COMBINATORIAL DESCRIPTIONS OF HOMOTOPY GROUPS OF CERTAIN SPACES

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Abstract. We give a combinatorial description of the homotopy groups of the suspension of a $K(\pi, 1)$ and of wedges of 2-spheres. In particular, all of the homotopy groups of the 2-sphere are given as the centres of certain combinatorially described groups.

1. Introduction

In this article, we study homotopy groups and some related problems by using simplicial homotopy theory. The point of view here is that combinatorial aspects of group theory provide further information about homotopy groups. This view does not yet admit computational applications to higher homotopy groups, but it does provide a somewhat different approach than those given classically. In particular, in this paper, homotopy groups are given as centres of groups that admit natural combinatorial descriptions.

The homotopy groups of the 3-sphere, the suspension of a $K(\pi, 1)$ and wedges of 2-spheres are shown to be the centres of certain groups with specific generators and specific relations. We list these group-theoretical descriptions as follows. Let G be a group and let $[x, y] = x^{-1}y^{-1}xy$ in G.

Definition 1.1. A bracket arrangement of weight n in a group G is a map $\beta^n\colon G^n \to G$ which is defined inductively as follows:

$$\beta^1 = \mathrm{id}_G,\ \beta^2(a_1, a_2) = [a_1, a_2]$$

for any $a_1, a_2 \in G$. Suppose that the bracket arrangements of weight k are defined for $1 \leq k < n$ with $n \geq 3$. A map $\beta^n\colon G^n \to G$ is called a bracket arrangement of weight n if β^n is the composite

$$G^n = G^k \times G^{n-k} \xrightarrow{\beta^k \times \beta^{n-k}} G \times G \xrightarrow{\beta^2} G$$

for some bracket arrangements β^k and β^{n-k} of weight k and $n-k$, respectively, with $1 \leq k < n$.

For instance, if $n = 3$, there are two bracket arrangements given by $[[a_1, a_2], a_3]$ and $[a_1, [a_2, a_3]]$.

Convention 1.2. Given an (ordered) finite set $\{y_0, y_1, \cdots, y_n\}$ in a group G we write y_{-1} for $(y_0 y_1 \cdots y_n)^{-1}$, the inverse of their product.

Notations 1.3. Let G be a group and let S be a subset of G. Let $\langle S\rangle$ denote the normal subgroup generated by S. Let H_j be a sequence of subgroups of G for $1 \leq j \leq k$. Let $[[H_1, \ldots, H_k]]$ denote the subgroup of G generated by all of the commutators $\beta^t(h^{(1)}_{i_1}, \ldots, h^{(t)}_{i_t})$, where

1) $1 \leq i_s \leq k$;
2) all integers in $\{1, 2, \cdots, k\}$ appear as at least one of the integers i_s;

3) $h_j^{(s)} \in H_j$;

4) for each $t \geq k$, β^t runs over all of the bracket arrangements of weight t.

Let $\vee_{\alpha\in J} S^2$ be a wedge of the 2-sphere with index set J. A group-theoretical description of the homotopy groups $\pi_*(\vee_{\alpha\in J} S^2)$ is as follows.

Theorem 1.4. *$\pi_{n+2}(\vee_{\alpha\in J} S^2)$ is isomorphic to the center of the group generated by $y_j^{(\alpha)}$ for $\alpha \in J$ and $0 \leq j \leq n$ modulo the normal subgroup*

$$[[\langle y_{-1}^{(\alpha_{-1})} | \alpha_{-1} \in J\rangle, \langle y_0^{(\alpha_0)} | \alpha_0 \in J\rangle, \cdots, \langle y_n^{(\alpha_n)} | \alpha_n \in J\rangle]]$$

with $y_{-1}^{(\alpha)} = (y_0^{(\alpha)} y_1^{(\alpha)} \cdots y_n^{(\alpha)})^{-1}$.

By the equality $\pi_{n+2}(S^3) = \pi_{n+2}(S^2)$ for $n \geq 1$, we have the following important corollary. Let $F(y_0, y_1, \cdots, y_n)$ be the group freely generated by $y_0, \cdots, y_n$.

Corollary 1.5. *For $n \geq 1$, $\pi_{n+2}(S^3)$ is isomorphic to the centre of*

$$F(y_0, y_1, \cdots, y_n)/[[\langle y_{-1}\rangle, \langle y_0\rangle, \ldots, \langle y_n\rangle]].$$

Remark 1.6. The subgroup $[[\langle y_{-1}\rangle, \langle y_0\rangle, \ldots, \langle y_n\rangle]]$ is generated by the commutators

$$\beta^t(x_1^{\epsilon_1}, x_2^{\epsilon_2}, \cdots, x_t^{\epsilon_t}),$$

where

1) $\epsilon_j = \pm 1$;

2) $x_s \in \{y_{-1}, y_0, y_1, \cdots, y_n\}$;

3) all elements in $\{y_{-1}, y_0, y_1, \cdots, y_n\}$ appear as at least one of the x_j;

4) for each $t \geq n+2$, β^t runs over all of the commutator bracket arrangements of weight t.

This gives specific generators and relations for the combinatorial group $F(y_0, y_1, \cdots, y_n)/[[\langle y_{-1}\rangle, \langle y_0\rangle, \ldots, \langle y_n\rangle]]$. An alternative combinatorial description of $\pi_*(S^3)$ is as follows.

Theorem 1.7. *Let $n \geq 1$ and let $F(y_0, \ldots, y_n)$ be the free group generated by $y_0, \ldots, y_n$. Then there is an isomorphism of groups*

$$([[\langle y_0\rangle, \langle y_1\rangle, \ldots, \langle y_n\rangle]] \cap \langle y_{-1}\rangle)/[[\langle y_{-1}\rangle, \langle y_0\rangle, \ldots, \langle y_n\rangle]] \cong \pi_{n+2}(S^3).$$

The method of the proofs of the theorems above is to study the Moore chain complex of Milnor's $F(K)$-construction [22] and to use the modified Moore-Postnikov system (Section 2). A group theoretical description of the homotopy groups $\pi_*(K(\pi, 1))$ for general π is as follows.

Observe that coproducts exist in the category of groups and they are given by free products. We write $\coprod_\alpha^{\text{groups}} G_\alpha$ for the free product of the groups G_α. Given an integer $n \geq 1$, let π be any group and let $\{x^{(\alpha)} | \alpha \in J\}$ be a set of generators for π. Let $(\pi)_j$ be a copy of π with generators $\{x_j^{(\alpha)} | \alpha \in J\}$ for $1 \leq j \leq n$. Let $y_j^{(\alpha)} = x_j^{(\alpha)} {x_{j+1}^{(\alpha)}}^{-1}$ for $1 \leq j \leq n-1$, $y_n^{(\alpha)} = x_n^{(\alpha)}$ and $y_{-1}^{(\alpha)} = {x_0^{(\alpha)}}^{-1} = (y_0^{(\alpha)} \cdots y_n^{(\alpha)})^{-1}$.

Theorem 1.8. *Let π be any group with a set of generators $\{x^\alpha | \alpha \in J\}$. If $n \neq 1$, then $\pi_{n+2}(\Sigma K(\pi, 1))$ is isomorphic to the centre of the quotient group of the free product*

$$\coprod_{0\leq j\leq n}^{\text{groups}} (\pi)_j$$

by the normal subgroup

$$[[\langle y_{-1}^{(\alpha_{-1})} | \alpha_{-1} \in J\rangle, \langle y_0^{(\alpha_0)} | \alpha_0 \in J\rangle, \cdots, \langle y_n^{(\alpha_n)} | \alpha_n \in J\rangle]].$$

In particular, for $\pi = \mathbb{Z}/m$, we have

Corollary 1.9. *$\pi_{n+2}(\Sigma K(\mathbb{Z}/m, 1))$ is isomorphic to the centre of the group with generators $x_0, \ldots, x_n$ modulo the normal subgroup generated by:*

I) *x_j^m for $0 \leq j \leq n$ and*

II) *$[[\langle y_{-1}\rangle, \langle y_0\rangle, \ldots, \langle y_n\rangle]]$.*

Theorem 1.8 is general in the following sense. By Kan-Thurston Theorem [19], for any connected space X, there exist a group π and a map $f\colon K(\pi, 1) \to X$ such that $f_*\colon H_*(K(\pi, 1)) \to H_*(X)$ is an isomorphism, that is, f is a homology equivalence. Thus $\Sigma f\colon \Sigma K(\pi, 1) \to \Sigma X$ is a (weak) homotopy equivalence. Hence Theorem 1.8 gives a combinatorial description of the homotopy groups of any suspension.

One of the features of Corollary 1.5 and Theorem 1.8 is that homotopy groups embed in certain 'enveloping groups'. These 'enveloping groups' have systematic and uniform structure. The centers of these groups are of course more complicated to analyze. Corollary 1.5 and Theorem 1.7 are the first descriptions, but not explicit calculations, of the general homotopy groups of the 3-sphere. These descriptions give new information about the homotopy groups of the 3-sphere. Also these descriptions allow us to calculate the homotopy groups of Cohen's K-construction of the 1-sphere.

The homotopy groups of $\Sigma K(\pi, 1)$ have been studied by many people from different point of view (see, for examples, [2, 3, 4, 5, 12, 13, 14, 15, 30, 28]). Brown and Loday [4] give a complete description of $\pi_3(\Sigma K(\pi, 1))$, which leads to calculations of $\pi_3(\Sigma K(\pi, 1))$. In [3], Baues and Conduché give a different method to find Brown-Loday results on $\pi_3(\Sigma K(\pi, 1))$ and information on the group $\pi_4(\Sigma K(\pi, 1))$. Theorem 1.8 gives a description of the general higher homotopy groups. So far calculations on the higher homotopy groups are still out of reach, but the combination of these different techniques for describing the higher homotopy groups has been pursued [26] after Theorem 1.8 was first announced in author's thesis.

We emphasize that descriptions here are NOT calculations. In Corollary 1.5, we give an explicit group which occurs naturally for which the homotopy group of the 3-sphere is the center. These structures do not immediately give any new or informative calculation on the higher homotopy groups in the classical sense, but they give a different view of the underlying structure that occurs in a natural way. It is an attempt to consider these groups in a systematic way.

Theorem 2.19 states that the torsion component of any homotopy group of a simply connected space is the torsion component of the center of certain nilpotent group. In

Proposition 4.10, we give an explicit finitely presented nilpotent group for which the higher homotopy group of the 2-sphere is the torsion component of the center.

There is certain similarity between the combinatorial description of the homotopy groups of the 2-sphere given in Theorem 1.7 and the combinatorial description of J. H. C. Whitehead's conjecture given in [6]. Observe that the homotopy groups of the 2-sphere and the Whitehead conjecture are different problems on low-dimensional CW-complexes. This 'incidental' relation can be seen in combinatorial group theory, in that both combinatorial descriptions ask the same combinatorial question, namely, how to understand the quotient groups $(R \cap S)/[R, S]$ for certain subgroups R and S of a finitely generated free group. In Theorem 1.7, particular subgroups R and S are given.

