observe that Φ restricted to the open subset $(X \smallsetminus A) \times I$ is continuous as $u(x) > 0$ for $x \in X \smallsetminus A$. Let $(x,t) \in A \times I$. Let U be any open neighborhood of x. Then

$$\{x\} \times I \subseteq h^{-1}(U)$$

as $h(x,s) = x$ for $x \in A$ and $0 \le s \le 1$. There exists an open neighborhood V of x such that

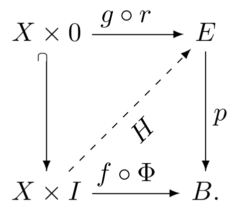
$$V \times I \subseteq h^{-1}(U)$$

because $h^{-1}(U)$ is open in $X \times I$ containing $\{x\} \times I$. Note that $\Phi(y,s) = h(y,s')$ for some s' depending on y and s. It follows that

$$\Phi(V \times I) \subseteq h(V \times I) \subseteq U.$$

and so Φ is continuous.

Now let $H\colon X\times I\to E$ be a solution to the lifting problem



Define $\tilde f\colon X\to E$ by setting

$$\tilde f(x)=H(x,u(x)).$$

Then

$$\begin{array}{rcl} p\circ\tilde f(x) & = & p\circ H(x,u(x)) \\ & = & f\circ\Phi(x,u(x)) \\ & = & \begin{array}{lll} f\circ h(x,0)=f(x) & \text{if} & u(x)>0 \\ f\circ h(x,1)=f(x) & \text{if} & u(x)=0 \end{array} \\ & = & f(x) \end{array}$$

and, for $x\in A$, we have $u(x)=0$ and $r(x)$. Thus

$$\begin{array}{rcl} \tilde f(x) & = & H(x,u(x)) \\ & = & H(x,0) \\ & = & g\circ r(x) \\ & = & g(x) \end{array}$$

for $x\in A$ and so $\tilde f$ is a solution to the lifting problem. The proof is finished. □

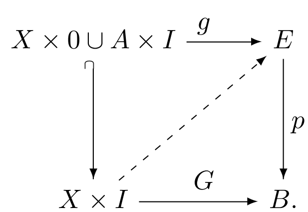
Note. The technical part in the proof is that the function Φ is continuous although the function

$$\theta(s,t)=\begin{cases} 1 & \text{if} \quad s=0 \\ 1-\frac{t}{s} & \text{if} \quad 0\le t<s \\ 0 & \text{if} \quad t\ge s \end{cases}$$

is NOT continuous on $I\times I$.

A *homotopy lifting problem for a pair of spaces* (X,A) can be symbolized by a commutative diagram

(7)



Lemma 6.34 has the following important consequence.

THEOREM 6.35. *Let $p\colon E\to B$ be a fibration and let (X,A) be a pair of spaces. Suppose that*

(1). *X is compactly generated.*
(2). *A is a closed subspace of X.*
(3). *the inclusion $A\to X$ is a fibration.*

Then every homotopy lifting problem

$$\begin{array}{ccc} X \times 0 \cup A \times I & \xrightarrow{g} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ X \times I & \xrightarrow{G} & B. \end{array}$$

has a solution.

Proof. By the assumptions, $(X \times I, X \times 0 \cup A \times I)$ is a DR-pair. The assertion follows from Lemma 6.34. □

Theorem 6.36. *Let $p\colon E \to B$ be a fibration. Suppose that $(P_p, *)$ is an NDR-pair, where $* = (x_0, \omega_0)$ is the basepoint of P_p. Then $P_p \simeq F$ as pointed spaces.*

The following proof is a modification of the proof of Theorem 6.5.7 in Maunder's book [**14**, Theorem 6.5.7]. The careful reader may find out that the assumption that $(P_p, *)$ is an NDR-pair is needed for having *pointed* homotopies constructed in Maunder's book.

Proof. Consider the sequence

$$P_p \xrightarrow{p_1} E \xrightarrow{p} B.$$

Define the map

$$j\colon F \longrightarrow P_p$$

by $j(x) = (x, \omega_0)$, where $\omega_0(t) = b_0$ is the constant path.

The composite $p \circ p_1\colon P_p \to B$ is null homotopic under the homotopy $G\colon P_p \times I \to B$ given by

$$G(x, \lambda, t) = \lambda(t).$$

Since

$$G(x_0, \omega_0, t) = \omega_0(t) = b_0$$

for $0 \leq t \leq 1$, G is a pointed homotopy. Since $(P_p, *)$ is an NDR-pair, $(P_p \times I, P_p \times 0 \cup * \times I)$ is a DR-pair. Thus there exists a pointed homotopy $H\colon P_p \times I \to E$ such that the diagram

$$\begin{array}{ccc} (P_p \times 0) \times ((x_0, \omega_0) \times I) & \xrightarrow{p_1} & E \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ P_p \times I & \xrightarrow{G} & B \end{array}$$

commutes.

Since $p(H_1(P_p)) = G_1(P_p) = b_0$, $H_1(P_p) \subseteq F$. Regard H_1 as a map into F by μ, that is $\mu\colon P_p \to F$ such that $i \circ \mu = H_1$, where $i\colon F \to E$ is the inclusion.

Consider the composite

$$F \times I \xrightarrow{j \times \mathrm{id}} P_p \times I \xrightarrow{H} E \xrightarrow{p} B.$$

For $(x,t) \in F \times I$,

$$p \circ H(j(x),t) = p \circ H(x,\omega_0,t) = G(x,\omega_0,t) = \omega_0(t) = b_0.$$

Thus the composite

$$F \times I \xrightarrow{j \times \mathrm{id}} P_p \times I \xrightarrow{H} E$$

maps into the subspace F with

$$H(g(x_0),t) = H(x_0,\omega_0,t) = x_0$$

for $0 \le t \le 1$, which gives a pointed homotopy between id_F and $\mu \circ j$.

Now define $J\colon P_p \times I \to PB$ by setting

$$J(x,\lambda,s)(t) = \lambda(s + t(1-s)).$$

This is continuous because under the association map it corresponds to a continuous map

$$(P_p \times I) \wedge I \longrightarrow B.$$

Now

$$(H,J)\colon P_p \times I \longrightarrow E \times PB$$

maps into P_p because

$$J(x,\lambda,s)(0) = \lambda(s) = G(x,\lambda,s) = p(H(x,\lambda,s)).$$

Thus it gives a homotopy $(H,J)\colon P_p \times I \to P_p$ with

$$(H,J)(x_0,\omega_0,s) = (H(x_0,\omega_0,s), J(x_0,\omega_0,s)) = (x_0,\omega_0).$$

Note that

$$(H,J)(x,\lambda,0) = (H(x,\lambda,0), J(x,\lambda,0)) = (x,\lambda) = \mathrm{id}_{P_f}(x,\lambda)$$

and

$$(H,J)(x,\lambda,1) = (H_1(x,\lambda), J(x,\lambda,1)) = (H_1(x,\lambda),\omega_0) = j \circ \mu(x,\lambda).$$

We obtain $j \circ \mu \simeq \mathrm{id}_{P_p}$ and hence the result. □

3.3. Serre Fibration. A map $p\colon E \to B$ is called *Serre fibration* if it has the homotopy lifting property with respect to the disks D^n for all $n \ge 0$. In other words, $p\colon E \to B$ is a Serre fibration if and only if every homotopy lifting problem

$$\begin{array}{ccc} D^n \times 0 & \xrightarrow{g} & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ D^n \times I & \xrightarrow{G} & B \end{array}$$

has a solution.

PROPOSITION 6.37 (Path-lifting Property). *Let $p\colon E \to B$ be a Serre fibration. Then for any point $x \in E$ and any path $\lambda\colon I \to B$ such that $\lambda(0) = p(x)$. Then there is a lifting path $\tilde{\lambda}\colon I \to E$ such that $\tilde{\lambda}(0) = x$ and $p \circ \tilde{\lambda} = \lambda$.*

PROOF. The assertion follows from a solution to the homotopy lifting problem

$$\begin{array}{ccc} 0 & \xrightarrow{\ g\ } & E \\ \big\downarrow & \overset{\tilde{\lambda}}{\nearrow} & \big\downarrow p \\ I & \xrightarrow[\ \lambda\ ]{} & B, \end{array}$$

where $g(0) = x$. □

Note. For covering spaces, the lifting path is unique. But for Serre fibrations, the lifting path may not be unique in general.

PROPOSITION 6.38. *If $p\colon E \to B$ is a Serre fibration, then it has the homotopy lifting property with respect to the pairs $(D^n, \partial D^n)$ for all $n \geq 0$.*

PROOF. The assertion follows from the fact that

$$(D^n \times I, D^n \times 0) \cong (D^n \times I, D^n \times 0 \cup \partial D^n \times I).$$

□

More generally, we have the following theorem for Serre fibrations.

