be the functorial map defined by

$$\tilde{f}_W = \alpha_n \colon T_n(W) = W^{\otimes n} \to T_n(W) = W^{\otimes n}$$

for $n \geq 0$. Then one can check that:

1) $\tilde{f}$ is a functorial map of coalgebras over $\mathbf{K}$;

(1). $\tilde{f}$ is an extension of f.

The assertion follows. □

8. The Functor $A^{\min}$ over a Field of Characteristic $p > 0$

In this section, the ground field $\mathbf{k}$ is of characteristic $p > 0$.

8.1. An upper bound on the size of $A^{\min}(V)$.

Lemma 8.1. *Let $m > 1$ such that $(m, p) = 1$. Suppose that the polynomial*

$$x^m - 1$$

splits in $\mathbf{k}[x]$. Then there exists a functorial map of coalgebras

$$\phi_V \colon T(V) \to T(V)$$

for any $\mathbf{k}$-module V such that:

(1). $\phi_V \circ \phi_V = \phi_V \colon T(V) \to T(V)$;

(2). Let $\alpha \in P(T_n(V))$ be a primitive element of tensor length n. Then

$$\phi_V(\alpha) = \begin{cases} \alpha & \text{if} \quad n = m, 2m, 3m, \dots ; \\ \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\zeta \in \mathbf{k}$ be a primitive mth root of 1. Let

$$T(\zeta) \colon T(V) \to T(V)$$

be the algebra map determined by

$$T(\zeta)(v) = \zeta v$$

for $x \in V$. Then $T(\zeta) \colon T(V) \to T(V)$ is a functorial map of Hopf algebras. Let $\chi \colon T(V) \to T(V)$ be the conjugation and let

$$f_V = T(\zeta) * \chi \colon T(V) \to T(V)$$

be the convolution product. Then $f_V \colon T(V) \to T(V)$ is a functorial map of coalgebras.

Let $\alpha \in P(T_n(V))$ be a primitive element of tensor length n. Then

$$f_V(\alpha) = (\zeta^n - 1)\alpha.$$

Let $D(V)$ be the colimit

$$D(V)=\operatorname{colim}_{f_V} T(V).$$

Let $r_V\colon T(V)\to D(V)$ be the canonical map to the colimit. By Theorem 4.5, $D(V)$ is a natural coalgebra retract of $T(V)$ and the functorial map r_V is a coalgebra retraction. Notice that $\zeta^n-1=0$ if and only if $n=mt$ for some $t\geq 0$. Thus

$$r_V\colon P(T_n(V))\to P(D_n(V))$$

is an isomorphism if n is not divisible by m and is zero if n is divisible by m.

By Lemma 5.2, there is a natural coalgebra decomposition

$$T(V)\cong B(V)\otimes D(V).$$

Let $r'_V\colon T(V)\to B(V)$ be a functorial coalgebra retraction and let $j_V\colon B(V)\to T(V)$ be the coalgebra injection. Let $\phi_V\colon T(V)\to T(V)$ be the composite

$$T(V)\xrightarrow{r'_V} B(V)\xrightarrow{j_V} T(V).$$

Then ϕ_V is an idempotent functorial coalgebra map. Let $\alpha\in P(T_n(V))$ be a primitive element of tensor length n. Notice that

$$P(T_n(V))\cong P(B_n(V))$$

if n is divisible by m and

$$P(B_n(V))=0$$

if n is not divisible by m. The assertion follows. □

Corollary 8.2. *Let $m>1$ such that $(m,p)=1$. Suppose that the*

$$x^m-1$$

splits in $\mathbf{k}[x]$. Let V be any $\mathbf{k}$-module. Then in tensor length mk we have

$$P(A^{\min}_{mk}(V))=0$$

for $k\geq 1$.

Proof. By Lemma 8.1, one has a natural coalgebra decomposition

$$T(V)\cong B(V)\otimes D(V)$$

such that $P(D_{mk}(V))=0$ for $k\geq 1$. Notice that there is a functorial inclusion

$$V\subseteq D(V).$$

Thus $A^{\min}(V)$ is a natural coalgebra retract of $D(V)$ by Theorem 4.12. The assertion follows. □

Theorem 8.3. *If the ground field* $\mathbf{k}$ *is of characteristic* p*, then*

$$P(A_n^{\min}(V)) = 0$$

if n *is not a power of* p*.*

Proof. Let m be a positive integer with $m > 1$ and $(m, p) = 1$. It suffices to show the following statement:

Statement: $P(A_{mk}^{\min}(V)) = 0$ for $k \geq 1$.