These combinatorial methods extend in a second direction which we now describe. It is well known that any space is weakly homotopy equivalent to a minimal simplicial set. An important feature of minimal simplicial sets is that these are the smallest (fibrant) simplicial sets with which we can still do homotopy theory. There have been uses of computers for studying homotopy groups. Minimal simplicial sets become very important in computer homotopy theory. A minimal simplicial group means a minimal simplicial set that is also a simplicial group. The group structure in a minimal simplicial group helps us to control the homotopy groups. Moore's problem is whether a loop space has the (weak) homotopy type of a minimal simplicial group. In [1], Adams proved that if X is a two-stage Postnikov system X with the only possible non-trivial homotopy groups being $\pi_n(X)$ and $\pi_{n+1}(X)$, then X has the homotopy type of a minimal simplicial group. In [32], we proved that a two-stage Postnikov system X, with the only non-trivial homotopy groups being $\pi_n(X) = \mathbb{Z}/2$ and $\pi_{n+i}(X) = \mathbb{Z}/2$ with $n, i > 0$, has the homotopy type of a minimal simplicial group if and only if the Postnikov invariant is trivial or Sq^{i+1}. Milnor proved that the three stage Postnikov system obtained by taking the first three non-trivial homotopy groups of S^n with $n > 1$ does not have the homotopy type of a minimal simplicial group. Unfortunately , this important example was never published. We will use Cohen's K-construction on pointed simplicial sets [8] to reprove Milnor's example in this paper. (See Proposition 6.19.)

Cohen's K-construction arose in studying self-maps of the loop space of a suspension. This construction is an important tool in studying the exponent problem in homotopy theory. Also important relations between Cohen's construction and Goodwillie's tower of the identity functor were found by Bill Dwyer and other people recently. Cohen [8] showed that the homotopy classes of all self maps of $\Omega\Sigma X$ that are natural in X form a progroup which is built up from the Moore complex of Milnor's construction $F(S^1)$. Thus Corollary 4.6 provides detailed information about this progroup. On the other hand, the homotopy classes of all functorial self-maps of the loop spaces of a double suspension form a progroup which is built up by the Moore complex of Cohen's construction $K(S^1)$ instead of $F(S^1)$ [8]. By using Corollary 1.5, we determine the general homotopy groups of $K(S^1)$. (See Theorem 6.7.)

Furthermore, the Lie structure of $\pi_*(K(S^1))$ is given. (See Proposition 6.11.) We find out that in addition to considerable applications of the homotopy groups of $K(S^1)$ to mathematical physics, the simplicial group $K(S^1)$ itself plays an important role classifying certain type of minimal simplicial groups. (See Theorem 6.13.). Theorems 6.7 and 6.13 are used to show that the three-stage Postnikov tower is not homotopy equivalent to a minimal simplicial group. In addition, Cohen's K-construction is extensively used to study functorial self coalgebra maps of primitively generated tensor algebras with applications to functorial decompositions of loop suspensions. (See [29].) It is known (unpublished) that all functorial self coalgebra maps of tensor algebras (without assuming that the Hopf structure is primitively generated) over the prime field $\mathbb{F}_p$ form a progroup which is built up from the p-completion groups of the Moore complex of $F(S^1)$. This raises the possibility of studying the homotopy by using representations of the Moore complex into the convolution algebra of self linear maps of tensor algebras.

The article is organized as follows. In Section 2, we study some general properties of simplicial groups, and examine central extensions in the Moore-Postnikov systems. In Section 3, we study the intersection of certain subgroups in free groups. The proofs of Theorems 1.4 and 1.7 are given in Section 4. Theorem 1.8 is proved in Section 5. In Section 6, we give some applications of our descriptions. One example is to compute the homotopy groups of Cohen's construction on the 1-sphere.

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2. Central Extensions in simplicial group theory

In this section, we study some general properties of simplicial groups. A simplicial set K is called a Kan complex if it satisfies the *extension condition*, i.e, any simplicial map $f\colon \Lambda^k[n] \to K$ has an extension $g\colon \Delta[n] \to K$, where $\Delta[n]$ is the standard n simplex, $\Lambda^k[n]$ is the subcomplex of $\Delta[n]$ generated by all $d_i(\sigma_n)$ for $i \neq k$ with σ_n as the nondegenerate n simplex in $\Delta[n]$ and d_j as one of the face functions [16, 11]. Recall that any simplicial group is a Kan complex [23]. A space (or simplicial set) X is called n-simple if the fundamental group $\pi_1(X)$ acts trivially on $\pi_n(X)$. X is called simple if $\pi_1(X)$ acts trivially for $\pi_n(X)$ for all $n \geq 1$. Given a simplicial group G, the Moore chain complex (NG, d_0) is defined by $NG_n = \bigcap_{j\neq 0} \mathrm{Ker}(d_j)$ together with $d_0\colon NG_n \to NG_{n-1}$. The classical Moore Theorem is that $\pi_n(G) \cong H_n(NG)$ (see [23, Theorem 4], [11, Theorem 3.7] or [17, Proposition 5.4]]. Now let $\mathcal{Z}_n = \mathcal{Z}_n(G) = \bigcap_j \mathrm{Ker}(d_j)$ denote the cycles and let $\mathcal{B}_n = \mathcal{B}G_n = d_0(NG_{n+1})$ denote the boundaries. It is easy to check that $\mathcal{B}_n$ is a normal subgroup of G_n for any simplicial group G.

Let G be a simplicial group and let $\alpha \in G_0$. The simplicial automorphism $\phi_\alpha\colon G \to G$ is defined by $\phi_\alpha(x) = (s_0^n\alpha)^{-1}xs_0^n\alpha$ for $x \in G_n$, where s_j is a degeneracy function. Suppose that α is homotopic to β, that is, there exists an element $\gamma \in G_1$ such that $d_0\gamma = \alpha$ and $d_1\gamma = \beta$. Then it is easy to check that ϕ_α is homotopic to ϕ_β. Thus $\pi_0(G)$ acts on $\pi_*(G)$. Observe that $\pi_0(G)$ acts on itself by conjugation.

Lemma 2.1. *Let G be a simplicial group and let $n \geq 0$. Suppose that $\pi_0(G)$ acts trivially on $\pi_n(G)$. Then the homotopy group $\pi_n(G)$ is contained in the centre of $G_n/\mathcal{B}G_n$.*

Proof. If $n = 0$, then $\pi_0(G) \cong G_0/\mathcal{B}G_0$, which is abelian if $\pi_0(G)$ acts trivially on itself. Thus we may assume that $n \geq 1$. Note $\pi_n(G) \cong \mathcal{Z}G_n/\mathcal{B}G_n$. It suffices to show that the commutator $[x, y] \in \mathcal{B}G_n$ for any $x \in \mathcal{Z}G_n$ and $y \in G_n$.

Now let $x \in \mathcal{Z}G_n$ and let $y \in G_n$. Let z denote $s_0^n d_0^n(y^{-1}) \cdot y$. Observe that x is a cycle. There is a simplicial map $f_x\colon S^n \to G$ such that $f_x(\sigma_n) = x$, where S^n is the standard n-sphere with a nondegenerate n-simplex σ_n. Let the simplicial map $f_z\colon \Delta[n] \to G$ be the representative of z, i.e, $f_z(\tau_n) = z$ for the nondegenerate n-simplex τ_n. Consider the simplicial map

$$[f_x, f_z]\colon S^n \times \Delta[n] \to G$$

defined by $[f_x, f_z](a,b) = [f_x(a), f_z(b)] = (f_x(a))^{-1}(f_z(b))^{-1}f_x(a)f_z(b)$, the commutator of $f_x(a)$ and $f_z(b)$. Notice the equality

$$f_z(d_0^n(\tau)) = d_0^n s_0^n d_0^n(y^{-1}) \cdot d_0^n y = 1.$$

Let v be the simplicial subset of $\Delta[n]$ generated by the vertex $d_0^n\tau$ and let $*$ be the base-point of S^n. Then $[f_x, f_z]$ is trivial restricted to the simplicial subset $S^n \vee \Delta[n] = (S^n \times v) \cup (* \times \Delta[n])$. Thus $[f_x, f_z]$ factors through the quotient simplicial set $S^n \wedge \Delta[n]$. Let ϕ be the composite

$$S^n \xrightarrow{j} S^n \wedge \Delta[n] \xrightarrow{[f_x, f_z]} G$$

with $j(\sigma_n) = \sigma_n \wedge \tau_n$. Notice that we have $\phi(\sigma_n) = [x,z]$. Let $\{[x,z]\}$ denote the homotopy class in $\pi_n(G)$ represented by the cycle $[x,z]$. Then the map $\phi\colon S^n \to G$ is a representative of the cycle $[x,z]$ and $\phi_*(\iota_n) = \{[x,z]\}$, where the homomorphism $\phi_*\colon \pi_*(S^n) \to \pi_*(G)$ is induced by the map ϕ, where ι_n is a generator for $\pi_n(S^n)$. Observe that $S^n \wedge \Delta[n]$ is contractible. Thus $\{[x,z]\} = 0$ in $\pi_n(G)$ and so $[x,z] \in \mathcal{B}G_n$.

Let $\alpha = d_0^n y$. Then $\{(s_0^n\alpha)^{-1} x s_0^n \alpha\} = \{x\}$ since $\pi_0(G)$ acts trivially on $\pi_n(G)$. Thus $[x, s_0^n\alpha] \in \mathcal{B}G_n$ and so $[x,y] = [x, s_0^n\alpha z] \in \mathcal{B}G_n$ by the Witt-Hall identity that $[a,bc] = [a,c][a,b][[a,b],c]$. The assertion follows. □

Remark 2.2. 1) Let G be a simplicial group. Then $\pi_0(G)$ acts trivially on $\pi_n(G)$ if and only if the classifying space BG is $n+1$-simple;

2) As the referee pointed out, this lemma can also be derived from a theorem of Conduché [9] extending a property of 1-cat groups as defined by Loday.

Corollary 2.3. *Let G be a simplicial group. Suppose that BG is simple, for example, if G is connected. Then the homotopy group $\pi_n(G)$ is contained in the center of $G_n/\mathcal{B}G_n$ for each n.*

Now we consider the Moore-Postnikov systems of simplicial groups.

Definition 2.4. Let G be a simplicial group. The simplicial subgroup R_nG is defined by setting

$$(R_nG)_q = \{x \in G_q | f_x(\Delta[q])^{[n]} = 1\},$$

where f_x is the representative of x and $X^{[n]}$ is the n-skeleton of the simplicial set X. The simplicial subgroup $\overline{R}_nG$ is defined by setting

$$(\overline{R}_nG)_q = \{x \in G_q | f_x((\Delta[q])_n) \subseteq \mathcal{B}G_n\}.$$

Let P_nG denote G/R_nG and let $\overline{P}_nG$ denote $G/\overline{R}_nG$.