THEOREM 6.39. *If $p\colon E \to B$ is a Serre fibration, then it has the homotopy lifting property with respect to any pairs (X, A) of CW-complexes.*

PROOF. Given a homotopy lifting problem

$$\begin{array}{ccc} X \times 0 \cup A \times I & \xrightarrow{\ g\ } & E \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ X \times I & \xrightarrow[\ G\ ]{} & B. \end{array}$$

We are going to construct a sequence of homotopies $H^n\colon (\mathrm{sk}_n(X) \cup A) \times I \to E$ such that each H^n is a solution of the homotopy lifting problem

$$\begin{array}{ccc} (\mathrm{sk}_n(X) \cup A) \times 0 \cup (\mathrm{sk}_{n-1}(X) \cup A) \times I & \xrightarrow{\ (g|_{\mathrm{sk}_n(X)\cup A}) \cup H^{n-1}\ } & E \\ \big\downarrow & \overset{H^n}{\nearrow} & \big\downarrow p \\ (\mathrm{sk}_n(X) \cup A) \times I & \xrightarrow[\ G|_{(\mathrm{sk}_n(X)\cup A)\times I}\ ]{} & B \end{array}$$

by induction on n. The assertion will follow from this statement by setting

$$H = \bigcup_n H^n\colon X \times I = \bigcup_n (\mathrm{sk}_n(X) \cup A) \times I \longrightarrow E.$$

When $n = 0$, then $\mathrm{sk}_0(X) \cup A$ is the union of A with a discrete set of points. For each $x \in \mathrm{sk}_0(X) \cup A \smallsetminus A$, there is a lifting path $\tilde{\lambda}\colon x \times I \to E$ such that $\tilde{\lambda}(x, 0) = g(x)$ and $p\tilde{\lambda}(x, t) = G(x, t)$. This defines a map

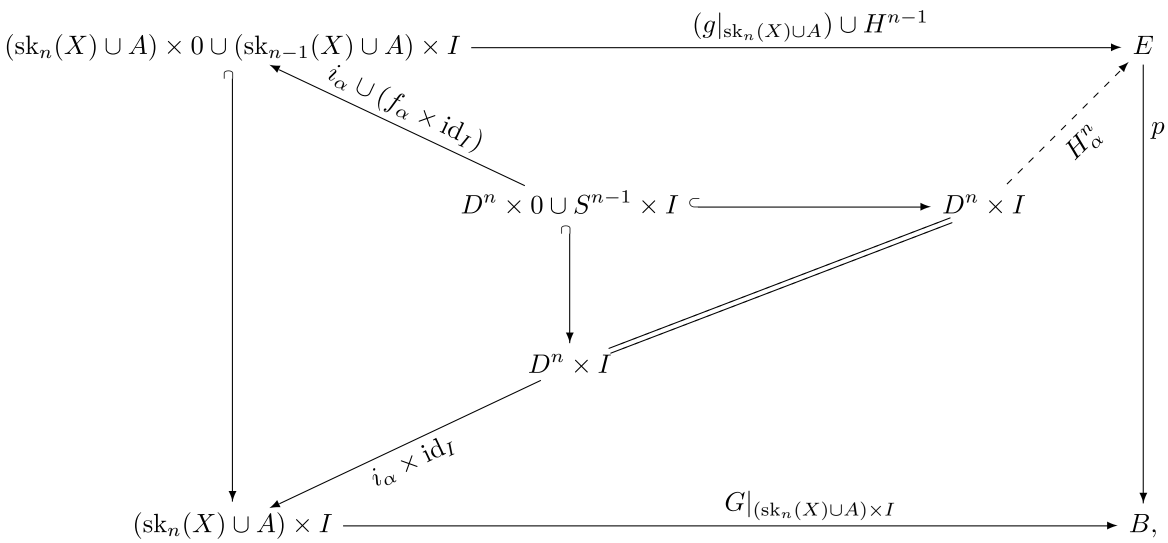
$$H^0\colon (\mathrm{sk}_0(X) \cup A) \times I \longrightarrow E$$

which is a solution of the above diagram in the case $n = 0$.

Suppose that H^{n-1} has constructed with the desired property. Let e^n_α be any n-cell such that

$$e^n \cap (\mathrm{sk}_{n-1}(X) \cup A) = \partial(e^n) \cap (\mathrm{sk}_{n-1}(X) \cup A),$$

that e^n_α is any n-cell that does not lie in $\mathrm{sk}_{n-1}(X) \cup A$. Let H^n_α be a solution of the following homotopy lifting problem



where

$$(i_\alpha, f_\alpha)\colon (D^n, S^{n-1}) \longrightarrow (\mathrm{sk}_n(X) \cup A, \mathrm{sk}_{n-1}(X) \cup A)$$

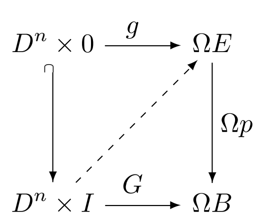
is the characteristic map of the n-cell e^n_α. The maps $\{H^n_\alpha\}$ induce a map

$$H^n\colon (\mathrm{sk}_n(X) \cup A) \times I \to E$$

with the desired property. The induction is finished and hence the result. $\square$

Proposition 6.40. *Let $p\colon E \to B$ be a Serre fibration with fibre F. Then $\Omega p\colon \Omega E \to \Omega B$ is a Serre fibration with fibre ΩF. Thus $\Omega^n p\colon \Omega^n E \to \Omega^n B$ is a Serre fibration with fibre $\Omega^n F$ for any $n \geq 0$.*

Proof. By taking the adjoint, the homotopy lifting problem



is equivalent to the homotopy lifting problem

$$\begin{array}{ccc} (D^n\times 0)\wedge S^1 & \xrightarrow{g'} & E \\ \downarrow & \nearrow & \downarrow p \\ (D^n\times I)\wedge S^1 & \xrightarrow{G'} & B, \end{array} \tag{8}$$

where g' and G' are the adjoint maps of g and G, respectively. Note that

$$(D^n\times I)\wedge S^1 = (D^n\times I\times I)/((*\times 0\times I)\cup(D^n\times I\times\partial I)).$$

Let $\bar{G}$ be the composite

$$\bar{G}\colon D^n\times I\times I \xrightarrow{\text{pinch}} (D^n\times I)\wedge S^1 \xrightarrow{G'} B,$$

that is $\bar{G}(x,s,t)=G'(x,s,t)$. Then $\bar{G}$ maps the subspace $(*\times 0\times I)\cup(D^n\times I\times\partial I)$ to the basepoint b_0 of B. Define the map

$$\bar{g}\colon (D^n\times 0\times I)\cup(D^n\times I\times\partial I)\longrightarrow E$$

by setting

$$\bar{g}(x)=\begin{cases} g'(x) & \text{if} \quad x\in D^n\times 0\times I,\\ e_0 & \text{if} \quad x\in D^n\times I\times\partial I,\end{cases}$$

where e_0 is the basepoint of E with $p(e_0)=b_0$. The map $\bar{g}$ is well-defined (and so continuous) because for

$$x\in(D^n\times 0\times I)\cap(D^n\times I\times\partial I)=D^n\times 0\times\partial I$$

we have $g'(x)=e_0$ as g' is pointed map from $(D^n\times 0)\wedge S^1$ to E. Let $\bar{H}$ be a solution to the homotopy lifting problem

$$\begin{array}{ccc} (D^n\times I\times 0)\cup(D^n\times\partial I)\times I\cong(D^n\times 0\times I)\cup(D^n\times I\times\partial I) & \xrightarrow{\bar{g}} & E \\ \downarrow & \overset{\bar{H}}{\nearrow} & \downarrow p \\ D^n\times I\times I\cong D^n\times I\times I & \xrightarrow{\bar{G}} & B, \end{array}$$

where the homeomorphisms in the left column are given by switching the last two coordinates. Then $\bar{H}$ maps the subspace

$$(*\times 0\times I)\cup(D^n\times I\times\partial I)$$

to the basepoint e_0 and so it induces a map

$$H'\colon (D^n\times I)\wedge S^1\longrightarrow E$$

which is a solution to the homotopy lifting problem in Diagram (8). This finishes the proof. $\square$

Given a Serre fibration $p\colon E\to B$ with fibre F. We are going to define a boundary map

$$\partial_{n+1}\colon \pi_{n+1}(B)\to\pi_n(F)$$

for each $n\geq 0$. Recall that $\pi_{n+1}(B)=\pi_1(\Omega^n B)$. Let $[\lambda]\in\pi_1(\Omega^n B)$. By Proposition 6.40, $\Omega^n p\colon \Omega^n E\to\Omega^n B$ is a Serre fibration. Thus there is a lifting path

$$\tilde{\lambda}\colon I\to E$$

such that $\tilde{\lambda}(0) = e_0$ and $p \circ \tilde{\lambda} = \lambda$. In particular.

$$p(\tilde{\lambda}(1)) = \lambda(1) = b_0$$

and so

$$\tilde{\lambda}(1) \in \Omega^n F.$$

Note $\pi_n(F) = \pi_0(\Omega^n F) = \Omega^n F/\simeq$ is the set of path-connected components of $\Omega^n F$. Define

$$\partial_{n+1}([\lambda]) = [\tilde{\lambda}(1)] \in \pi_0(\Omega^n F).$$

LEMMA 6.41. *Let $p\colon E \to B$ be a Serre fibration with fibre F. Then*

$$\partial_{n+1}\colon \pi_{n+1}(B) \to \pi_n(F)$$

is a well-defined function for each $n \geq 0$. Moreover ∂_{n+1} is natural in the following sense: If

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} \\ E' & \xrightarrow{p'} & B' \end{array}$$

is a commutative diagram with the property that p and p' are Serre fibrations with fibres F and F', respectively, then there is a commutative diagram

$$\begin{array}{ccc} \pi_{n+1}(B) & \xrightarrow{\partial^p_{n+1}} & \pi_n(F) \\ \downarrow{\scriptstyle \bar{f}_*} & & \downarrow{\scriptstyle f|_{F*}} \\ \pi_{n+1}(B') & \xrightarrow{\partial^{p'}_{n+1}} & \pi_n(F'). \end{array}$$

PROOF. The second statement follows immediately from the construction of the boundary ∂_{n+1} whence it is well-defined.