The proof of this statement will be given by induction on k. Let $k = 1$. There is a natural coalgebra decomposition

$$T(V) \cong T(L_m(V)) \otimes D(V)$$

for some natural subcoalgebra $D(V) \subseteq T(V)$. (See Corollary 6.7 or [9].) Notice that there is a functorial inclusion $V \subseteq D(V)$. Thus $A^{\min}(V)$ is a natural coalgebra retract of $D(V)$. Observe that the primitive submodule $P(T(V))$ is the free restricted Lie algebra $L^{\mathrm{res}}(V)$. Thus $P(T_m(V)) = L_m(V)$ and so $P(D_m(V)) = 0$. Therefore $P(A_m^{\min}(V)) = 0$.

Now suppose that

$$P(A_{mt}^{\min}(V)) = 0$$

for $t < k$ with $k > 1$ and let $n = mk$.

Goal 1: *The overall plan is to construct a natural coalgebra retract* $\overline{\overline{D}}(V)$ *such that* $V \subseteq \overline{\overline{D}}(V)$ *but* $P(\overline{\overline{D}}_{mk}(V)) = 0$.

By Theorem 6.5, there is a natural coalgebra decomposition

$$T(V) \cong \otimes_{j=2}^{\infty} A^{\min}(V; L_j^{\max}) \otimes A^{\min}(V).$$

Thus there is a functorial isomorphism of $\mathbf{k}$-modules

$$P(T(V)) \cong \oplus_{j=2}^{\infty} P(A^{\min}(V; L_j^{\max})) \oplus P(A^{\min}(V)).$$

Let

$$B(V) = \otimes_{j=2}^{n-1} A^{\min}(V; L_j^{\max})$$

be the partial tensor products in the natural decomposition for $T(V)$. By induction, $P(A_{mt}^{\min}(V)) = 0$ for $t < k$. Notice that $P(A_{mt}^{\min}(V; L_j^{\max})) = 0$ for $j \geq n = mk > mt$ by Corollary 6.4. Thus

$$P(T_{mt}(V)) \cong P(B_{mt}(V))$$

for $t < k$. Let $r_V \colon T(V) \to B(V)$ be a functorial coalgebra retraction, let $j_V \colon B(V) \to T(V)$ be a functorial coalgebra injection and let f_V denote the composite

$$f_V = j_V \circ r_V \colon T(V) \to T(V).$$

Then $f_V\colon T(V)\to T(V)$ is an idempotent functorial coalgebra map with a functorial isomorphism

$$\mathrm{Im}(f_V\colon T(V)\to T(V))\cong B(V).$$

Goal 2: *The goal in the following is to change the ground field to one containing the mth roots of 1.*

Let $\mathbf{K}$ be an extension field of the ground field $\mathbf{k}$ such that the polynomial

$$x^m-1$$

splits in $\mathbf{K}[x]$. By Lemma 7.8, there exists a functorial map of $\mathbf{K}$-coalgebras

$$\tilde f_W\colon T(W)\to T(W)$$

such that $\tilde f$ is an extension of f. Let $\tilde B(W)$ denote the colimit

$$\tilde B(W)=\mathrm{colim}_{\tilde f_W}T(W).$$

By Lemmas 5.2, 5.3 and Theorem 4.5, there is a natural $\mathbf{K}$-coalgebra decomposition

$$T(W)\cong\tilde B(W)\otimes_{\mathbf{K}}D(W)$$

for any finite dimensional $\mathbf{K}$-module W and so for any $\mathbf{K}$-module W, where $D(W)$ is given by the cotensor product

$$D(W)=\mathbf{K}\,\square_{\tilde B(W)}T(W).$$

Let $A^{\min,\mathbf{K}}$ denote the functor $A^{\min}$ over the extension field $\mathbf{K}$. Notice that there is a natural inclusion of $\mathbf{K}$-modules

$$W\subseteq D(W).$$

Thus $A^{\min,\mathbf{K}}(W)$ is a natural coalgebra retract of $D(W)$ and so one has a further natural $\mathbf{K}$-coalgebra decomposition

$$T(W)\cong\tilde B(W)\otimes D'(W)\otimes A^{\min,\mathbf{K}}(W)$$

for any $\mathbf{K}$-module W, where $D'(W)$ is given by the cotensor product

$$D'(W)=\mathbf{K}\,\square_{A^{\min,\mathbf{K}}(W)}D(W).$$

Goal 3: *The goal in the following is to give a further decomposition of $D'(W)$ by using Lemma 8.1.*

Let $r'_W\colon T(W)\to D'(W)$ be a functorial $\mathbf{K}$-coalgebra retraction and let $j'_W\colon D'(W)\to T(W)$ be a functorial $\mathbf{K}$-coalgebra injection. By Lemma 8.1, there exists an idempotent functorial map of $\mathbf{K}$-coalgebras

$$\phi_W\colon T(W)\to T(W)$$

such that the induced map on primitive elements

$$\phi_W\colon P(T_n(W)) \to P(T_n(W))$$

is the identity if $n = mt$ for some $t \geq 1$ and is zero if n is not divisible by m. Let the map $\phi'_W\colon T(W) \to T(W)$ denote the composite

$$T(W) \xrightarrow{r'_W} D'(W) \xrightarrow{j'_W} T(W) \xrightarrow{\phi_W} T(W).$$