It is easy to check that both R_nG and $\overline{R}_nG$ are normal simplicial subgroups of G. Thus both P_nG and $\overline{P}_nG$ are quotient simplicial groups of G. Observe that

$$R_nG \subseteq \overline{R}_nG \subseteq R_{n-1}G.$$

There is a cofiltration

$$G \to \cdots \to P_nG \to \overline{P}_nG \to P_{n-1}G \to \cdots \to \overline{P}_0G.$$

By checking the definition of Moore-Postnikov systems of a simplicial set [11, 23, 24, 25], we have

Lemma 2.5. *The quotient simplicial group P_nG is the standard nth stage of the Moore-Postnikov system of the simplicial group G.*

The quotient simplicial group $\overline{P}_nG$ has the same homotopy type as P_nG.

Lemma 2.6. *The quotient simplicial homomorphism $q_n\colon P_nG \to \overline{P}_nG$ is a homotopy equivalence for each n.*

Proof. The Moore chain complex of P_nG is as follows:

$$N(P_nG)_q = \begin{cases} 1 & \text{for} \quad q > n+1, \\ NG_{n+1}/\mathcal{Z}G_{n+1} & \text{for} \quad q = n+1, \\ NG_q & \text{for} \quad q < n+1. \end{cases}$$

The Moore chain complex of $\overline{P}_nG$ is as follows:

$$N(\overline{P}_nG)_q = \begin{cases} 1 & \text{for} \quad q > n, \\ NG_n/\mathcal{B}G_n & \text{for} \quad q = n, \\ NG_q & \text{for} \quad q < n. \end{cases}$$

Thus $(q_n)_*\colon \pi_*(P_nG) \to \pi_*(\overline{P}_nG)$ is an isomorphism and the assertion follows. □

Definition 2.7. A simplicial group is *minimal* if it is also a minimal simplicial set.

Let $\overline{F}_n$ denote the kernel of the quotient simplicial homomorphism $r_n\colon \overline{P}_nG \to P_{n-1}G$ for $n > 0$.

Lemma 2.8. *The simplicial group $\overline{F}_n$ is the minimal simplicial group $K(\pi_nG, n)$ for each $n > 0$.*

Proof. The Moore chain complex of $\overline{F}_nG$ is as follows:

$$N(\overline{F}_nG)_q = \begin{cases} 1 & \text{for} \quad q \neq n, \\ \pi_nG & \text{for} \quad q = n. \end{cases}$$

The assertion follows. □

Lemma 2.9. *Let G be a simplicial group. Suppose that $\pi_0(G)$ acts trivially on $\pi_n(G)$. Then the short exact sequence of simplicial groups*

$$0 \to \overline{F}_nG \to \overline{P}_nG \to P_{n-1}G \to 0$$

is a central extension.

Proof. Consider the relative commutator simplicial subgroup $[\overline{F}_nG, \overline{P}_nG]$. Notice the equalities

$$\overline{F}_nG_n = \mathcal{Z}G_n/\mathcal{B}G_n \cong \pi_n(G)$$

and

$$\overline{P}_nG_n = G_n/\mathcal{B}G_n.$$

By Lemma 2.1, $[\overline{F}_nG,\overline{P}_nG]_n=1$. Observe that $[\overline{F}_nG,\overline{P}_nG]$ is a simplicial subgroup of the minimal simplicial group $\overline{F}_nG\cong K(\pi_nG,n)$. Thus $[\overline{F}_nG,\overline{P}_nG]$ is a minimal simplicial group of type $K(\pi,n)$ and so $[\overline{F}_nG,\overline{P}_nG]=1$. The assertion follows. □

Corollary 2.10. *Let G be a simplicial group. Suppose that BG is simple, for example, if G is connected. Then the short exact sequence of simplicial groups*

$$0\to\overline{F}_nG\to\overline{P}_nG\to P_{n-1}G\to 0$$

is a central extension for each n.

By Lemmas 2.6, 2.8 and 2.9, we have

Theorem 2.11. *Let G be a simplicial group. Suppose that BG is simple. Then there is a cofiltration*

$$G\to\cdots\to P_nG\to\overline{P}_nG\to P_{n-1}G\to\cdots\to\overline{P}_0G.$$

of G such that:

1) *the quotient simplicial homomorphism $q_n\colon P_nG\to\overline{P}_nG$ is a homotopy equivalence for each n;*
2) *the kernel $\overline{F}_nG$ of the quotient simplicial homomorphism $r_n\colon\overline{P}_nG\to P_{n-1}G$ is the minimal simplicial group $K(\pi_n(G),n)$ for each n;*
3) *the short exact sequence of simplicial groups*

$$0\to\overline{F}_nG\to\overline{P}_nG\to P_{n-1}G\to 0$$

is a central extension for each n;

4) *G is the inverse limit of the cofiltration;*

5) *G is the homotopy inverse limit of the cofiltration.*

Corollary 2.12. *Let G be a connected minimal simplicial group and let $\{P_nG\}$ be the Moore-Postnikov systems of G. Then the short exact sequence of simplicial groups*

$$0 \to F_nG \to P_nG \to P_{n-1}G \to 0$$

is a central extension for each n.

In some cases, the homotopy group $\pi_n(G)$ is the same as the center of $G_n/\mathcal{B}G_n$.

Definition 2.13. A simplicial group G is said to be r-*centerless* if the center $Z(G_n)$ is trivial for $n \geq r$.

Proposition 2.14. *Let G be a connected r-centerless simplicial group. Then*

$$\pi_n(G) \cong Z(G_n/\mathcal{B}G_n)$$

for $n \geq r+1$.

Proof. By Lemma 2.1, $\mathcal{Z}G_n/\mathcal{B}G_n \subseteq Z(G_n/\mathcal{B}G_n)$. So it suffices to show that

$$Z(G_n/\mathcal{B}G_n) \subseteq \mathcal{Z}G_n/\mathcal{B}G_n$$

for $n \geq r+1$. Now let $\tilde{x} \in Z(G_n/\mathcal{B}G_n)$. Choose $x \in G_n$ with $p(x) = \tilde{x}$, where $p\colon G_n \to G_n/\mathcal{B}G_n$ is the quotient homomorphism. To check $\tilde{x} \in \mathcal{Z}G_n/\mathcal{B}G_n$, it suffices to show that $x \in \mathcal{Z}G_n$ or $d_jx = 1$ for all j. Since $Z(G_{n-1}) = \{1\}$, $d_jx = 1$ if and only if $[d_jx, y] = 1$ for all $y \in G_{n-1}$. Now $[d_jx, y] = d_j[x, s_{j-1}y]$ for $j > 0$ and $[d_0x, y] = d_0[x, s_0y]$. Since $\tilde{x} \in Z(G_n/\mathcal{B}G_n)$, $[x, z] \in \mathcal{B}G_n \subseteq \mathcal{Z}G_n$ for all $z \in G_n$ and therefore $[d_jx, y] = 1$ for all $y \in G_{n-1}$. We finish the proof. □

The proof also gives

Proposition 2.15. *Let G be a connected r-centerless simplicial group. Then*

$$Z(G_n/\mathcal{Z}G_n) = \{1\}$$

for $n \geq r + 1$.

A simplicial group G is called *reduced* if $G_0 = \{1\}$.

Lemma 2.16. *Let G be a reduced simplicial group such that G_n is cyclic or centerless for each n. Let γ_G be the smallest n such that we have $G_n \neq \{1\}$. Then $Z(G_n) = \{1\}$ for $n > \gamma_G$.*

This proof, which shortens our original proof, was provided and suggested by the referee.

Proof. The assertion follows from two easy remarks:

a) If G_n is not abelian, any G_q for $q \geq n$ is also non-abelian because it contains copies of G_n via degeneracies.

b) If the simplicial group G is abelian and $G_n = NG_n$ is cyclic for $n = \gamma_G$ with $\gamma_G \geq 1$, then by the Dold-Kan theorem G_{n+1} contains $n+1$ different copies of G_n. So it cannot be cyclic.

□

Corollary 2.17. *Let G be a reduced simplicial group such that G_n is cyclic or centerless for each n. Then $\pi_n(G) \cong Z(G_n/\mathcal{B}G_n)$ for $n \neq \gamma_G + 1$, where γ_G is defined as above.*

Theorem 2.18. *Let G be a reduced simplicial group such that G_n is a free group for each n. Then there exits a unique integer $\gamma_G > 0$ such that $G_n = \{1\}$ for $n < \gamma_G$ and $\mathrm{rank}(G_n) \geq 2$ for $n > \gamma_G$. Furthermore, $\pi_n(G) \cong Z(G_n/\mathcal{B}G_n)$ for $n \neq \gamma_G + 1$.*

Let G be a group and let $\Gamma_n G$ be the descending central series of G starting with $\Gamma_1 = G$. Let $\mathrm{Tor}(G) = \{x \in G | x^q = 1 \quad \text{for some} \quad q > 0\}$. If G is an abelian group, then $\mathrm{Tor}(G)$ is the torsion subgroup of G. For a general group G, $\mathrm{Tor}(G)$ is not a subgroup of G. Let G be a connected free simplicial group. Then the torsion component of the homotopy groups $\pi_*(G)$ can be described as follows. Recall that a simplicial group G is called free if each G_n is a free group and there is a choice of generators for each G_n such that the degeneracies send generators to generators [11, 21].

Theorem 2.19. *Let G be a connected free simplicial group, let $q \geq 0$ be an integer and let $s \geq 2^{n-q}$. Suppose that $\pi_j(G) = 0$ for $j \leq q$. Then $\mathrm{Tor}(G_n/\langle \mathcal{B}G_n, \Gamma_s \rangle)$ is a* *subgroup of* $G_n/\langle \mathcal{B}G_n, \Gamma_s\rangle$. *Furthermore, there are isomorphisms of groups*

$$\mathrm{Tor}(\pi_n(G)) \cong \mathrm{Tor}(Z(G_n/\langle \mathcal{B}G_n, \Gamma_s\rangle)) \cong \mathrm{Tor}(G_n/\langle BG_n, \Gamma_s\rangle),$$

with $\Gamma_s = \Gamma_s G_n$.

Proof. Notice that $\Gamma_s G = \{\Gamma_s G_n\}_{n\geq 0}$ is a normal simplicial subgroup of G. Let $\phi\colon G \to G/\Gamma_s G$ be the quotient simplicial homomorphism. Then $N(\phi)\colon NG_n \to N(G/\Gamma_s G)_n$ is an epimorphism for each $n \geq 0$ and so $\mathcal{B}(\phi)\colon \mathcal{B}G_n \to \mathcal{B}(G/\Gamma_s G)_n$ is an epimorphism for each $n \geq 0$. Thus there is a canonical isomorphism of groups

$$\frac{(G/\Gamma_s G)_n}{\mathcal{B}(G/\Gamma_s G)_n} \cong G_n/\langle \mathcal{B}G_n, \Gamma_s\rangle.$$

By Curtis Theorem [10], we have $\pi_j(\Gamma_s G) = 0$ for $j \leq n$ and so

$$\phi_*\colon \pi_j(G) \to \pi_j(G/\Gamma_s G)$$

is an isomorphism for $j \leq n$. By Lemma 2.1, the composite

$$\pi_n(G) = \mathcal{Z}G_n/\mathcal{B}G_n \xrightarrow[\cong]{\phi_*} \mathcal{Z}(G/\Gamma_s G)_n/\mathcal{B}(G/\Gamma_s G)_n \hookrightarrow Z(G_n/\langle \mathcal{B}G_n, \Gamma_s G_n\rangle)$$

is a monomorphism. Thus the composite

$$\mathrm{Tor}(\pi_n(G)) \longrightarrow \mathrm{Tor}(Z(G_n/\langle \mathcal{B}G_n, \Gamma_s G_n\rangle)) \hookrightarrow \mathrm{Tor}(G_n/\langle \mathcal{B}G_n, \Gamma_s G_n\rangle)$$

is one-to-one. It suffices to show that this composite is an epimorphism.