For proving the first statement, let $\lambda, \lambda'\colon S^1 \to \Omega^n B$ be loops such that $\lambda \simeq \lambda' \operatorname{rel} b_0$. Then there is a homotopy

$$G\colon I \times I \longrightarrow \Omega^n B$$

such that $G(s,0) = \lambda(s)$, $G(s,1) = \lambda'(s)$, $G(0,t) = G(1,t) = b_0$ for $0 \leq s,t \leq 1$. Let $\tilde{\lambda}, \tilde{\lambda}'\colon I \to \Omega^n E$ be lifting paths of λ and λ', respectively, starting from e_0. Let

$$g\colon (0 \times I) \cup (I \times \partial I) \longrightarrow \Omega^n E$$

be the map defined by $g(0,t) = e_0$, $g(s,0) = \tilde{\lambda}(s)$ and $g(s,1) = \tilde{\lambda}'(s)$. Let H be a solution to the homotopy lifting problem

$$\begin{array}{ccc} (0 \times I) \cup (I \times \partial I) & \xrightarrow{g} & \Omega^n E \\ \big\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle \Omega^n p} \\ I \times I & \xrightarrow{G} & \Omega^n B. \end{array}$$

Since
$$\Omega^n pH(1,t) = G(1,t) = b_0,$$
we have a path $t \mapsto H(1,t)$ in $\Omega^n F$ starting from $H(1,0) = \tilde{\lambda}(1)$ and ending with $H(1,1) = \tilde{\lambda}'(1)$. Thus
$$[\tilde{\lambda}(1)] = [\tilde{\lambda}'(1)]$$
in $\pi_0(\Omega^n F)$ and hence ∂_{n+1} is well-defined. □

PROPOSITION 6.42. *Let $p\colon E \to B$ be a Serre fibration. Then the boundary*
$$\partial_{n+1}\colon \pi_{n+1}(B) \to \pi_n(F)$$
is a group homomorphism for $n \geq 1$.

PROOF. Consider the commutative diagram
$$\begin{array}{ccccc}
\Omega^n F \times \Omega^n F & \hookrightarrow & \Omega^n E \times \Omega^n E & \xrightarrow{\Omega^n p \times \Omega^n p} & \Omega^n B \times \Omega^n B \\
\downarrow \mu & & \downarrow \mu & & \downarrow \mu \\
\Omega^n F & \hookrightarrow & \Omega^n E & \xrightarrow{p} & \Omega^n B,
\end{array}$$
where $\mu\colon \Omega^n Z \times \Omega^n Z \to \Omega^n Z$ is the multiplication on $\Omega^n Z = \mathrm{Map}_*(S^n, Z)$ induced by the comultiplication of S^n. Since $\Omega^n p\colon \Omega^n E \to \Omega^n B$ is a Serre fibration, so is
$$\Omega^n p \times \Omega^n p\colon \Omega^n E \times \Omega^n E \longrightarrow \Omega^n B \times \Omega^n B.$$

Let $[\lambda_i]$ be an element in $\pi_{n+1}(B) = \pi_1(\Omega^n B)$ for $i = 1, 2$. Let $\tilde{\lambda}_i\colon I \to \Omega^n E$ be a lifting path of λ_i starting from e_0. Then the path
$$t \mapsto (\tilde{\lambda}_1(t), \tilde{\lambda}_2(t))$$
in $\Omega^n E \times \Omega^n E$ is a lifting of the loop
$$(\lambda_1, \lambda_2)\colon S^1 \longrightarrow \Omega^n B \times \Omega^n B.$$
By the definition of the boundary, we have
$$\begin{array}{rcl}
\partial^{p\times p}_{n+1}([\lambda_1], [\lambda_2]) & = & [(\tilde{\lambda}_1(1), \tilde{\lambda}_2(1))] \\
& = & ([\tilde{\lambda}_1(1)], [\tilde{\lambda}_2(1)]) \\
& = & (\partial_{n+1}([\lambda_1]), \partial_{n+1}([\lambda_2]))
\end{array}$$
in $\pi_0(\Omega^n F \times \Omega^n F) = \pi_0(\Omega^n F) \times \pi_0(\Omega^n F)$. From the commutative diagram
$$\begin{array}{ccc}
\pi_{n+1}(B \times B) = \pi_1(\Omega^n B \times \Omega^n B) & \xrightarrow{\partial^{p\times p}_{n+1}} & \pi_0(\Omega^n F \times \Omega^n F) \\
\downarrow \mu_* & & \downarrow \mu_* \\
\pi_{n+1}(B) = \pi_1(\Omega^n B) & \xrightarrow{\partial_{n+1}} & \pi_0(\Omega^n F) = \pi_n(F),
\end{array}$$

we have

$$\begin{aligned}\partial_{n+1}([\lambda_1][\lambda_2]) &= \partial_{n+1}\circ\mu_*([\lambda_1],[\lambda_2])\\ &= \mu_*\circ\partial_{n+1}^{p\times p}([\lambda_1],[\lambda_2])\\ &= \mu_*(\partial_{n+1}([\lambda_1]),\partial_{n+1}([\lambda_2]))\\ &= \partial_{n+1}([\lambda_1])\partial_{n+1}([\lambda_2])\end{aligned}$$

and hence the result. □

THEOREM 6.43. *Suppose that $p\colon E\to B$ is a Serre fibration with fibre F. Then there is a long exact sequence*

$$\cdots \xrightarrow{\partial_{n+1}} \pi_n(F)\xrightarrow{i_*}\pi_n(E)\xrightarrow{p_*}\pi_n(B)\xrightarrow{\partial_n}\pi_{n-1}(F)\longrightarrow\cdots\longrightarrow\pi_0(F)\xrightarrow{i_*}\pi_0(E)\xrightarrow{p_*}\pi_0(B),$$

where $i\colon F\to E$ is the inclusion.

PROOF. (1). $\pi_n(F)\xrightarrow{i_*}\pi_n(E)\xrightarrow{p_*}\pi_n(B)$ is exact.

Since $p\colon E\to B$ is a Serre fibration, so is $\Omega^n p\colon \Omega^n E\to\Omega^n B$. Let $x\in\Omega^n E$ such that $\Omega^n p_*(x)$ is trivial in $\pi_0(\Omega^n B)$. Then there is a path λ in B starting from $\Omega^n p(x)$ and ending with the basepoint. Let $\tilde\lambda$ be a lifting path of λ starting from x. Then $\tilde\lambda(1)\Omega^n F$ because

$$\Omega^n p(\tilde\lambda(1))=\lambda(1)=*.$$

It follows that

$$i_*([\tilde\lambda(1)])=[x]$$

because x and $\tilde\lambda(1)$ are connected by the path $\tilde\lambda$ in $\Omega^n E$.

(2). $\pi_{n+1}(E)\xrightarrow{p_*}\pi_{n+1}(B)\xrightarrow{\partial_{n+1}}\pi_n(F)$ is exact.

Let $[\lambda]\in\pi_1(\Omega^n B)$ and let $\tilde\lambda\colon I\to\Omega^n E$ be a lifting path of λ with $\tilde\lambda(0)=*$. Suppose that

$$\partial_{n+1}([\lambda])=[\tilde\lambda(1)]$$

is trivial in $\pi_0(\Omega^n F)$. Then there is a path

$$\mu\colon I\to\Omega^n F$$

with $\mu(0)=\tilde\lambda(1)$ and $\mu(1)=*$. Define

$$\bar\lambda=\tilde\lambda*\mu$$

be the path product in $\Omega^n E$. Then $\bar\lambda(0)=\tilde\lambda(0)=*$ and $\bar\lambda(1)=\mu(1)=*$. Hence $\bar\lambda$ is a loop in $\Omega^n E$. Now

$$\begin{aligned}p_*([\bar\lambda]) &= p_*([\tilde\lambda]*[\mu])\\ &= [\Omega^n p(\tilde\lambda)]*[\Omega^n p(\mu)]\\ &= [\lambda]*[\Omega^n p(\mu)]\\ &= [\lambda]\end{aligned}$$

because $\Omega^n p(\mu)$ is the constant loop in $\Omega^n B$ as μ is a path in the fibre $\Omega^n F$. It follows that $[\lambda]\in\mathrm{Im}(p_*)$.

(3). $\pi_{n+1}(B)\xrightarrow{\partial_{n+1}}\pi_n(F)\xrightarrow{i_*}\pi_n(E)$ is exact.

Let $x\in\Omega^n F$ be a point such that $i_*([x])$ is trivial in $\pi_0(E)$. Then there is a path λ in $\Omega^n E$ from the basepoint $*$ to x. Then

$$\bar\lambda=\Omega^n p\circ\lambda$$

is a loop in $\Omega^n B$. Since λ is a lifting path of $\bar\lambda$, by the definition of ∂_{n+1}

$$\partial_{n+1}([\bar\lambda])=[\lambda(1)]=[x]$$

and so $[x] \in \operatorname{Im}(\partial_{n+1})$. The proof is finished now. $\square$

Let $p\colon E \to B$ be a Serre fibration. Then there is a commutative diagram

$$\begin{array}{ccccc} P_p & \hookrightarrow & \tilde{P}_p & \xrightarrow{q} & B \\ \uparrow{\scriptstyle j'} & & \uparrow{\scriptstyle j} & & \| \\ F & \hookrightarrow & E & \xrightarrow{p} & B, \end{array}$$

where j is a homotopy equivalence and q is a fibration.