Let $D''(W)$ denote the colimit

$$D''(W) = \operatorname{colim}_{\phi'_W} T(W).$$

Then $D''(W)$ is a natural $\mathbf{K}$-coalgebra retract of $D'(W)$. By Lemma 5.2, there is a natural $\mathbf{K}$-coalgebra decomposition

$$D'(W) \cong D''(W) \otimes D'''(W).$$

Notice that by Lemma 5.3

$$D''(W) \cong \mathbf{K}\,\square_{D'''(W)}\, D'(W)$$

for any finite dimensional $\mathbf{K}$-module W and so for any $\mathbf{K}$-module W. Thus we may view $D''(W) = \mathbf{K}\,\square_{D'''(W)}\, D'(W)$ as a subcoalgebra of

$$D'(W) \subseteq D(W) \subseteq T(W).$$

Consider the further natural $\mathbf{K}$-coalgebra decomposition

$$T(W) \cong \tilde{B}(W) \otimes D''(W) \otimes D'''(W) \otimes A^{\min,\mathbf{K}}(W).$$

There is a functorial isomorphism of $\mathbf{K}$-modules

$$P(T_l(W)) \cong P(\tilde{B}_l(W)) \oplus P(D''_l(W)) \oplus P(D'''_l(W)) \oplus P(A^{\min,\mathbf{K}}_l(W))$$

for any $l \geq 1$.

Goal 4: *The goal in the following is to show that $D''_j(W) = 0$ for $0 < j < mk$ and that $D''_{mk}(W)$ is a functorial $\mathbf{K}$-submodule of $L^{\mathbf{K}}_{mk}(W)$, where we write $L^{\mathbf{K}}$ for the free Lie algebra functor over the field $\mathbf{K}$.*

Now we determine the primitive elements $P(D''_l(W))$ up to tensor length $n = mk$. Let $\{x_i\}$ be a $\mathbf{K}$-basis for W and let W' be the $\mathbf{k}$-module generated by $\{x_i\}$. Then there is an isomorphism of $\mathbf{K}$-modules

$$\mathbf{K} \otimes_{\mathbf{k}} W' \cong W.$$

As noted earlier, there is an isomorphism of $\mathbf{k}$-modules

$$P(T_{mt}(W')) \cong P(B_{mt}(W'))$$

for $t < k$ and so, by Lemma 7.8, there is an isomorphism of $\mathbf{K}$-modules

$$P(T_{mt}(\mathbf{K} \otimes_{\mathbf{k}} W') \cong P(\tilde{B}_{mt}(\mathbf{K} \otimes_{\mathbf{k}} W')) = \mathbf{K} \otimes_{\mathbf{k}} P(B_{mt}(W'))$$

for $t < k$. In particular there is an isomorphism of $\mathbf{K}$-modules

$$P(T_{mt}(W)) \cong P(\tilde{B}_{mt}(W))$$

for $t < k$ and so

$$P(D''_{mt}(W)) \subseteq P(D''_{mt}(W)) \oplus P(D'''_{mt}(W)) \oplus P(A^{min,\mathbf{K}}_{mt}(W)) = 0$$

for $t < k$. Notice that $D''(W)$ is a natural $\mathbf{K}$-coalgebra retract of

$$\mathrm{Im}(\phi_W \colon T(W) \to T(W)) \cong \mathrm{colim}_{\phi_W} T(W).$$

Thus

$$P(D''_l(W)) \subseteq \mathrm{colim}_{\phi_W} P(T_l(W)) = 0$$

if l is not divisible by m. Therefore

$$P(D''_j(W)) = 0$$

for $j < mk$. Thus

$$D''_j(W) = 0$$

for $0 < j < mk$ and so one has

$$D''_{mk}(W) = P(D''_{mk}(W)).$$

Notice that the map

$$\phi_W \colon P(T_{mk}(W)) \to P(T_{mk}(W))$$

is the identity map. Thus there is a functorial isomorphism of $\mathbf{K}$-modules

$$P(D'_{mk}(W)) \cong P(D''_{mk}(W)) = D''_{mk}(W)$$

and so

$$P(D'''_{mk}(W)) = 0$$

for any $\mathbf{K}$-module W. By Corollary 8.2, one has

$$P(A^{\mathrm{min},\mathbf{K}}_{mk}(W)) = 0$$

for any $\mathbf{K}$-module W. Thus there are functorial isomorphisms of $\mathbf{K}$-modules