Let $x \in \mathrm{Tor}(G_n/\langle \mathcal{B}G_n, \Gamma_s\rangle)$ and let $\alpha \in (G/\Gamma_s G)_n$ such that $x = \alpha\mathcal{B}(G/\Gamma_s G)_n$. Then there exists $k > 0$ such that $\alpha^k \in \mathcal{B}(G/\Gamma_s G)_n \subseteq \mathcal{Z}(G/\Gamma_s G)_n$. Thus, for each $0 \leq j \leq n$, we have

$$(d_j(\alpha))^k = d_j(\alpha^k) = 1$$

in the group $(G/\Gamma_s G)_{n-1} = G_{n-1}/\Gamma_s G_{n-1}$. Observe that $(G/\Gamma_s G)_{n-1} = G_{n-1}/\Gamma_s G_{n-1}$ is a torsion-free group. Thus

$$d_j(\alpha) = 1$$

for each $0 \leq j \leq n$ and so $\alpha \in \mathcal{Z}G_n$. Hence

$$x \in \frac{\mathcal{Z}(G/\Gamma_s G)_n}{\mathcal{B}(G/\Gamma_s G)_n} \cap \mathrm{Tor}(G_n/\langle \mathcal{B}G_n, \Gamma_s G_n\rangle),$$

and the assertion follows. □

Remark 2.20. Let X be a simply connected reduced simplicial set and let GX be the Kan-construction for X. Then GX is a free simplicial group in the sense of [11]. Thus this theorem shows that the torsion component of any homotopy group of a simply connected space is the torsion component of the center of a certain nilpotent group.

Example 2.21. A combinatorial calculation of $\pi_3(S^2)$ has been computed by D. Kan [17]. A combinatorial calculation of $\pi_{n+1}(S^n)$ for $n \geq 2$ is given as follows. The following calculation also gives combinatorial representatives for the generator of $\pi_{n+1}(S^n)$, that is the η-elements.

Let $G = F(S^n)$, Milnor's F-construction on the standard n-sphere for $n \geq 1$. Then $G_n \cong F(\sigma) \cong \mathbb{Z}(\sigma)$, the free abelian group generated by σ, and also $G_{n+1} \cong F(s_0\sigma, s_1\sigma, \ldots, s_n\sigma)$ and $G_{n+2} \cong F(s_is_j\sigma | 0 \leq j < i \leq n)$. It is easy to check that $\Gamma_2 G_{n+1} = \mathcal{Z}_{n+1}$. By Lemma 2.1, $\Gamma_3 G_{n+1} = [\mathcal{Z}G_{n+1}, G_{n+1}] \subseteq \mathcal{B}G_{n+1}$.

For $n = 1$, it will be shown that $\Gamma_3 G_{n+1} = \mathcal{B}G_{n+1}$ in Section 4, and therefore $\pi_3(S^2) \cong \pi_2(F(S^1)) \cong \mathbb{Z}$, generated by $[s_0\sigma, s_1\sigma]$.

Suppose that $n > 1$. Consider the equations for $i + 1 < j \leq n$:

$$d_k([s_{j-1}s_i\sigma, s_{j+1}s_j\sigma]) = \begin{cases} 1 & \text{for} \quad k \neq j, \\ [s_i\sigma, s_j\sigma] & \text{for} \quad k = j; \end{cases}$$

$$d_k[s_{i+2}s_{i+1}\sigma, s_{i+3}s_i\sigma] = \begin{cases} [s_{i+1}\sigma, s_{i+2}\sigma] & \text{for} \quad k = i+1, \\ [s_{i+1}\sigma, s_i\sigma] & \text{for} \quad k = i+3, \\ 1 & \quad\quad \text{otherwise}; \end{cases}$$

and

$$d_k[s_{i+2}s_i\sigma, s_{i+3}s_{i+1}\sigma] = \begin{cases} [s_{i+1}\sigma, s_{i+2}\sigma] & \text{for} \quad k = i+1, \\ [s_i\sigma, s_{i+2}\sigma] & \text{for} \quad k = i+2, \\ [s_i\sigma, s_{i+1}\sigma] & \text{for} \quad k = i+3, \\ 1 & \quad\quad \text{otherwise}. \end{cases}$$

By the Homotopy Addition Theorem [11, Theorem 2.4],

$$[s_i\sigma, s_j\sigma] \in \mathcal{B}_{n+1} \quad \text{for} \quad i + 1 < j,$$

$$[s_{i+1}\sigma, s_{i+2}\sigma] \equiv [s_i\sigma, s_{i+1}\sigma] \mod \mathcal{B}_{n+1},$$

$$0 \equiv [s_i\sigma, s_{i+2}\sigma] \equiv [s_i\sigma, s_{i+1}\sigma] + [s_{i+1}\sigma, s_{i+2}\sigma] \equiv 2[s_i\sigma, s_{i+1}\sigma] \quad \text{if} \quad i+2 \leq n.$$

Direct calculation shows that $[s_0\sigma, s_1\sigma] \notin \mathcal{B}(\Gamma_2 G_{n+1}/\Gamma_3 G_{n+1})$. Thus

$$[s_0\sigma, s_1\sigma] \notin \mathcal{B}G_{n+1}$$

and, by the relations above,

$$\pi_{n+2}(S^{n+1}) \cong \pi_{n+1}(G) \cong \mathcal{Z}G_{n+1}/\mathcal{B}G_{n+1} \cong \mathbb{Z}/2$$

for $n \geq 2$, which can be represented by $[s_i\sigma, s_{i+1}\sigma]$ for $0 \leq i \leq n-1$.

3. Intersections of certain subgroups in free groups

Definition 3.1. let S be a set and let $T \subseteq S$ be a subset. The *projection homomorphism*

$$\pi\colon F(S) \to F(T)$$

is defined by

$$\pi(x) = \begin{cases} x & \text{if} \quad x \in T, \\ 1 & \text{if} \quad x \notin T. \end{cases}$$

Now let $\pi\colon F(S) \to F(T)$ be the projection homomorphism and let R be the kernel of π. Define subsets of the free group $F(S)$ as follows.

For $x \in S - T$ and y a reduced word in $F(T)$ define $\mu(x,y)$ by induction on the length of y:

$\mu(x,y) = x$ if y is the empty word;

$$\mu(x,y)=[\mu(x,y'),z^{\epsilon}] \text{ if } y=y'z^{\epsilon} \text{ with } z\in T \text{ and } \epsilon=\pm 1.$$

Let $\mathcal{A}_T$ be the set of $\mu(x,y)$. Let $\mathcal{B}_T$ be the set of $y^{-1}xy$ for $x\in S-T$ and $y\in F(T)$. By the classical Kurosch-Schreier theorem (see [20, Theorem 18.1]), we have

Proposition 3.2. *The subgroup R is a free group freely generated by $\mathcal{B}_T$.*

In fact $\mathcal{A}_T$ is also a set of free generators for R.

Proposition 3.3. *The subgroup R is a free group freely generated by $\mathcal{A}_T$.*

This proof, which shortens our original long proof, was provided and suggested by the referee.

Proof. The proof follows from the following two steps:
a) Any finite part of $\mathcal{B}_T$ can be replaced, using Tietze moves (see [20]) by a finite part of $\mathcal{A}_T$. Thus $\mathcal{A}_T$ generates the subgroup R.
b) In the same way any finite part of $\mathcal{A}_T$ can be replaced by a finite part of $\mathcal{B}_T$. Thus $\mathcal{A}_T$ freely generates R.

□

Now let's consider the intersection of kernels of projection homomorphisms. Let S be a set and let T_j be a subset of S for $1\leq j\leq k$. Let $\pi_j\colon F(S)\to F(T_j)$ be the projection homomorphism for $1\leq j\leq k$. We construct a subset $\mathcal{A}(T_1,\cdots,T_k)$ of $F(S)$ by induction on k as follows.

$$\mathcal{A}(T_1)=\mathcal{A}_{T_1},$$

where $\mathcal{A}_T$ is defined in Definition 3.1. Let

$$T_2^{(2)} = \{w \in \mathcal{A}(T_1) | w = [[x, y_1^{\epsilon_1}], \cdots], y_t^{\epsilon_t}] \text{ with } x, y_j \in T_2 \text{ for all } j\}$$

and define

$$\mathcal{A}(T_1, T_2) = \mathcal{A}(T_1)_{T_2^{(2)}}.$$

Suppose that $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ is well-defined such that all of the elements in $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ are written down as certain commutators in $F(S)$ in terms of elements in S. Let $T_k^{(k)}$ be the subset of $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ defined by

$$T_k^{(k)} = \{w \in \mathcal{A}(T_1, T_2, \cdots, T_{k-1}) | \ w = [x_1^{\epsilon_1}, \cdots, x_l^{\epsilon_l}] \text{ with } x_j \in T_k \text{ for all } j\},$$

where $[x_1^{\epsilon_1}, \cdots, x_l^{\epsilon_l}]$ are the elements in $\mathcal{A}(T_1, T_2, \cdots, T_{k-1})$ that are written down as commutators. Then let $\mathcal{A}(T_1, T_2, \cdots, T_k)$ be defined by

$$\mathcal{A}(T_1, T_2, \cdots, T_k) = \mathcal{A}(T_1, T_2, \cdots, T_{k-1})_{T_k^{(k)}}.$$

Theorem 3.4. *Let S be a set and let T_j be a subset of S for $1 \leq j \leq k$. Let $\pi_j \colon F(S) \to F(T_j)$ be the projection homomorphism for $1 \leq j \leq k$. Then the intersection $\bigcap_{j=1}^k \mathrm{Ker}(\pi_j)$ is a free group freely generated by $\mathcal{A}(T_1, T_2, \cdots, T_k)$.*