Proposition 6.44. *Let $p\colon E \to B$ be a Serre fibration. Then $j'\colon F \to P_p$ is a weak homotopy equivalence, that is*

$$j'_*\colon \pi_n(F) \longrightarrow \pi_n(P_p)$$

is an isomorphism for each n.

Proof. Since $q\colon \tilde{P}_p \to B$ is a fibration, it is a Serre fibration. Thus there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & \pi_{n+1}(\tilde{P}_p) & \xrightarrow{q_*} & \pi_{n+1}(B) & \xrightarrow{\partial_{n+1}} & \pi_n(P_p) & \longrightarrow & \pi_n(\tilde{P}_p) & \xrightarrow{q_*} & \pi_n(B) & \xrightarrow{\partial_n} & \cdots \\ & & \cong\uparrow{\scriptstyle j_*} & & \| & & \uparrow{\scriptstyle j'_*} & & \cong\uparrow{\scriptstyle j_*} & & \| & & \\ \cdots & \longrightarrow & \pi_{n+1}(E) & \xrightarrow{p_*} & \pi_{n+1}(B) & \xrightarrow{\partial_{n+1}} & \pi_n(F) & \xrightarrow{i_*} & \pi_n(E) & \xrightarrow{p_*} & \pi_n(B) & \xrightarrow{\partial_n} & \cdots . \end{array}$$

Thus

$$j'_*\colon \pi_n(F) \longrightarrow \pi_n(P_p)$$

is an isomorphism for $n \geq 1$ by the Five Lemma.

We show that $j'_*\colon \pi_0(F) \to \pi_0(P_p)$ is one-to-one. Let $x_1, x_2 \in F$ such that $j'_*([x_1]) = j'_*([x_2])$, that is there is a path $\tilde{\lambda}\colon I \to P_p$ such that $\tilde{\lambda}(0) = x_1$ and $\tilde{\lambda}(1) = x_2$. Recall that

$$P_p = \{(x, \lambda) \in E \times PB \mid p(x) = \lambda(0)\}$$

with $p_1\colon P_p \to E$ given by $p_1(x, \lambda) = x$ and $p_2\colon P_p \to PB$ given by $p_2(x, \lambda) = \lambda$. Observe that

$$j'(x) = (x, \omega_0)$$

for $x \in F$, where $\omega_0(t) = b_0$ for $0 \leq t \leq 1$. Let

$$\lambda = p_1 \circ \tilde{\lambda}\colon I \to E.$$

Then $\lambda(0) = x_1$ and $\lambda(1) = x_2$. By taking the adjoint of the path $p_2 \circ \tilde{\lambda}\colon I \to PB$, we have the map

$$G\colon I \times I \to B$$

with $G(s,t) = p_2(\tilde{\lambda}(s))(t)$. Then the following conditions hold

(1). $G(s, 0) = p_2(\tilde{\lambda}(s))(0) = p(\lambda(s))$ for $0 \leq s \leq 1$.
(2). $G(0, t) = p_2(\tilde{\lambda}(0))(t) = p_2(x_1, \omega_0)(t) = b_0$ for $0 \leq t \leq 1$.
(3). $G(1, t) = p_2(\tilde{\lambda}(1))(t) = p_2(x_2, \omega_0)(t) = b_0$ for $0 \leq t \leq 1$.
(4). $G(s, 1) = p_2(\tilde{\lambda}(s))(1) = b_0$ for $0 \leq t \leq 1$.

Define $g\colon I\times 0\cup(\partial I\times I)\to E$ by setting $g(s,0)=\lambda(s)$, $g(0,t)=x_1$, $g(1,t)=x_2$. Let H be a solution to the homotopy lifting problem

$$\begin{array}{ccc} I\times 0\cup(\partial I\times I) & \xrightarrow{\ g\ } & E \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ I\times I & \xrightarrow{\ G\ } & B. \end{array}$$

Then $pH(s,1)=G(s,1)=b_0$ for $0\le s\le 1$ with $H(0,1)=g(0,1)=x_1$ and $H(1,1)=g(1,1)=x_2$. Thus $H(s,1)$ is a path in F from x_1 to x_2 and so $[x_1]=[x_2]$. This proves that j'_* is one-to-one.

Now we show that $j'_*\colon \pi_0(F)\to\pi_0(P_p)$ is onto. Let

$$(x,\lambda)\in P_p$$

be a point, where $x\in E$ and $\lambda\in PB$ with $\lambda(0)=p(x)$ and $\lambda(1)=b_0$. Since $p\colon E\to B$ is a Serre fibration, there is a lifting path $\tilde\lambda\colon I\to E$ such that $\tilde\lambda(0)=x$ and $p\circ\tilde\lambda=\lambda$. Then $y=\tilde\lambda(1)\in F$ because

$$p\tilde\lambda(1)=\lambda(1)=b_0.$$

Define $\theta\colon I\times I\to B$ by setting

$$\theta(s,t)=\lambda((1-t)s+t).$$

Then

(1). $\theta(s,1)=\lambda(1)=b_0$ and so $\theta(s,-)\in PB$.
(2). $\theta(s,0)=\lambda(s)=p(\tilde\lambda(s))$.
(3). $\theta(1,t)=\lambda(1)=b_0$ and so $\theta(1,-)=\omega_0$.
(4). $\theta(0,t)=\lambda(t)$ and so $\theta(0,-)\lambda$.

Now define a path $\bar\lambda\colon I\to P_p$ by

$$\bar\lambda(s)=(\tilde\lambda(s),\theta(s,-)).$$

Then $\bar\lambda(0)=(x,\lambda)$ and $\bar\lambda(1)=(y,\omega_0)=j'(y)$. Thus $j'(y)\simeq(x,\lambda)$. This proves that

$$j'_*\colon \pi_0(F)\to\pi_0(P_p)$$

is onto and hence the result. □

By Theorems 6.15 and 6.23, we have the following:

Proposition 6.45. *Suppose that $p\colon E\to B$ is a Serre fibration with fibre F. Then there is long exact sequence*

$$\cdots\xrightarrow{\Omega\partial_*}[X,\Omega F]\xrightarrow{i_*}[X,\Omega E]\xrightarrow{\Omega p_*}[X,\Omega B]\xrightarrow{\partial_*}[X,F]\xrightarrow{i_*}[X,E]\xrightarrow{p_*}[X,B]$$

for any pointed CW-complexes X. □

3.4. Fibre Sequences. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called a *fibre sequence* if there exists a a fibration $p\colon E \to B$ with fibre F with a homotopy commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\theta} \\ F & \xrightarrow{i} & E & \xrightarrow{p} & B \end{array}$$

such that the columns are (weak) homotopy equivalences.

Lemma 6.46 (Cohen-Moore-Neisendorfer Lemma). *A homotopy commutative diagram*

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ A_2 & \xrightarrow{f_2} & B_2 \end{array}$$

can be embedded into a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow & & \downarrow \\ G_1 & \longrightarrow & A_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow & & \downarrow \\ G_2 & \longrightarrow & A_2 & \longrightarrow & B_2 \end{array}$$

in which the rows and columns are fibre sequences.

Proof. First we replace g_2 and f_2 by fibrations as in the following diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{f_1} & B_1 & \underset{\simeq}{\overset{j_1}{\hookrightarrow}} & \tilde{P}_{g_2} \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & \swarrow{\scriptstyle \tilde{g}_2} & \\ A_2 & \xrightarrow{f_2} & B_2 & & \\ \simeq\ \downarrow{\scriptstyle j_2} & \nearrow{\scriptstyle \tilde{f}_2} & & & \\ \tilde{P}_{f_2}, & & & & \end{array}$$

where $\tilde{f}_2$ and $\tilde{g}_2$ are fibrations with homotopy equivalences j_1 and j_2.

Now let E be the pull-back in the following diagram

$$\begin{array}{ccc} E & \xrightarrow{f_2'} & \tilde{P}_{g_2} \\ {\scriptstyle g_2'}\downarrow & & \downarrow{\scriptstyle \tilde{g}_2} \\ \tilde{P}_{f_2} & \xrightarrow{\tilde{f}_2} & B_2, \end{array}$$

that is

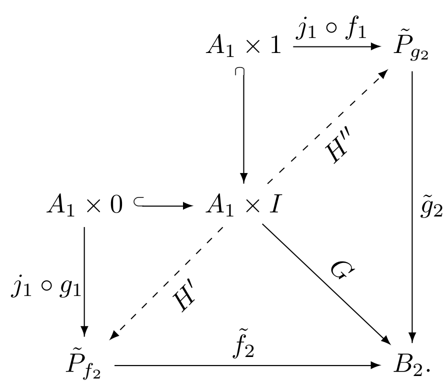
$$E = \{(x, y) \in \tilde{P}_{f_2} \times \tilde{P}_{g_2} \mid \tilde{f}_2(x) = \tilde{g}_2(y)\}.$$

Since $\tilde{f}_2$ and $\tilde{g}_2$ are fibrations, so are f_2' and g_2'.

Let

$$G\colon A_1 \times I \longrightarrow B_2$$

be a homotopy such that $G_0 = f_2 \circ g_1 = \tilde{f}_2 \circ (j_2 \circ g_1)$ and $G_1 = g_2 \circ f_1 = \tilde{g}_2 \circ (j_1 \circ f_1)$. Then there are solutions H' and H'' for the homotopy lifting problem:



Define the map $H\colon A_1 \times I \to E$ by setting

$$H(x, t) = (H'(x, t), H''(x, t)).$$

This map is well-defined because $\tilde{g}_2 \circ H'' = \tilde{f}_2 \circ H' = G$. The map H has a decomposition

$$A_1 \times I \underset{\simeq}{\xrightarrow{j_3}} \tilde{P}_H \xrightarrow{\tilde{H}} E$$

with $\tilde{H}$ a fibration. Now we obtain a commutative diagram of fibrations

$$\begin{array}{ccc} \tilde{P}_H & \xrightarrow{f_2' \circ \tilde{H}} & \tilde{P}_{g_2} \\ {\scriptstyle g_2' \circ \tilde{H}}\downarrow & & \downarrow{\scriptstyle \tilde{g}_2} \\ \tilde{P}_{f_2} & \xrightarrow{\tilde{f}_2} & B_2, \end{array}$$

where $f_2'\circ\tilde H_2$ and $g_2'\circ\tilde H$ are fibrations because they are the compositions of fibrations. Since the composites

$$A_1\times 1 \underset{\simeq}{\hookrightarrow} A_1\times I \underset{\simeq}{\overset{j_3}{\hookrightarrow}} \tilde P_H \xrightarrow{f_2'\circ\tilde H} \tilde P_{g_2}$$

$$A_1\times 0 \underset{\simeq}{\hookrightarrow} A_1\times I \underset{\simeq}{\overset{j_3}{\hookrightarrow}} \tilde P_H \xrightarrow{g_2'\circ\tilde H} \tilde P_{f_2}$$

are homotopic to $j_1\circ f_1$ and $j_2\circ g_1$, respectively with homotopy equivalences j_1 and j_2. The above diagram is a replacement of the homotopy commutative diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{f_1} & B_1 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ A_2 & \xrightarrow{f_2} & B_2. \end{array}$$

Now we obtain a commutative diagram

$$\begin{array}{ccccc} F & \hookrightarrow & F_1 & \longrightarrow & F_2 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i_1} \\ G_1 & \hookrightarrow & \tilde P_H & \xrightarrow{f_2'\circ\tilde H} & \tilde P_{g_2} \\ \downarrow & & \downarrow{\scriptstyle g_2'\circ\tilde H} & & \downarrow{\scriptstyle \tilde g_2} \\ G_2 & \overset{i_2}{\hookrightarrow} & \tilde P_{f_2} & \xrightarrow{\tilde f_2} & B_2 \end{array}$$

of fibrations, where the top row is a fibration because it is the induced fibration via the inclusion $i_1\colon F_2\to\tilde P_{g_2}$ and the left column is a fibration as it is induced by $i_2\colon G_2\hookrightarrow\tilde P_{f_2}$. The proof is finished. □

3.5. Fibre Bundles. A *bundle* means a triple (E,p,B), where $p\colon E\to B$ is a (continuous) map. The space B is called the *base space*, the space E is called the *total space*, and the map p is called the *projection* of the bundle. For each $b\in B$, $p^{-1}(b)$ is called the *fibre* of the bundle over $b\in B$.

Intuitively, a bundle can be thought as a union of fibres $f^{-1}(b)$ for $b\in B$ parameterized by B and *glued together* by the topology of the space E. Usually a Greek letter (ξ,η,ζ,λ, etc) is used to denote a bundle; then $E(\xi)$ denotes the total space of ξ, and $B(\xi)$ denotes the base space of ξ.

A *morphism* of bundles $(\phi, \bar{\phi})\colon \xi \to \xi'$ is a pair of (continuous) maps $\phi\colon E(\xi) \to E(\xi')$ and $\bar{\phi}\colon B(\xi) \to B(\xi')$ such that the diagram

$$\begin{array}{ccc} E(\xi) & \xrightarrow{\phi} & E(\xi') \\ \downarrow{\scriptstyle p(\xi)} & & \downarrow{\scriptstyle p(\xi')} \\ B(\xi) & \xrightarrow{\bar{\phi}} & B(\xi') \end{array}$$

commutes.

The trivial bundle is the projection of the Cartesian product:

$$p\colon B \times F \to B, \qquad (x, y) \mapsto x.$$

Roughly speaking, a *fibre bundle* $p\colon E \to B$ is a "locally trivial" bundle with a "fixed fibre" F. More precisely, for any $x \in B$, there exists an open neighborhood U of x such that $p^{-1}(U)$ is a trivial bundle, in other words, there is a homeomorphism $\phi_U\colon p^{-1}(U) \to U \times F$ such that the diagram

$$\begin{array}{ccc} U \times F & \xrightarrow{\phi_x} & p^{-1}(U) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle p} \\ U & = & U \end{array}$$

commutes, that is, $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$.

Similar to manifolds, we can use "chart" to describe fibre bundles. A *chart* (U, ϕ) for a bundle $p\colon E \to B$ is (1) an open set U of B and (2) a homeomorphism $\phi\colon U \times F \to p^{-1}(U)$ such that $p(\phi(x', y)) = x'$ for any $x' \in U$ and $y \in F$. An *atlas* is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ is an open covering of B.

Proposition 6.47. *A bundle $p\colon E \to B$ is a fibre bundle with fibre F if and only if it has an atlas.*

Proof. Suppose that $p\colon E \to B$ is a fibre bundle. Then the collection $\{(U(x), \phi_x) | x \in B\}$ is an atlas.

Conversely suppose that $p\colon E \to B$ has an atlas. For any $x \in B$ there exists α such that $x \in U_\alpha$ and so U_α is an open neighborhood of x with the property that $p|_{p^{-1}(U)}\colon p^{-1}(U_\alpha) \to U_\alpha$ is a trivial bundle. Thus $p\colon E \to B$ is a fibre bundle. □

Theorem 6.48. *Let $p\colon E \to B$ be a fibre bundle. Then p is a Serre fibration.*

A theorem of Huebsch and Hurewicz proved in Spanier's book [**21**, Section 2.7] says that fibre bundles over paracompact base spaces are fibrations.

Proof. We show that $p\colon E \to B$ has the homotopy lifting property for disks, or equivalently, cubes. Given a homotopy lifting problem

$$\begin{array}{ccc} I^n \times 0 & \xrightarrow{\ g\ } & E \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow p \\ I^n \times I & \xrightarrow{\ G\ } & B. \end{array}$$

Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas. Since $\{U_\alpha\}$ is an open cover of B, $\{G^{-1}(U_\alpha)\}$ is an open cover of $I^n \times I$, that is

$$\bigcup_{\alpha} G^{-1}(U_\alpha) = I^n \times I.$$

Since $I^n \times I$ is compact, there exist finite open sets $G^{-1}(U_{\alpha_1}), \ldots G^{-1}(U_{\alpha_K})$ such that

$$\bigcup_{k=1}^{K} G^{-1}(U_{\alpha_k}) = I^n \times I.$$

Subdivide I^n into small cubes C and I into intervals $I_j = [t_j, t_{j+1}]$, $0 \le j \le m$, so that each product $C \times I_j$ is mapped by G into U_{α_k} for some $1 \le k \le K$. Let the small cubes C be ordered by $C_1, C_2, \ldots, C_T$. Define an order on $\{C_i \times I_j\}$ by the dictionary order:

$$C_1 \times I_0, C_2 \times I_0, \ldots, C_T \times I_0, C_1 \times I_1, C_2 \times I_1, \ldots, C_T \times I_1, \ldots, C_1 \times I_m, C_2 \times I_m, \ldots, C_T \times I_m.$$

Let $G(C_i \times I_j) \subseteq U_{\alpha_{k_{i,j}}}$ for some $1 \le k_{i,j} \le K$. The homotopy lifting problem can be solved by constructing $H|_{C_i \times I_j}$ with desired properties by induction. In the first case, from the commutative diagram

$$\begin{array}{ccccc} C_1 \times 0 & \xrightarrow{\ g|_{C_1}\ } & p^{-1}(U_{\alpha_{k_{1,0}}}) \subseteq & & E \\ \big\downarrow & \overset{H|_{C_1 \times I_0}}{\nearrow} & \big\downarrow p & & \big\downarrow p \\ C_1 \times I_0 & \xrightarrow{\ G|_{C_1 \times I_0}\ } & U_{\alpha_{k_{1,0}}} & \subseteq & B, \end{array}$$

the lifting $H|_{C_1 \times I_0}$ exists because $p\colon p^{-1}(U_{\alpha_{k_{1,0}}}) \to U_{\alpha_{k_{1,0}}}$ is the trivial bundle. Suppose that $H_{C_{i'} \times I_{j'}}$ are constructed for all $C_{i'} \times C_{j'}$ preceding to $C_i \times I_j$. Then there is a subcomplex A of ∂C_i such that the map H is already defined on $C_i \times t_j$ and $A \times I_j$. Since

$$p^{-1}(U_{\alpha_{k_{i,j}}}) \to U_{\alpha_{k_{i,j}}}$$

is the trivial bundle, there is a solution $H|_{C_i \times I_j}$ to the homotopy lifting problem

$$\begin{array}{ccccc} (C_i \times t_j) \cup (A \times I_j) & \longrightarrow & p^{-1}(U_{\alpha_{k_{i,j}}}) \subseteq & & E \\ \big\downarrow & \overset{H|_{C_i \times I_j}}{\nearrow} & \big\downarrow p & & \big\downarrow p \\ C_i \times I_j & \xrightarrow{\ G|_{C_i \times I_j}\ } & U_{\alpha_{k_{i,j}}} & \subseteq & B. \end{array}$$