Proof. The proof is given by induction on k. If $k = 1$, the assertion follows from Proposition 3.3. Suppose that $\bigcap_{j=1}^{k-1} \mathrm{Ker}(\pi_j) = F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1}))$, and consider $\pi_k \colon F(S) \to F(T_k)$. Then

$$\bigcap_{j=1}^{k} \mathrm{Ker}(\pi_j) = \mathrm{Ker}(\bar{\pi}_k \colon F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) \to F(T_k)),$$

where $\bar{\pi}_k$ is π_k restricted to the subgroup $F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1}))$. Let $w = [x_1^{\epsilon_1}, \cdots, x_l^{\epsilon_l}] \in \mathcal{A}(T_1, T_2, \cdots, T_{k-1})$. If $w \notin T_k^{(k)}$, then $x_j \notin T_k$ for some j and $\bar{\pi}_k(w) = 1$. Thus $\bar{\pi}_k$ factors through $F(T_k^{(k)})$, that is, there is a homomorphism $j\colon F(T_k^{(k)}) \to F(T_k)$ such that $\bar{\pi}_k = j \circ \pi$, where $\pi\colon F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) \to F(T_k^{(k)})$ is the projection homomorphism. We claim that $j\colon F(T_k^{(k)}) \to F(T_k)$ is a monomorphism. Consider the commutative diagram

$$\begin{array}{ccccc}
F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) & \xrightarrow{\pi_k} & F(T_k) & \hookrightarrow & F(S) \\
\uparrow & & \uparrow j & & \uparrow \\
F(T_k^{(k)}) & \xrightarrow{=} & F(T_k^{(k)}) & \hookrightarrow & F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})),
\end{array}$$

where $F(T_k^{(k)}) \to F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1}))$ and $F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) \to F(S)$ are inclusions of subgroups. Thus $j\colon F(T_k^{(k)}) \to F(T_k)$ is a monomorphism and

$$\mathrm{Ker}(\bar{\pi}_k) = \mathrm{Ker}(F(\mathcal{A}(T_1, T_2, \cdots, T_{k-1})) \to F(T_k^{(k)}) = F(\mathcal{A}(T_1, T_2, \cdots, T_k)).$$

The assertion follows. $\square$

Corollary 3.5. *Let π_j be the projection homomorphisms as in Theorem 3.4. If S is the union of the sets T_j, then the intersection subgroup $\bigcap_{j=1}^k \mathrm{Ker}(\pi_j)$ equals the commutator subgroup $[[\langle T_1\rangle, \cdots, \langle T_k\rangle]]$ defined in Notations 1.3.*

4. On the Homotopy Groups of Wedges of 2-Spheres

In this section, we give a combinatorial description of $\pi_*(\vee_{\alpha \in J} S^2)$. The proofs of Theorems 1.4 and 1.7 are given in this section. Recall that Milnor's construction

$F(K)$ for a (pointed) simplicial set K is a simplicial group model for $\Omega\Sigma|K|$, see [22]. In particular, $F(\vee_{\alpha\in J}S^1)$ is a simplicial group model for $\Omega(\vee_{\alpha\in J}S^2)$. The combinatorial description of $\pi_*(\vee_{\alpha\in J}S^2)$ will follow from our general results on simplicial groups given in Section 2 together with determination of the Moore chain complex of Milnor's construction $F(\vee_{\alpha\in J}S^1)$.

Recall that $(\vee_{\alpha\in J}S^1)_0 = *$ and $(\vee_{\alpha\in J}S^1)_1 = \{\sigma_\alpha, *|\alpha\in J\}$ and
$(\vee_{\alpha\in J}S^1)_{n+1} = \{s_n\cdots\hat{s}_i\cdots s_0\sigma_\alpha, *|\alpha\in J, 0\leq i\leq n\}$.
Let $x_i^{(\alpha)}$ denote $s_n\cdots\hat{s}_i\cdots s_0\sigma_\alpha$. Then

$$F(\vee_{\alpha\in J}S^1)_{n+1} = F(x_0^{(\alpha)}, x_1^{(\alpha)},\cdots,x_n^{(\alpha)}|\alpha\in J).$$

Let $y_j^{(\alpha)}$ denote $x_j^{(\alpha)}\cdot(x_{j+1}^{(\alpha)})^{-1}$ for $-1\leq j\leq n$ with $x_{-1}^\alpha = x_{n+1}^\alpha = 1$ in $F(\vee_{\alpha\in J}S^1)_{n+1} = F(x_0^{(\alpha)}, x_1^{(\alpha)},\cdots,x_n^{(\alpha)}|\alpha\in J)$. By direct calculation, we have

Lemma 4.1. *$F(\vee_{\alpha\in J}S^1)_{n+1} = F(y_j^{(\alpha)}|0\leq j\leq n, \alpha\in J)$ with*

$$d_j(y_k^{(\alpha)}) = \begin{cases} y_{k-1}^{(\alpha)} & \textit{for} \quad j\leq k,\\ 1 & \textit{for} \quad j=k+1,\\ y_k^{(\alpha)} & \textit{for} \quad j>k+1,\end{cases}$$

and

$$s_j(y_k^{(\alpha)}) = \begin{cases} y_{k+1}^{(\alpha)} & \textit{for} \quad j\leq k,\\ y_k^{(\alpha)}\cdot y_{k+1}^{(\alpha)} & \textit{for} \quad j=k+1,\\ y_k^{(\alpha)} & \textit{for} \quad j>k+1,\end{cases}$$

for $0\leq j\leq n+1$ with $y_{-1}^{(\alpha)} = (y_0^{(\alpha)}\cdots y_{n-1}^{(\alpha)})^{-1}$ in $F(\vee_{\alpha\in J}S^1)$.

Let C_{n+1}^J denote the subgroup of $F(\vee_{\alpha\in J}S^1)_{n+1}$ generated by all of the commutators of the form

$$[y_{i_1}^{(\alpha_1)\epsilon_1},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}],$$

where

1) $\epsilon=\pm 1$;
2) $0\le i_s\le n$;
3) $\alpha_j\in J$;
4) all integers in $\{0,1,\cdots,n\}$ appear as at least one of the integers i_s;
5) for each $t\ge n+1$, $[\cdots]$ runs over all of the commutator bracket arrangements of weight t.

We write C_{n+1} for C_{n+1}^J if the index set J consists of one element. Observe that C_{n+1} is a subgroup of $F(S^1)$.

Lemma 4.2. $C_{n+1}^J\subseteq NF(\vee_{\alpha\in J}S^1)_{n+1}$.

Proof. For each $1\le j\le n+1$, there exists some $i_s=j-1$. Thus $d_j(y_{i_s}^{(\alpha_s)\epsilon_s})=1$ for some i_s and therefore

$$d_j([y_{i_1}^{(\alpha_1)\epsilon_1},\cdots,y_{i_t}^{(\alpha_t)\epsilon_t}])=1$$

for each $j>0$. The assertion follows. □

Lemma 4.3. *For each* $1\le j\le n+1$*,*

$$\mathrm{Ker}(d_j)\cap F(\vee_{\alpha\in J}S^1)_{n+1}=\langle y_{j-1}^{(\alpha)}|\alpha\in J\rangle,$$

the normal subgroup generated by $y_{j-1}^{(\alpha)}$ *with* $\alpha \in J$*.*

Proof. By the definition of d_j, there is a commutative diagram

$$\begin{array}{ccc} F(y_j^{(\alpha)}|0\le j\le n,\alpha\in J) & \xrightarrow{p} & F(y_0^{(\alpha)}\cdots\hat{y}_{j-1}^{(\alpha)}\cdots y_n^{(\alpha)}|\alpha\in J) \\ \Big\downarrow d_j & & \cong\Big\downarrow \bar{d}_j \\ F(y_j^{(\alpha)}|0\le j\le n-1,\alpha\in J) & \xrightarrow{=} & F(y_j^{(\alpha)}|0\le j\le n-1,\alpha\in J), \end{array}$$

where p is the projection and

$$\bar{d}_j y_k^{(\alpha)} = \begin{cases} y_{k-1}^{(\alpha)} & \text{for} \quad j\le k, \\ y_k^{(\alpha)} & \text{for} \quad j>k+1. \end{cases}$$

The assertion follows. □

Theorem 4.4. *Let* C_{n+1}^J *be defined as above. Then*

$$NF(\vee_{\alpha\in J}S^1)_{n+1} = C_{n+1}^J$$

Proof. By lemma 4.1, each d_j with $j>0$ is a projection homomorphism. Thus, by Theorem 3.4,

$$NF(\vee_{\alpha\in J}S^1)_{n+1} = \bigcap_{j=1}^{n+1}\mathrm{Ker}(d_j) = F(\mathcal{A}(T_0,T_1,\cdots,T_n))$$

where $T_j=\{y_0^{(\alpha)},\cdots,\hat{y}_j^{(\alpha)},\cdots,y_n^{(\alpha)}|\alpha\in J\}$. It suffices to show that

$$\mathcal{A}(T_0,T_1,\cdots,T_n)\subseteq C_{n+1}^J.$$

This follows from the next lemma. □

Lemma 4.5. *For* $0 \leq j \leq n$*, let* $W = [y_{i_1}^{(\alpha_1)\epsilon_1}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}] \in \mathcal{A}(T_0, T_1, \cdots, T_j)$*. Then the set* $\{0, 1, \cdots, j\}$ *is contained in the set* $\{i_1, \cdots, i_t\}$*.*

Proof. The proof is by induction on j for $0 \leq j \leq n$. Observe that, by the construction, each element in $\mathcal{A}(T_0, T_1, \cdots, T_j)$ is written as a certain commutator. If $j = 0$, then

$$\mathcal{A}(T_0) = \{y_0^{(\alpha)}, [[y_0^{(\alpha)}, y_{i_1}^{(\alpha_1)\epsilon_1}], \cdots,], y_{i_t}^{(\alpha_t)\epsilon_t}]\}$$

where $\alpha, \alpha_j \in J$, $\epsilon_j = \pm 1$ and $y_{i_1}^{(\alpha_1)\epsilon_1} \cdots y_{i_t}^{(\alpha_t)\epsilon_t}$ runs over all of the nontrivial reduced words in $F(y_j^{(\alpha)} | \alpha \in J, 1 \leq j \leq n)$. Thus the assertion holds for $j = 0$. Suppose that the assertion holds for $j - 1$ with $j \leq n$. Recall that

$$T_j^{(j)} = \{W \in \mathcal{A}(T_0, T_1, \cdots, T_{j-1}) |\ W = [y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}] \text{ with } y_{i_s}^{(\alpha_s)} \in T_j\}.$$

Note that $y_{i_s}^{(\alpha_s)} \in T_j$ if and only if $i_s \neq j$. Let

$$W = [y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}] \in \mathcal{A}(T_0, \cdots, T_{j-1}) - T_j^{(j)}.$$

Then there exists some i_s with $1 \leq s \leq t$ such that $i_s = j$. By the induction hypothesis, the set $\{0, \cdots, j-1\}$ is contained in the set $\{i_1, \cdots, i_t\}$. Thus $\{0, 1, \cdots, j\} \subseteq \{i_1, i_2, \cdots, i_t\}$. Recall that, by construction,

$$\mathcal{A}(T_0, \cdots, T_j) = \mathcal{A}_{T_j^{(j)}}.$$

Thus we have

$$\{0, 1, \cdots, j\} \subseteq \{i_1, i_2, \cdots, i_t\}$$

for any $W = [y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}] \in \mathcal{A}(T_0, \cdots, T_j)$. This completes the proof.