The induction is finished and hence the result. □

COROLLARY 6.49. *Suppose that $p\colon E\to B$ is a Serre fibration with fibre F. Then there is a long exact sequence*

$$\cdots \xrightarrow{\partial_{n+1}} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial_n} \pi_{n-1}(F) \longrightarrow \cdots \longrightarrow \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B),$$

where $i\colon F\to E$ is the inclusion. □

3.5.1. *Further Properties of Fibre Bundles.* Let ξ be a fibre bundle with fibre F and an atlas $\{(U_\alpha,\phi_\alpha)\}$. The composite

$$\phi_\alpha^{-1}\circ\phi_\beta\colon (U_\alpha\cap U_\beta)\times F \xrightarrow{\phi_\beta} p^{-1}(U_\alpha\cap U_\beta) \xrightarrow{\phi_\alpha^{-1}} (U_\alpha\cap U_\beta)\times F$$

has the property that

$$\phi_\alpha^{-1}\circ\phi_\beta(x,y)=(x,g_{\alpha\beta}(x,y))$$

for any $x\in U_\alpha\cap U_\beta$ and $y\in F$. Consider the continuous map $g_{\alpha\beta}\colon U_{\alpha\beta}\times F\to F$. Fixing any x, $g_{\alpha\beta}(x,-)\colon F\to F$, $y\mapsto g_{\alpha\beta}(x,y)$ is a homeomorphism with inverse given by $g_{\beta\alpha}(x,-)$. This gives a *transition function*

$$g_{\alpha\beta}\colon U_\alpha\cap U_\beta \longrightarrow \mathrm{Homeo}(F,F),$$

where $\mathrm{Homeo}(F,F)$ is the group of all homeomorphisms from F to F.

EXERCISE 3.1. *Prove that the transition functions $\{g_{\alpha\beta}\}$ satisfy the following equation*

$$g_{\alpha\beta}(x)\circ g_{\beta\gamma}(x)=g_{\alpha\gamma}(x) \qquad x\in U_\alpha\cap U_\beta\cap U_\gamma. \tag{9}$$

By choosing $\alpha=\beta=\gamma$, $g_{\alpha\alpha}(x)\circ g_{\alpha\alpha}(x)=g_{\alpha\alpha}(x)$ and so

$$g_{\alpha\alpha}(x)=x \qquad x\in U_\alpha \tag{10}$$

.

By choosing $\alpha=\gamma$, $g_{\alpha\beta}(x)\circ g_{\beta\alpha}(x)=g_{\alpha\alpha}(x)=x$ and so

$$g_{\beta\alpha}(x)=g_{\alpha\beta}(x)^{-1} \qquad x\in U_\alpha\cap U_\beta. \tag{11}$$

We need to introduce a *topology* on $\mathrm{Homeo}(F,F)$ such that the transition functions $g_{\alpha\beta}$ are continuous. The topology on $\mathrm{Homeo}(F,F)$ is given by *compact-open topology* briefly reviewed as follows:

Let X and Y be topological spaces. Let $\mathrm{Map}(X,Y)$ denote the set of all continuous maps from X to Y. Given any compact set K of X and any open set U of Y, let

$$W_{K,U}=\{f\in\mathrm{Map}(X,Y)\mid f(K)\subseteq U\}.$$

Then the *compact-open topology* is generated by $W_{K,U}$, that is, an open set in $\mathrm{Map}(X,Y)$ is an arbitrary union of a finite intersection of subsets with the form $W_{K,U}$.

$\mathrm{Map}(F,F)$ be the set of all continuous maps from F to F with compact open topology. Then $\mathrm{Homeo}(F,F)$ is a subset of $\mathrm{Map}(F,F)$ with subspace topology.

PROPOSITION 6.50. *If* $\mathrm{Homeo}(F,F)$ *has the compact-open topology, then the transition functions* $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\to\mathrm{Homeo}(F,F)$ *are continuous.*

Proof. Given $W_{K,U}$, we show that $g_{\alpha\beta}^{-1}(W_{K,U})$ is open in $U_\alpha \cap U_\beta$. Let $x_0 \in U_\alpha \cap U_\beta$ such that $g_{\alpha\beta}(x_0) \in W_{(K,U)}$. We need to show that there is a neighborhood V is x_0 such that $g_{\alpha\beta}(V) \subseteq W_{K,U}$, or $g_{\alpha\beta}(V \times K) \subseteq U$. Since U is open and $g_{\alpha\beta}\colon (U_\alpha \cap U_\beta) \times F \to F$ is continuous, $g^{-1}(U)$ is an open set of $(U_\alpha \cap U_\beta) \times F$ with $x_0 \times K \subseteq g_{\alpha\beta}^{-1}(U)$. For each $y \in K$, there exist open neighborhoods $V(y)$ of x and $N(y)$ of y such that $V(x) \times N(y) \subseteq g_{\alpha\beta}^{-1}(U)$. Since $\{N(y) \mid y \in K\}$ is an open cover of the compact set K, there is a finite cover $\{N(y_1), \ldots, N(y_n)\}$ of K. Let $V = \bigcap_{j=1}^{n} V(y_j)$. Then $V \times K \subseteq g_{\alpha\beta}^{-1}(U)$ and so $g_{\alpha\beta}(V) \subseteq W_{K,U}$. □

Proposition 6.51. *If F regular and locally compact, then the composition and evaluation maps*

$$\mathrm{Homeo}(F,F) \times \mathrm{Homeo}(F,F) \longrightarrow \mathrm{Homeo}(F,F) \qquad (g,f) \mapsto f \circ g$$

$$\mathrm{Homeo}(F,F) \times F \longrightarrow F \qquad (f,y) \mapsto f(y)$$

are continuous.

Proof. Suppose that $f \circ g \in W_{K,U}$. Then $f(g(K)) \subseteq U$, or $g(K) \subseteq f^{-1}(U)$, and the latter is open. Since F is regular and locally compact, there is an open set V such that

$$g(K) \subseteq V \subseteq \bar{V} \subseteq f^{-1}(U)$$

and the closure $\bar{V}$ is compact. If $g' \in W_{K,V}$ and $f' \in W_{\bar{V},U}$, then $f' \circ g' \in W_{K,U}$. Thus $W_{K,V}$ and $W_{\bar{V},U}$ are neighborhoods of g and f whose composition product lies in $W_{K,U}$. This implies that $\mathrm{Homeo}(F,F) \times \mathrm{Homeo}(F,F) \to \mathrm{Homeo}(F,F)$ is continuous.

Let U be an open set of F and let $f_0(y_0) \in U$ or $y_0 \in f_0^{-1}(U)$. Since F is regular an locally compact, there is a neighborhood V of y_0 such that $\bar{V}$ is compact and $y_0 \in V \subseteq \bar{V} \subseteq g_0^{-1}(U)$. If $g \in W_{\bar{V},U}$ and $y \in V$, then $g(y) \in U$ and so the evaluation map $\mathrm{Homeo}(F,F) \times F \to F$ is continuous. □

Proposition 6.52. *If F is compact Hausdorff, then the inverse map*

$$\mathrm{Homeo}(F,F) \longrightarrow \mathrm{Homeo}(F,F) \qquad f \mapsto f^{-1}$$

is continuous.

Proof. Suppose that $g_0^{-1} \in W_{K,U}$. Then $g_0^{-1}(K) \subseteq U$ or $K \subseteq g_0(U)$. It follows that

$$F \smallsetminus K \supseteq F \smallsetminus g_0(U) = g_0(F \smallsetminus U)$$

because g_0 is a homeomorphism. Note that $F \smallsetminus U$ is compact, $F \smallsetminus K$ is open and $g_0 \in W_{F\smallsetminus U, F\smallsetminus K}$. If $g \in W_{F\smallsetminus U, F\smallsetminus K}$, then, from the above arguments, $g^{-1} \in W_{K,U}$ and hence the result. □

Note. If F is regular and locally compact, then $\mathrm{Homeo}(F,F)$ is a topological monoid, namely compact-open topology only fails in the continuity of g^{-1}. A modification on compact-open topology eliminates this defect [**2**].

Let G be a topological group and let X be a space. A right G-action on X means a(continuous) map $\mu\colon X \times G \to X, (x,g) \mapsto x \cdot g$ such that $x \cdot 1 = x$ and $(x \cdot g) \cdot h = x \cdot (gh)$. In this case, we call X a (right) G-space. Let X and Y be (right) G-spaces. A continuous map $f\colon X \to Y$ is called a G-map if $f(x \cdot g) = f(x) \cdot g$ for any $x \in X$ and $g \in G$. Let X/G be the set of G-orbits xG, $x \in X$, with quotient topology.

Proposition 6.53. *Let X be a G-space.*

1) *For fixing any $g \in G$, the map $x \mapsto x \cdot g$ is a homeomorphism.*
2) *The projection $\pi\colon X \to X/G$ is an open map.*

PROOF. (1). The inverse is given by $x \mapsto x \cdot g^{-1}$.

(2) If U is an open set of X,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} U \cdot g$$

is open because it is a union of open sets, and so $\pi(U)$ is open by quotient topology. Thus π is an open map. □

We are going to find some conditions such that $\pi\colon X \to X/G$ has canonical fibre bundle structure with fibre G. Given any point $\bar{x} \in X/G$, choose $x \in X$ such that $\pi(x) = \bar{x}$. Then

$$\pi^{-1}(\bar{x}) = \{x \cdot g \mid g \in G\} = G/H_x,$$

where $H_x = \{g \in G \mid x \cdot g = x\}$.

For having constant fibre G, we need to assume that the G-action on X is free, namely

$$x \cdot g = x \quad \Longrightarrow \quad g = 1$$

for any $x \in X$. This is equivalent to the property that

$$x \cdot g = x \cdot h \quad \Longrightarrow \quad g = h$$

for any $x \in X$. In this case we call X a *free G-space*.

Since a fibre bundle is locally trivial (locally Cartesian product), there is always a local cross-section from the base space to the total space. Our second condition is that the projection $\pi\colon X \to X/G$ has local cross-sections. More precisely, for any $\bar{x} \in X/G$, there is an open neighborhood $U(\bar{x})$ with a continuous map $s_{\bar{x}}\colon U(\bar{x}) \to X$ such that $\pi \circ s_{\bar{x}} = \mathrm{id}_{U(\bar{x})}$.

(**Note.** For every point $\bar{x}$, we can always choose a pre-image of π, the local cross-section means the pre-images can be chosen "continuously" in a neighborhood. This property depends on the topology structure of X and X/G.)

Assume that X is a (right) free G-space with local cross-sections to $\pi\colon X \to X/G$. Let $\bar{x}$ be any point in X/G. Let $U(\bar{x})$ be a neighborhood of $\bar{x}$ with a (continuous) crosse-section $s_{\bar{x}}\colon U(\bar{x}) \to X$. Define

$$\phi_{\bar{x}}\colon U(\bar{x}) \times G \longrightarrow \pi^{-1}(U(\bar{x})) \qquad (\bar{y}, g) \longrightarrow s_{\bar{x}}(\bar{y}) \cdot g$$

for any $y \in U(\bar{x})$.

EXERCISE 3.2. Let X be a (right) free G-space with local cross-sections to $\pi\colon X \to X/G$. Then the continuous map $\phi_{\bar{x}}\colon U(\bar{x}) \times G \to \pi^{-1}(U(\bar{x}))$ is one-to-one and onto. □

We need to find the third condition such that $\phi_{\bar{x}}$ is a homeomorphism. Let

$$X^* = \{(x, x \cdot g) \mid x \in X, g \in G\} \subseteq X \times X.$$

A function

$$\tau\colon X^* \longrightarrow G$$

such that

$$x \cdot \tau(x, x') = x' \qquad \text{for all } (x, x') \in X$$

is called a *translation function*. (**Note.** If X is a free G-space, then translation function is unique because, for any $(x, x') \in X^*$, there is a unique $g \in G$ such that $x' = x \cdot g$, and so, by definition, $\tau(x, x') = g$.)

Proposition 6.54. *Let X be a (right) free G-space with local cross-sections to $\pi\colon X \to X/G$. Then the following statements are equivalent each other:*

1) *The translation function $\tau\colon X^* \to G$ is continuous.*
2) *For any $\bar{x} \in X/G$, the map $\phi_{\bar{x}}\colon U(\bar{x}) \times G \to \pi^{-1}(U(\bar{x}))$ is a homeomorphism.*
3) *There is an atlas $\{(U_\alpha, \phi_\alpha\}$ of X/G such that the homeomorphisms*

$$\phi_\alpha\colon U_\alpha \times G \longrightarrow \pi^{-1}(U_\alpha)$$

satisfy the condition $\phi_\alpha(\bar{y}, gh) = \phi_\alpha(\bar{y}, g) \cdot h$, that is ϕ_α is a homeomorphism of G-spaces.

Proof. (1) $\Longrightarrow$ (2). Consider the (continuous) map

$$\theta\colon \pi^{-1}(U(\bar{x})) \longrightarrow U(\bar{x}) \times G \qquad z \mapsto (\pi(z), \tau(s_{\bar{x}}(\pi(z)), z)).$$

Then

$$\theta \circ \phi_{\bar{x}}(\bar{y}, g) = \theta(s_{\bar{x}}(\bar{y}) \cdot g) = (\bar{y}, \tau(s_{\bar{x}}(\bar{y}), s_{\bar{x}}(\bar{y}) \cdot g)) = (\bar{y}, g),$$

$$\phi_{\bar{x}} \circ \theta(z) = \phi_{\bar{x}}(\pi(z), \tau(s_{\bar{x}}(\pi(z)), z)) = s_{\bar{x}}(\pi(z)) \cdot \tau(s_{\bar{x}}(\pi(z)), z) = z.$$

Thus $\phi_{\bar{x}}$ is a homeomorphism.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Note that the translation function is unique for free G-spaces. It suffices to show that the restriction

$$\tau(X)\colon X^* \cap (\pi^{-1}(U_\alpha) \times \pi^{-1}(U_\alpha)) = \left(\pi^{-1}(U_\alpha)\right)^* \longrightarrow G$$

is continuous. Consider the commutative diagram

$$\begin{array}{ccc}
(U_\alpha \times G)^* & \xrightarrow[\cong]{\phi_\alpha^*} & \left(\pi^{-1}(U_\alpha)\right)^* \\
\Big\downarrow{\scriptstyle \tau(U_\alpha \times G)} & & \Big\downarrow{\scriptstyle \tau(X)} \\
G & = \!= \!= & G.
\end{array}$$

Since

$$\tau(U_\alpha \times G)((\bar{y}, g), (\bar{y}, h)) = g^{-1}h$$

is continuous, the translation function restricted to $\left(\pi^{-1}(U_\alpha)\right)^*$

$$\tau(X) = \tau(U_\alpha \times G) \circ ((\phi_\alpha)^*)^{-1}$$

is continuous for each α and so $\tau(X)$ is continuous. □

Now we give the definition. A *principal G-bundle* is a free G-space X such that

$$\pi\colon X \to X/G$$

has local cross-sections and one of the (equivalent) conditions in the above Proposition holds.

Example. Let Γ be a topological group and let G be a closed subgroup. Then the action of G on Γ given by $(a, g) \mapsto ag$ for $a \in \Gamma$ and $g \in G$ is free. Then translation function is given by $\tau(a, b) = a^{-1}b$, which is continuous. Thus $\Gamma \to \Gamma/G$ is principal G-bundle if and only if it has local cross-sections.

3.5.2. *The Associated Principal G-Bundles of Fibre Bundles.* We come back to look at fibre bundles ξ given by $p\colon E \to B$ with fibre F. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas and let

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{Homeo}(F,F)$$

be the transition functions. A topological group G is called a *group of the bundle* ξ if

1) There is a group homomorphism
$$\theta\colon G \longrightarrow \mathrm{Homeo}(F,F).$$
2) There exists an atlas of ξ such that the transition functions $g_{\alpha\beta}$ lift to G via θ, that is, there is commutative diagram
$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & \mathrm{Homeo}(F,F) \\ \| & & \uparrow \theta \\ U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & G, \end{array}$$
(where we use the same notation $g_{\alpha\beta}$.)
3) The transition functions
$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$$
are continuous.
4) The G-action on F via θ is continuous, that is, the composite
$$G \times F \xrightarrow{\theta\times \mathrm{id}_F} \mathrm{Homeo}(F,F)\times F \xrightarrow{\text{evaluation}} F$$
is continuous.

We write $\bar{\xi} = \{(U_\alpha, g_{\alpha\beta})\}$ for the set of transition functions to the atlas $\{(U_\alpha, \phi_\alpha)\}$.

Note. In Steenrod's definition [**24**, p.7], θ is assume to be a monomorphism (equivalently, the G-action on F is effective, that is, if $y \cdot g = y$ for all $y \in F$, then $g = 1$.).

We are going to construct a principal G-bundle $\pi\colon E^G \to B$. Then prove that the total space $E = F \times_G E^G$ and $p\colon E \to B$ can be obtained canonically from $\pi\colon E^G \to B$. In other words, all fibre bundles can obtained through principal G-bundles through this way. Also the topological group G plays an important role for fibre bundles. Namely, by choosing different topological groups G, we may get different properties for the fibre bundle ξ. For instance, if we can choose G to be trivial (that is, $g_{\alpha\beta}$ lifts to the trivial group), then fibre bundle is trivial. We will see that the bundle group G for n-dimensional vector bundles can be chosen as the general linear group $\mathrm{GL}_n(\mathbb{R})$. The vector bundle is *oriented* if and only if the transition functions can left to the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of $n \times n$ matrices whose determinant is positive. If $n = 2m$, then $\mathrm{GL}_m(\mathbb{C}) \subseteq \mathrm{GL}_{2m}(\mathbb{R})$. The vector bundle admits (almost) complex structure if and only if the transition functions can left to $\mathrm{GL}_m(\mathbb{C})$. (For manifolds, one can consider the structure on the tangent bundles. For instance, an oriented manifold means its tangent bundle is oriented.)

Proposition 6.55. *If $\bar{\xi}$ is the set of transition functions for the space B and topological group G, then there is a principal G-bundle ξ^G given by*

$$\pi\colon E^G \longrightarrow B$$

and an atlas $\{(U_\alpha, \phi_\alpha)\}$ such that $\bar{\xi}$ is the set of transition functions to this atlas.