$\square$

Corollary 4.6. $NF(S^1)_{n+1} = C_{n+1}$.

Corollary 4.7. *The Moore chain complex*

$$NF(\vee_{\alpha\in J} S^1)_{n+1} \subseteq \Gamma_{n+1} F(\vee_{\alpha\in J} S^1)_{n+1}$$

for $n \geq 0$, where $\Gamma_q G$ is the qth term in the lower central series of a group G starting with $\Gamma_1 G = G$.

Proof of Theorem 1.4. Evidently $\pi_{n+2}(\vee_{\alpha\in J} S^2) \cong \pi_{n+1}(F(\vee_{\alpha\in J} S^1))$. By the theorem above, $\mathcal{B}_{n+1}$ is generated by

$$[y_{i_1}^{(\alpha_1)\epsilon_1}, y_{i_2}^{(\alpha_2)\epsilon_2}, \cdots, y_{i_t}^{(\alpha_t)\epsilon_t}]$$

By Proposition 2.14, the assertion holds for $n \geq 1$. For $n = 0$, $\mathcal{B}_1 = \Gamma_2(F(y_0^{(\alpha)} | \alpha \in J))$ and

$$F(y_0^{(\alpha)} | \alpha \in J)/\mathcal{B}_1 \cong \oplus_{\alpha\in J}\mathbb{Z} \cong \pi_2(\vee_{\alpha\in J} S^2).$$

Hence assertion holds for all n. $\square$

Let ${C'}_{n+1}$ be the subgroup of $F(S^1)_{n+1}$ generated by all commutators of the form $[[y_{i_1}^{\epsilon_1}, y_{i_2}^{\epsilon_2}], \cdots, y_{i_t}^{\epsilon_t}]$, where

1) $\epsilon = \pm 1$;

2) $0 \leq i_s \leq n$;

3) all elements in $\{y_0, y_1, \cdots, y_n\}$ appear as one of the elements y_{i_s}.

Proposition 4.8. *There is an isomorphism of groups*

$$NF(S^1)_{n+1}/\left(\Gamma_s \cap NF(S^1)_{n+1}\right) \cong C'_{n+1}/\Gamma_s \cap C'_{n+1}$$

for each s, where $\Gamma_s = \Gamma_s F(S^1)_{n+1}$ is the s-term in the lower central series of $F(S^1)_{n+1}$.

Proof. Observe that we have $C'_{n+1} \subseteq NF(S^1)_{n+1}$. The induced homomorphism

$$f_s \colon C'_{n+1}/\left(\Gamma_s \cap C'_{n+1}\right) \to NF(S^1)_{n+1}/\left(\Gamma_s \cap NF(S^1)_{n+1}\right)$$

is a monomorphism. We check that f is an epimorphism. It suffices to show that, for each $w \in NF(S^1)$, there exists a sequence of elements $\{x_j\}$ such that $x_j \in \Gamma_j \cap C'_{n+1}$ and $wx_1x_2\cdots x_s \in \Gamma_{s+1}$ for each s. In fact, if this statement holds, then

$$wx_1x_2\cdots x_{s-1} \equiv 1 \ \ mod \ \ \Gamma_s \cap NF(S^1)_{n+1}$$

for each s and

$$w = (wx_1\cdots x_{s-1})\cdot(x_1x_2\cdots x_{s-1})^{-1} \in C'_{n+1} \ \ mod \ \ \Gamma_s.$$

Now we construct x_j by induction, which depends on w. Observe that, for $n \geq 1$, we have $NF(S^1)_{n+1} \subseteq \Gamma_{n+1} \subseteq \Gamma_2$. Choose $x_j = 1$ for $j \leq n$. Suppose that there are

$x_1, \cdots, x_{s-1}$ such that $x_j \in \Gamma_j \cap C'_{n+1}$ and $wx_1 \cdots x_{s-1} \in \Gamma_s$. Note that $C'_{n+1} \subseteq NF(S^1)_{n+1}$. Thus

$$wx_1 \cdots x_{s-1} \in \Gamma_s \cap NF(S^1)_{n+1}$$

and

$$d_j(wx_1 \cdots x_{s-1}) = 1$$

for $j > 1$. Let $\pi\colon \Gamma_s \to \Gamma_s/\Gamma_{s+1}$ be the quotient homomorphism. Then the element $\pi(wx_1 \cdots x_{s-1})$ is a linear combination of basic Lie products. We claim that $\pi(wx_1 \cdots x_{s-1})$ is a linear combination of basic Lie products in which each y_j appears in the Lie product for $0 \leq j \leq n$. If not, there exists j such that $\pi(wx_1 \cdots x_{s-1}) = b + c$, where b is a nontrivial linear combination of basic Lie products in which y_j does not appear, and c is a linear combination of basic Lie products in which y_j appears. Now the face homomorphism

$$d_{j+1}\colon F(y_0, \cdots, y_n) \to F(y_0, \cdots, y_{n-1})$$

induces a homomorphism of abelian groups

$$\bar{d}_{j+1}\colon \Gamma_s F(y_0, \cdots, y_n)/\Gamma_{s+1} F(y_0, \cdots, y_n) \to \Gamma_s F(y_0, \cdots, y_{n-1})/\Gamma_{s+1} F(y_0, \cdots, y_{n-1})$$

and

$$1 = \bar{d}_{j+1}\pi(wx_1 \cdots x_{s-1}) = \bar{d}_{j+1}(b) + \bar{d}_{j+1}(c) = \bar{d}_{j+1}(b).$$

Observe that

$$d_{j+1}|_{F(y_0,\cdots,y_{j-1},y_{j+1},\cdots,y_n)}\colon F(y_0,\cdots,y_{j-1},y_{j+1},\cdots,y_n)\to F(y_0,\cdots,y_{n-1})$$

is an isomorphism. Thus $b=1$. This contradicts the fact that b is a nontrivial linear combination of a basis. So $\pi(wx_1\cdots x_{s-1})$ is a linear combination of basic Lie products in which all of the y_j appear. By [20, Theorem 5.12], there exists x_s in C'_{n+1} such that $\pi(wx_1\cdots x_s)=1$, or $wx_1\cdots x_s\in\Gamma_{s+1}$. The induction is finished now and the assertion follows. □

Let $\mathcal{B}'_{n+1}$ be the subgroup of $F(S^1)_{n+1}$ generated by all commutators of the form $[[y_{i_1}^{\epsilon_1},y_{i_2}^{\epsilon_2}],\cdots,y_{i_t}^{\epsilon_t}]$, where

1) $\epsilon=\pm1$;
2) $-1\le i_s\le n$ with $y_{-1}=y_0y_1\cdots y_n$; and
3) all elements in $\{y_{-1},y_0,y_1,\cdots,y_n\}$ appear as one of the elements y_{i_s}.

Corollary 4.9. *There is an isomorphism of groups*

$$\mathcal{B}F(S^1)_{n+1}/(\Gamma_s\cap\mathcal{B}F(S^1)_{n+1})\cong\mathcal{B}'_{n+1}/(\Gamma_s\cap\mathcal{B}'_{n+1})$$

for each s.

Let D^s_{n+1} be the subgroup of $F(S^1)_{n+1}=F(y_0,y_1,\cdots,y_n)$ generated by all commutators of the form $[[y_{i_1}^{\epsilon_1},y_{i_2}^{\epsilon_2}],\cdots,y_{i_t}^{\epsilon_t}]$, where

1) $\epsilon=\pm1$;

2) $-1 \leq i_j \leq n$ with $y_{-1} = y_0 y_1 \cdots y_n$; and

3) if $t < s$, then all elements in $\{y_{-1}, y_0, y_1, \cdots, y_n\}$ appear as one of the elements y_{i_j} **or** $t \geq s$.

Recall that $\pi_n(S^2)$ is a finite group for $n \geq 4$. By Theorem 2.19, we have

Proposition 4.10. *Let $n \geq 2$. Then $\mathrm{Tor}(F(S^1)_{n+1}/D^{2^{n+1}}_{n+1})$ is a subgroup of $F(S^1)_{n+1}/D^{2^{n+1}}_{n+1}$. Furthermore there are isomorphisms of groups*

$$\pi_{n+2}(S^2) \cong \mathrm{Tor}\left(Z\left(F(S^1)_{n+1}/D^{2^{n+1}}_{n+1}\right)\right) \cong \mathrm{Tor}\left(F(S^1)_{n+1}/D^{2^{n+1}}_{n+1}\right).$$

Remark 4.11. The group $F(S^1)_{n+1}/D^{2^{n+1}}_{n+1}$ is a finitely presented nilpotent group with explicit generators and relations.

By Corollary 3.5, we have

Theorem 4.12. *In the free group $F(y_0, \cdots, y_n)$, we have the identifications of subgroups*

$$NF(S^1)_{n+1} = [[\langle y_0\rangle, \cdots, \langle y_n\rangle]],$$

$$\mathcal{B}F(S^1)_{n+1} = [[\langle y_{-1}\rangle, \langle y_0\rangle, \cdots, \langle y_n\rangle]].$$

Thus Corollary 1.5 can be rewritten as follows.

Theorem 4.13. *For $n \geq 1$ the free group $F(y_0, \cdots, y_n)$ has quotient with centre*

$$Z(F(y_0, \cdots, y_n)/([[\langle y_{-1}\rangle, \langle y_0\rangle, \cdots, \langle y_n\rangle]]) \cong \pi_{n+2}(S^3)$$

Proof of Theorem 1.7. By Lemma 4.1 and Theorem 3.4, $\mathrm{Ker}(d_0) = \langle x_0 \rangle = \langle y_{-1} \rangle$ and therefore

$$\mathcal{Z}_{n+1} = [[\langle y_0 \rangle, \cdots, \langle y_n \rangle]] \cap \langle y_{-1} \rangle.$$

The assertion follows. □

5. On the Homotopy Groups of $\Sigma K(\pi, 1)$

In this section, for any group π we give group theoretical descriptions for $\pi_*(\Sigma K(\pi, 1))$. The proof of Theorem 1.8 is given in this section, using the notation defined in Section 3. We need a simplicial group construction.

Definition 5.1. Let G be a simplicial group and let X be a pointed simplicial set with a base-point $*$. The simplicial group $\mathbf{F^G(X)}$ is defined by setting

$$F^G(X)_n = \coprod_{x \in X_n} (G_n)_x,$$

that is, the free product, modulo the relations $(G_n)_*$, where $(G_n)_x$ is a copy of G_n. The face and degeneracy homomorphisms in $F^G(X)$ are given in the canonical way by the universal property of the coproduct in the category of groups and group homomorphisms.

Lemma 5.2 ([7], Theorem 9, pp.88)**.** *Let G be a simplicial group and let X be a pointed simplicial set. Then the geometric realization $|F^G(X)|$ is homotopy equivalent to $\Omega(|X| \wedge B|G|)$.*

A generalization of this lemma by using fibrewise simplicial groups is given in [31].

Proof of Theorem 1.8. Since $\{x^{(\alpha)}|\alpha\in J\}$ is a set of generators for π, $F(x^{(\alpha)}|\alpha\in J)\to\pi$ is an epimorphism. Thus

$$F(\vee_{\alpha\in J}S^1)\cong F^{F(x^{(\alpha)}|\alpha\in J)}(S^1)\to F^{\pi}(S^1)$$

is an epimorphism. Hence

$$NF(\vee_{\alpha\in J}S^1)\to NF^{\pi}(S^1)$$

and

$$\mathcal{B}F(\vee_{\alpha\in J}S^1)\to \mathcal{B}F^{\pi}(S^1)$$

are epimorphisms. Now the assertion follows from Theorem 1.4, Lemma 5.2 and Proposition 2.14. □

Example 5.3. Let π be an abelian group. Then

$$\pi_3(\Sigma K(\pi,1))\cong\Gamma_2(\pi*\pi)/\Gamma_3(\pi*\pi)\cong\pi\otimes\pi,$$

where $G*H$ is the free product of the groups G and H.

Proof. Consider the simplicial group $F^{\pi}(S^1)$. Then $F^{\pi}(S^1)_1=\pi$ and $F^{\pi}(S^1)_2=\pi*\pi$. Since π is abelian, the commutator subgroup $\Gamma_2(\pi*\pi)$ is contained in the cycles

$\mathcal{Z}F^{\pi}(S^1)_2$. By Lemma 2.1, the subgroup $\Gamma_3(\pi * \pi)$ is contained in the boundaries $\mathcal{B}F^{\pi}(S^1)_2$. On the other hand, by Corollary 4.7 and Theorem 1.8, there are inclusions

$$\mathcal{Z}F^{\pi}(S^1)_2 \subseteq NF^{\pi}(S^1)_2 \subseteq \Gamma_2(\pi * \pi)$$

and

$$\mathcal{B}F^{\pi}(S^1)_2 \subseteq \Gamma_3(\pi * \pi).$$

Thus

$$\mathcal{Z}F^{\pi}(S^1)_2 = \Gamma_2(\pi * \pi)$$

and

$$\mathcal{B}F^{\pi}(S^1)_2 = \Gamma_3(\pi * \pi).$$

Hence the result. □

6. Applications

In this section, we study Cohen's K-construction. Our descriptions of the homotopy groups of the 3-sphere allow us to give a calculation for the K-construction of the 1-sphere.

Definition 6.1. Let X be a set. The group $\mathbf{K(X)}$ is the quotient group of the free group $F(X)$ by the normal subgroup generated by all of the commutators $[[\cdots[x_1, x_2], \cdots], x_t]$ with $x_i \in X$ and $x_i = x_j$ for some $1 \leq i < j \leq t$. Now let S be a pointed simplicial set. The simplicial group $\mathbf{K(S)}$ is defined to be the quotient simplicial group of $F(S)$ modulo the normal simplicial subgroup generated by all of the commutators $[[\cdots[x_1,x_2],\cdots],x_t]$ with $x_i\in S$ and $x_i=x_j$ for some $1\leq i<j\leq t$, where $F(S)$ is Milnor's $F(K)$-construction for the simplicial set S.

Definition 6.2. The group $\mathrm{Lie}(n)$ consists of the elements of weight n in the Lie algebra $\mathrm{Lie}(x_1,x_2,\cdots,x_n)$ which is the quotient Lie algebra of the free Lie algebra $L(x_1,x_2,\cdots,x_n)$ over $\mathbb{Z}$ by the two sided Lie ideal generated by the Lie elements $[[x_{i_1},x_{i_2}],\cdots,],x_{i_t}]$ with $i_l=i_k$ for some $1\leq l<k\leq t$.

The following lemmas are due to Fred Cohen.

Lemma 6.3 ([8]). $\Gamma_qK(x_1,x_2,\cdots,x_n)=\{1\}$ *for* $q\geq n$ *and*

$$\Gamma_nK(x_1,x_2,\cdots,x_n)\cong\mathrm{Lie}(n).$$

Lemma 6.4 ([8]). *In the group* $K(x_1,x_2,\cdots,x_n)$*, for each* $1\leq j\leq n$ *the normal subgroup generated by* x_j *is abelian for each* $1\leq j\leq n$*.*

Lemma 6.5 ([8]). *The set* $\{[[x_1,x_{\sigma(2)}],\cdots,x_{\sigma(n)}]|\sigma\in\Sigma_{n-1}\}$ *is a* $\mathbb{Z}$*-basis for* $\mathrm{Lie}(n)$*, where* Σ_{n-1} *acts on* $\{2,3,\cdots,n\}$ *by permutation.*

Recall that the simplicial 1-sphere S^1 is a free simplicial set generated by a 1-simplex σ. More precisely, $S^1_0=\{*\}$, $S^1_1=\{\sigma,*\}$ and $S^1_{n+1}=\{*,s_n\cdots\hat{s}_j\cdots s_0\sigma|0\leq j\leq n\}$. Let x_i denote $s_n\cdots\hat{s}_j\cdots s_0\sigma$. Then

Lemma 6.6. *The face functions* $d_i\colon S^1_{n+1} \to S^1_n$ *and the degenerate functions* $s_i\colon S^1_{n+1} \to S^1_{n+2}$ *are as follows:*

$$d_i x_j = \begin{cases} x_j & \text{for} \quad j < i, \\ x_{j-1} & \text{for} \quad j \geq i, \end{cases}$$

and

$$s_i x_j = \begin{cases} x_j & \text{for} \quad j < i, \\ x_{j+1} & \text{for} \quad j \geq i, \end{cases}$$

where we put $x_{-1} = *$ *and* $x_n = *$ *in* S^1_n.

Theorem 6.7. $\pi_n(K(S^1))$ *is isomorphic to* $\mathrm{Lie}(n)$.

Proof. Let $\pi\colon F(S^1) \to K(S^1)$ be the quotient homomorphism. Then $NF(S^1) \to NK(S^1)$ is an epimorphism. Recall from Corollary 4.6 that $NF(S^1)_{n+1}$ is generated by all of the commutators

$$[y_{i_1}, y_{i_2}, \cdots, y_{i_t}]$$

such that $\{i_1, i_2, \cdots, i_t\} = \{0, 1, \cdots, n\}$. by Corollary 4.6. Thus $NF(S^1)_{n+1} \subseteq \Gamma_{n+1} F(S^1)_{n+1}$ and therefore

$$NK(S^1)_{n+1} \subseteq \Gamma_{n+1} K(S^1)_{n+1}.$$

Observe that we have $K(S^1)_{n+1} \cong K(x_0, x_1, \cdots, x_n)$. Thus

$$\Gamma_{n+1} K(S^1)_{n+1} \cong \mathrm{Lie}(n+1)$$

and $\Gamma_{n+1}K(S^1)_n = \{1\}$. Thus

$$d_j|_{\Gamma_{n+1}K(S^1)_{n+1}}\colon \Gamma_{n+1}K(S^1)_{n+1} \to K(S^1)_n$$

is trivial for each $j \geq 0$. And therefore

$$NK(S^1)_{n+1} = \Gamma_{n+1}K(S^1)_{n+1} \cong \mathrm{Lie}(n+1)$$

with $d_0\colon NK(S^1)_{n+1} \to NK(S^1)_n$ is a trivial differential. The assertion follows. □

Remark 6.8. Let Σ_n act on $\mathrm{Lie}(n)$ by permuting letters. The $R(\Sigma_n)$-module $\mathrm{Lie}^R(n) = \mathrm{Lie}(n) \otimes R$ has important applications in representation Theory and mathematical physics.

The calculation of $\pi_n(K(S^1))$ for $n = 1, 2$ shows that $\pi_n(F(S^1)) \cong \pi_n(K(S^1))$ for $n = 1, 2$. Observe that $\pi_n(K(S^1)) \cong \mathrm{Lie}(n)$ is torsion-free by the theorem above, and $\pi_n(F(S^1))$ is a torsion group for $n \geq 3$. Thus we have

Corollary 6.9. *Let $\phi\colon F(S^1) \to K(S^1)$ be the quotient simplicial homomorphism. Then*

$$\phi_*\colon \pi_n(F(S^1)) \to \pi_n(K(S^1))$$

is an isomorphism for $n = 1, 2$ and zero for $n > 2$.

Now we consider the Samelson product in $\pi_*(K(S^1))$. As previously, let x_j denote $s_n \cdots \hat{s}_j \cdots s_0\sigma$ in S^1_{n+1}. The following lemma follows directly from Lemma 6.6.

Lemma 6.10. *Let $I=(i_1,i_2,\cdots,i_m)$ be a sequence with $i_1<i_2<\cdots<i_m$. Define $s_I=s_{i_m}\cdots s_{i_1}$ and $\bar{s}_I$ the order-preserving isomorphism. Then*

$$s_I\colon S^1_{n+1}-*\to S^1_{n+m+1}-*$$

is the composite

$$\{x_0,x_1,\cdots,x_n\}\xrightarrow{\bar{s}_I}\{x_0,x_1,\cdots,\hat{x}_{i_1},\cdots,\hat{x}_{i_2},\cdots,\hat{x}_{i_m},\cdots,x_{n+m}\}\hookrightarrow\{x_0,x_1,\cdots,x_{n+m}\}$$

Recall that, for $x\in\pi_n(G)$ and $y\in\pi_m(G)$, the Samelson product [10] is defined to be

$$<x,y>=\prod_{(a,b)}[s_bx,s_ay]^{\mathrm{sign}(a,b)},$$

where G is a simplicial group, $(a,b)=(a_1,\cdots,a_n,b_1,\cdots,b_m)$ runs over all permutations of $(0,1,\cdots,m+n-1)$ such that $a_1<a_2<\cdots<a_n$, $b_1<b_2<\cdots<b_m$, $\mathrm{sign}(a,b)$ is the sign of the permutation (a,b), the order of the product $\prod$ is right lexicographic on a and $s_a=s_{a_n}\cdots s_{a_1}$.