Proof. The proof is given by construction. Let

$$\bar{E} = \bigcup_\alpha U_\alpha \times G \times \alpha,$$

that is $\bar{E}$ is the disjoint union of $U_\alpha \times G$. Now define a relation on $\bar{E}$ by

$$(b, g, \alpha) \sim (b', g', \beta) \Longleftrightarrow b = b', g = g_{\alpha\beta}(b)g'.$$

This is an equivalence relation by Equations (9)-(11). Let $E^G = \bar{E}/\sim$ with quotient topology and let $\{b, g, \alpha\}$ for the class of (b, g, α) in E^G. Define $\pi\colon E^G \to B$ by

$$\pi\{b, g, \alpha\} = b,$$

then π is clearly well-defined (and so continuous). The right G-action on E^G is defined by

$$\{b, g, \alpha\} \cdot h = \{b, gh, \alpha\}.$$

This is well-defined (and so continuous) because if $(b', g', \beta) \sim (b, g, \alpha)$, then

$$(b', g'h, \beta) = (b, (g_{\alpha\beta}(b)g)h, \beta) = (b, g_{\alpha\beta}(b)(gh), \beta) \sim (b, gh, \alpha).$$

Define $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$ by setting

$$\phi_\alpha(b, g) = \{b, g, \alpha\},$$

then ϕ_α is continuous and satisfies $\pi \circ \phi_\alpha(b, g) = b$ and

$$\phi_\alpha(b, g) = \{b, 1 \cdot g, \alpha\} = \{b, 1, \alpha\} \cdot g$$

for $b \in U_\alpha$ and $g \in G$. The map ϕ_α is a homeomorphism because, for fixing α, the map

$$\coprod_\beta (U_\alpha \cap U_\beta) \times G \times \beta \longrightarrow U_\alpha \times G \qquad (b, g', \beta) \mapsto (b, g_{\alpha\beta}(b)g')$$

induces a map $\pi^{-1}(U_\alpha) \to U_\alpha \times G$ which the inverse of ϕ_α. Moreover,

$$\phi_\alpha(b, g_{\alpha\beta}(b)g) = \{b, g_{\alpha\beta}(b)g, \alpha\} = \{b, g, \beta\} = \theta_\beta(b, g)$$

for $b \in U_\alpha \cap U_\beta$ and $g \in G$. Thus the $\{(U_\alpha, g_{\alpha\beta})\}$ is the set of transition function to the atlas $\{(U_\alpha, \phi_\alpha)\}$. □

Let X be a right G-space and let Y be a left G-space. The product over G is defined by

$$X \times_G Y = X \times Y/(xg, y) \sim (x, gy)$$

with quotient topology. Note that the composite

$$\begin{array}{ccccc} X \times Y & \xrightarrow{\pi_X} & X & \xrightarrow{\pi} & X/G \\ (x, y) & \mapsto & x & \mapsto & \bar{x} \end{array}$$

factors through $X \times_G Y$. Let $p\colon X \times_G Y \to X/G$ be the resulting map. For any $\bar{x} \in X/G$, choose $x \in \pi^{-1}(\bar{x}) \subseteq X$, then

$$p^{-1}(\bar{x}) = \pi^{-1}(\bar{x}) \times_G Y = x \times Y/H_x,$$

where $H_x = \{g \in G \mid xg = x\}$. Thus if X is a free right G-space, then the projection $p\colon X \times_G Y \to X/G$ has the constant fibre Y.

PROPOSITION 6.56. *Let $\pi\colon X \to X/G$ be a (right) principal G-bundle and let Y be any left G-space. Then*

$$p\colon X \times_G Y \longrightarrow X/G$$

is a fibre bundle with fibre Y.

PROOF. Consider a chart (U_α, ϕ_α) for $\pi\colon X \to X/G$. Since the homeomorphism $\phi_\alpha\colon U_\alpha \times G \to \pi^{-1}(U_\alpha)$ is a G-map, there is a commutative diagram

$$\begin{array}{ccccccc}
U_\alpha \times Y & = & (U_\alpha \times G) \times_G Y & \xrightarrow[\cong]{\phi_\alpha} & \pi^{-1}(U_\alpha) \times_G Y & = & p^{-1}(U_\alpha) \\
\downarrow \pi_{U_\alpha} & & \downarrow \pi_{U_\alpha} & & \downarrow p & & \downarrow p \\
U_\alpha & = & U_\alpha & = & U_\alpha & = & U_\alpha
\end{array}$$

and hence the result. □

Let ξ be a (right) principal G-bundle given by $\pi\colon X \to X/G$. Let Y be any left G-space. Then fibre bundle

$$p\colon X \times_G Y \longrightarrow X/G$$

is called *induced fibre bundle* of ξ, denoted by $\xi[Y]$.

Now let $p\colon E \to B$ is a fibre bundle with fibre F and bundle group G. Observe that the action of $\mathrm{Homeo}(F, F)$ on F is a left action because $(f \circ g)(x) = f(g(x))$. Thus G acts by left on F via $\theta\colon G \to \mathrm{Homeo}(F, F)$.

A bundle morphism

$$\begin{array}{ccc}
E(\xi) & \xrightarrow{\phi} & E(\xi') \\
\downarrow p(\xi) & & \downarrow p(\xi') \\
B(\xi) & \xrightarrow{\bar{\phi}} & B(\xi')
\end{array}$$

is call an *isomorphism* if both ϕ and $\bar{\phi}$ are homeomorphisms. (**Note.** this means that $(\phi^{-1}, (\bar{\phi})^{-1})$ are continuous.) In this case, we write $\xi \cong \xi'$.

THEOREM 6.57. *Let ξ be a fibre bundle given by $p\colon E \to B$ with fibre F and bundle group G. Let ξ^G be the principal G-bundle constructed in Proposition 6.55 according to a set of transitions functions to ξ Then $\xi^G[F] \cong \xi$.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for ξ. We write $\tilde{\phi}_\alpha$ for ϕ_α in the proof of Proposition 6.55. Consider the map θ_α given by the composite:

$$\pi^{-1}(U_\alpha) \times_G F \xleftarrow[\cong]{\tilde{\phi}_\alpha \times \mathrm{id}_F} (U_\alpha \times G \times \alpha)_G \times F = U_\alpha \times F \xrightarrow[\cong]{\phi_\alpha} p^{-1}(U_\alpha).$$

From the commutative diagram

$$\begin{array}{ccc}
((U_\alpha \cap U_\beta) \times G \times \beta) \times_G F & \xrightarrow[((b,g',\beta),y) \mapsto ((b, g_{\alpha\beta}(b)g', \alpha), y)]{(b,g',y) \mapsto (b,g',g_{\alpha\beta}(b)(y))} & ((U_\alpha \cap U_\beta) \times G \times \alpha) \times_G F \\
\| & & \| \\
(U_\alpha \cap U_\beta) \times F & \xrightarrow{(b,y) \mapsto (b, g_{\alpha\beta}(b,y))} & (U_\alpha \cap U_\beta) \times F \\
\cong \downarrow \phi_\beta & & \cong \downarrow \phi_\alpha \\
p^{-1}(U_\alpha \cap U_\beta) & = & p^{-1}(U_\alpha \cap U_\beta),
\end{array}$$

the map θ_α induces a bundle map

$$\begin{array}{ccc}
E^G \times_G F & \xrightarrow{\theta} & E(\xi) \\
\downarrow & & \downarrow \\
B(\xi) & = & B(\xi).
\end{array}$$

This is an bundle isomorphism because θ is one-to-one and onto, and θ is a local homeomorphic by restricting each chart. The assertion follows. □

This theorem tells that any fibre bundle with a bundle group G is an induced fibre bundle of a principal G-bundle. Thus, for classifying fibre bundles over a fixed base space B, it suffices to classify the principal G-bundles over B. The latter is actually done by the homotopy classes from B to the classifying space BG of G. (There are few assumptions on the topology on B such as B is paracompact.) The theory for classifying fibre bundles is also called *(unstable) K-theory*, which is one of important applications of homotopy theory to geometry. Rough introduction to this theory is as follows:

There exists a *universal G-bundle* ω_G as $\pi\colon EG \to BG$. Given any principal G-bundle ξ over B, there exists a (continuous) map $f\colon B \to BG$ such that ξ, as a principal G-bundle, is isomorphic to the pull-back bundle $f^*\omega_G$ given by

$$\begin{array}{ccc}
E(f^*\omega_G) = \{(x,y) \in B \times EG \mid f(x) = \pi(y)\} & \longrightarrow & EG \\
\downarrow (x,y) \mapsto x & & \downarrow \pi \\
B & \xrightarrow{f} & BG.
\end{array}$$

Moreover, for continuous maps $f, g\colon B \to BG$, $f^*\omega_G \cong g^*\omega_G$ if and only if $f \simeq g$. In other words, the set of homotopy classes $[B, BG]$ is one-to-one correspondent to the set of isomorphic classes of principal G-bundles over G.

Seminar Topic: The classification of principal G-bundles and fibre bundles. (References: for instance [**9**, pp.48-58] Or [**17, 18**].)

4. Spectral Sequences

4.1. The Ideas for Obtaining Spectral Sequences.

4.2. Leray-Serre Spectral Sequences.

5. Localization and Completion of Spaces

6. Simplicial Homotopy Theory and Simplicial Groups

6.1. The Relations between Spaces and Simplicial Sets.

6.2. Simplicial Groups.

6.3. Seifert- Van Kampen Theorem for Simplicial Groups.

6.4. Constructing Simplicial Group Models for Loop Spaces.

7. Configuration Spaces and Combinatorial Models for Mapping Spaces

7.1. Configuration Spaces.

7.2. Configuration Spaces with Labels.

8. Homotopy Decompositions of Spaces

9. Cohen Groups

10. Homotopy Groups and the Exponent Problem in Homotopy Theory

10.1. EHP-Sequences.

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