Proposition 6.11. *The Samelson product in $\pi_*(K(S^1))$ is as follows:*

$$<[x_{\sigma(0)},x_{\sigma(1)},\cdots,x_{\sigma(n)}],[x_{\tau(0)},x_{\tau(1)},\cdots,x_{\tau(m)}]>$$

$$=\sum_{(I,J)}\mathrm{sign}(I,J)[[x_{i_{\sigma(0)}},\cdots,x_{i_{\sigma(n)}}],[x_{j_{\tau(0)}},\cdots,x_{j_{\tau(m)}}]]$$

for the commutators

$$[x_{\sigma(0)},x_{\sigma(1)},\cdots,x_{\sigma(n)}]\in\pi_{n+1}(KS^1))\cong\mathrm{Lie}(n+1)$$

and

$$[x_{\tau(0)}, x_{\tau(1)}, \cdots, x_{\tau(m)}] \in \pi_{m+1}(K(S^1)) \cong \mathrm{Lie}(m+1)$$

where

$$(I, J) = (i_0, i_1, \cdots, i_n, j_0, j_1, \cdots, j_m)$$

runs over all permutations on $(0, 1, \cdots, m+n+1)$ *such that* $i_0 < i_1 < \cdots < i_n$, $j_0 < j_1 < \cdots < j_m$, $\mathrm{sign}(I, J)$ *is the sign of the permutation* (I, J), $\sigma \in \Sigma_{n+1}$ *acts on* $\{0, 1, \cdots, n\}$ *and* $\tau \in \Sigma_{m+1}$ *acts on* $\{0, 1, \cdots, m\}$

Proof. Observe that we have $\{x_0, \cdots, \hat{x}_{j_0}, \cdots, \hat{x}_{j_m}, \cdots, x_{n+m+1}\} = \{x_{i_0}, \cdots, x_{i_n}\}$, and

$$\bar{s}_J\colon \{x_0, \cdots, x_n\} \to \{x_{i_0}, \cdots, x_{i_n}\}$$

is an ordered isomorphism. The assertion follows because

$$s_J([x_{\sigma(0)}, x_{\sigma(1)}, \cdots, x_{\sigma(n)}]) = [x_{i_{\sigma(0)}}, \cdots, x_{i_{\sigma(n)}}]$$

and

$$s_I([x_{\tau(0)}, x_{\tau(1)}, \cdots, x_{\tau(m)}]) = [x_{j_{\tau(0)}}, \cdots, x_{j_{\tau(m)}}].$$

□

Let $\mathrm{Lie}^R = \bigoplus_{n=1}^{\infty} \mathrm{Lie}(n) \otimes R$ be a graded module over a ring R with the graded pairing defined in Proposition 6.11. Then Lie^R is a graded Lie algebra.

Corollary 6.12. *There is an isomorphism of Hopf algebras*

$$H_*(K(S^1);\mathbb{Q}) \cong U(\mathrm{Lie}^{\mathbb{Q}}),$$

where $U(L)$ is the universal enveloping algebra of the Lie algebra L.

Recall that a simplicial group G is minimal if and only if the Moore chain complex NG is minimal [11].

Theorem 6.13. *The simplicial group $K(S^1)$ is the universal minimal simplicial quotient simplicial group of $F(S^1)$ in the following sense.*

1) *$K(S^1)$ is a minimal simplicial group.*
2) *Let G be a minimal simplicial group. Then every simplicial homomorphism $f\colon F(S^1) \to G$ factors uniquely through $K(S^1)$.*

Proof. By inspecting the proof of Theorem 6.7, $K(S^1)$ is a minimal simplicial group. The assertion (2) follows from the following lemma.

Lemma 6.14. *$K(S^1)$ is the quotient simplicial group of $F(S^1)$ by the normal simplicial subgroup generated by the boundaries.*

Proof. Let H denote the kernel of the quotient map $p\colon F(S^1) \to K(S^1)$ and let $\overline{B}$ denote the normal simplicial subgroup of $F(S^1)$ generated by the boundaries $\mathcal{B}F(S^1)$. Since $K(S^1)$ is a minimal simplicial group, $\overline{B}$ is contained in H. Let Q denote the quotient simplicial group $F(S^1)/\overline{B}$. Then Q is a minimal simplicial group. By Corollary 2.12, there is a central extension

$$0 \to K(\pi_{n+1}Q, n+1) \to P_{n+1}Q \to P_nQ \to 0,$$

where P_nQ is the nth stage of the Moore-Postnikov system of Q. Since

$$P_1Q = K(\pi_1(Q), 1) = K(\mathbb{Z}, 1),$$

$\Gamma_{n+2}P_{n+1}Q = 1$ by induction on n. Because $Q_{n+1} \cong (P_{n+1}Q)_{n+1}$, $\Gamma_{n+2}Q_{n+1} = 1$. Now we show that H is contained in $\overline{B}$ by induction on the dimension, starting with $H_1 = \overline{B}_1 = 1$. Suppose that $H_n \subseteq \overline{B}_n$ with $n > 0$. Note $F(S^1)_{n+1} = F(x_0, \cdots, x_n)$ and $K(S^1)_{n+1} = K(x_0, \cdots, x_n)$. Thus H_{n+1} is the normal subgroup of $F(x_0, \cdots, x_n)$ generated by the commutators

$$[[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}]$$

such that $i_p \neq i_q$ for $p < q$. Now let w be such a commutator.

If $t \geq n+1$, then $w \in \Gamma_{n+2}F(x_0, \cdots, x_n)$. Thus $w \in \overline{B}_{n+1}$ since $\Gamma_{n+2}Q_{n+1} = 1$.

If $t < n+1$, then there exists an index $j \in \{0, 1, \cdots, n\} - \{i_1, \cdots, i_t\}$. Recall that

$$s_i x_k = \begin{cases} x_k & k < i, \\ x_{k+1} & k \geq i, \end{cases}$$

for $x_k = s_{n-1} \cdots \hat{s}_k \cdots s_0 \sigma \in S^1_n$. Thus

$$s_j[[x_{i'_1}, x_{i'_2}], \cdots, x_{i'_t}], x_{i'_t}] = [[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}],$$

with $i'_k = i_k$ if $i_k < j$ and $i'_k = i_k - 1$ if $i_k > j$. By induction,

$$[[x_{i'_1}, x_{i'_2}], \cdots, x_{i'_t}], x_{i'_t}] \in \overline{B}.$$

Thus $w = [[x_{i_1}, x_{i_2}], \cdots, x_{i_t}], x_{i_t}] \in \overline{B}$. The induction is finished and the assertion follows. □

The simplicial group $K(S^1)$ is homotopy equivalent to a product of the Eilenberg-MacLane spaces with a different multiplication.

Proposition 6.15. $\Omega K(S^1)$ *with the loop multiplication is an abelian simplicial group. Therefore, as a simplicial set,* $K(S^1)$ *is homotopy equivalent to a product of the Eilenberg-MacLane spaces.*

Proof. Consider

$$d_0 \colon K(x_0, x_1, \cdots, x_n) \to K(x_0, x_1, \cdots, x_{n-1}),$$

$d_0(x_0) = 1$ and $d_0(x_j) = x_{j-1}$. Thus $\mathrm{Ker}(d_0) \cap K_{n+1}(S^1) \cong \langle x_0 \rangle$ is the normal subgroup generated by x_0 which is abelian by Lemma 6.4. Hence the result. □

Remark 6.16. Let $BK(S^1)$ be the classifying space of $K(S^1)$. Then $BK(S^1)$ is **NOT** a product of Eilenberg-Maclane space by Proposition 6.11, where the Whitehead product on $\pi_*(BK(S^1))$ is given by Proposition 6.11 and so the Gerstenhaber algebra $\pi_*(BK(S^1))$ is determined. But the cohomology of $BK(S^1)$ is still unknown even in rational case.

Finally, we give some applications of $K(S^1)$ to minimal simplicial groups. Let G be a simplicial group. We write G^{ab} for the abelianization of G.

Proposition 6.17. *Let G be a minimal simplicial group. If G^{ab} is a minimal simplicial group $K(\pi, 1)$ for a cyclic group π, then G is homotopy equivalent to a product of Eilenberg-MacLane spaces.*

Proof. Notice the equality $G_1 = \pi$. Let x be a generator for the cyclic group π and let $f_x\colon S^1 \to G$ be a representative map of x, that is, $f_x(\sigma) = x$. Let $g\colon F(S^1) \to G$ be the simplicial homomorphism induced by f_x. We need a lemma.

Lemma 6.18. *The simplicial homomorphism $g\colon F(S^1) \to G$ is a simplicial surjection.*

Proof. It suffices to show that the simplicial subgroup H of G generated by G_1 is G itself. This is given by induction on the dimensions starting with $H_1 = G_1$. Suppose that $H_{n-1} = G_{n-1}$ with $n > 1$. By [27, Proposition 1, pp.6], G_n is generated by the degenerate images of lower order Moore chain complex terms and NG_n. Thus G_n is generated by NG_n and H_n by induction. Since G is a minimal simplicial group, $NG_n = \mathcal{Z}G_n$, the cycles. By Lemma 2.1, $\mathcal{Z}G_n$ is contained in the center of G_n. Thus H_n is a normal subgroup of G_n and the composite

$$\phi\colon \pi_n G \cong \mathcal{Z}G_n \to G_n \to G_n/H_n$$

is an epimorphism. Thus G_n/H_n is an abelian group and so the quotient homomorphism $G_n \to G_n/H_n$ factors through G_n^{ab}. Since $G^{\mathrm{ab}} = K(\pi, 1)$, we have $N(G^{\mathrm{ab}})_n = 1$ for $n > 1$ and so the composite $\mathcal{Z}G_n = NG_n \to G_n \to G_n^{\mathrm{ab}}$ is trivial. Thus the homomorphism $\phi\colon \pi_n G \to G_n/H_n$ is trivial and therefore G_n/H_n is trivial. Hence $H_n = G_n$ as required. □

Continuation of the proof of Proposition 6.17. We find that G is minimal. The simplicial epimorphism $g\colon F(S^1) \to G$ factors through $K(S^1)$ by Theorem 6.13. By Proposition 6.15, $\Omega K(S^1)$ is an abelian simplicial group. Thus ΩG is an abelian simplicial group and so G is homotopy equivalent to a product of Eilenberg-MacLane spaces, which is the assertion. □

The following counter-example for minimal simplicial groups is due to J. W. Milnor (unpublished).

Proposition 6.19. *Let $S^{n+1}[n+1, n+2, n+3]$ be the 3-stage Postnikov system obtained by taking the first three nontrivial homotopy groups of S^{n+1}. Suppose $n \geq 1$. Then $\Omega(S^{n+1}[n+1, n+2, n+3])$ does not have the homotopy type of a minimal simplicial group.*

Proof. Suppose that there were a minimal simplicial group G such that

$$G \simeq \Omega(S^{n+1}[n+1, n+2, n+3]).$$

Let $f\colon F(S^1)\to\Omega^{n-1}G$ be a simplicial homomorphism such that $f(\sigma)$ is a generator of $(\Omega^{n-1}G)_1\cong G_n\cong\mathbb{Z}$. Then $f_*\colon\pi_3(F(S^1))=\mathbb{Z}/2\to\pi_3(\Omega^{n-1}G)=\mathbb{Z}/2$ is an isomorphism. Observe that $\Omega^{n-1}G$ is also a minimal simplicial group. The simplicial homomorphism $f\colon F(S^1)\to\Omega^{n-1}G$ factors through $K(S^1)$. Observe that

$$\pi_3(K(S^1))\cong \mathrm{Lie}(3)\cong\mathbb{Z}\oplus\mathbb{Z}.$$

It follows that $\mathbb{Z}/2$ is a summand of $\mathbb{Z}\oplus\mathbb{Z}$, which is the contradiction. Hence the assertion follows. $\square$

More examples and counter-examples for minimal simplicial groups will be given in [32]. It is known that there are many counter-examples of two-stage Postnikov systems for minimal simplicial groups [32].

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