$$P(T_{mk}(W)) \cong P(\tilde{B}_{mk}(W)) \oplus P(D''_{mk}(W)) = P(\tilde{B}_{mk}(W)) \oplus D''_{mk}(W);$$

$$P(D_{mk}(W)) = P(D'_{mk}(W)) = P(D''_{mk}(W)) = D''_{mk}(W).$$

Let $r''_W \colon T(W) \to D''(W)$ be a functorial $\mathbf{K}$-coalgebra retraction, let $j''_W \colon D''(W) \to T(W)$ be a functorial $\mathbf{K}$-coalgebra inclusion and let f''_W denote the composite

$$f''_W = j''_W \circ r''_W \colon T(W) \to T(W).$$

Notice that f''_W is a functorial map of $\mathbf{K}$-coalgebras with

$$f''_W = 0\colon T_j(W) \to T_j(W)$$

for $0 < j < mk$. By Corollary 2.9, there exists an element $\lambda \in \mathbf{K}(S_{mk})$ such that

$$f''_W|_{T_{mk}(W)} = \beta_{mk} \circ \lambda\colon T_{mk}(W) = W^{\otimes mk} \to T_{mk}(W) = W^{\otimes mk}.$$

Thus $D''_{mk}(W)$ is a functorial $\mathbf{K}$-submodule of $L^{\mathbf{K}}_{mk}(W)$ for any $\mathbf{K}$-module W, where we write $L^{\mathbf{K}}$ for the free Lie algebra functor over the field $\mathbf{K}$.

Goal 5: *The goal in the following is to construct the functor $\overline{\overline{D}}$ by using information about the functor D'' and to finish the proof.*

Now let V be any $\mathbf{k}$-module. Let $\bar{D}(V)$ denote the cotensor product

$$\bar{D}(V) = \mathbf{k} \Box_{B(V)} T(V).$$

Let $W = \mathbf{K} \otimes_{\mathbf{k}} V$. Then

$$D(W) = \mathbf{K} \otimes_{\mathbf{k}} \bar{D}(V).$$

Thus $D''(W)$ is a subcoalgebra of $\mathbf{K} \otimes \bar{D}(V)$. Recall that

$$D''_{mk}(W) = P(D_{mk}(W)).$$

Thus

$$D''_{mk}(W) = \mathbf{K} \otimes_{\mathbf{k}} P(\bar{D}_{mk}(V)) \subseteq \mathbf{K} \otimes_{\mathbf{k}} L^{\mathrm{res}}_{mk}(V).$$

Recall that

$$D''_{mk}(W) \subseteq L^{\mathbf{K}}_{mk}(W) = \mathbf{K} \otimes_{\mathbf{k}} L_{mk}(V).$$

Thus there are no restricted elements in $P(\bar{D}_{mk}(V))$ or equivalently

$$P(\bar{D}_{mk}(V)) \subseteq L_{mk}(V).$$

Let $\gamma(P(\bar{D}_{mk}))$ be the associated $\mathbf{k}(S_{mk})$-module of the functor $P(\bar{D}_{mk})$. (See Proposition 7.6.) Notice that $D''_{mk}(W)$ is a functorial retract of $W^{\otimes n}$ for any $\mathbf{K}$-module W. Thus

$$\mathbf{K} \otimes_{\mathbf{k}} \gamma(P(\bar{D}_{mk}))) = \gamma(D''_{mk})$$

is a projective $\mathbf{K}(S_{mk})$-module. By Corollary 7.7, there exists an element $\bar{\lambda} \in \mathbf{k}(S_{mk})$ such that

(1). $\beta_{mk} \circ \bar{\lambda} \circ \beta_{mk} \circ \bar{\lambda} = \beta_{mk} \circ \bar{\lambda}\colon V^{\otimes mk} \to V^{\otimes mk}$ for any $\mathbf{k}$-module V;
(2). there is a functorial isomorphism of $\mathbf{k}$-modules

$$P(\bar{D}_{mk}(V)) = \mathrm{Im}(\beta_{mk} \circ \bar{\lambda}\colon V^{\otimes mk} \to V^{\otimes mk}) \cong \mathrm{colim}_{\beta_{mk} \circ \bar{\lambda}} V^{\otimes mk}$$

for any $\mathbf{k}$-module V.

We write $\bar{L}_{mk}$ for $P(\bar{D}_{mk})$. Then $A^{\min}(V;\bar{L}_{mk})$ is a natural coalgebra retract of $\bar{D}(V)$ and so there is a natural decomposition of connected graded coalgebras

$$\bar{D}(V) \cong A^{min}(V;\bar{L}_{mk}) \otimes \overline{\overline{D}}(V).$$

By Lemma 6.2, $\bar{L}_{mk}(V)$ is a natural retract of $L^{\max}_{mk}(V)$ and so $A^{\min}(V;\bar{L}_{mk})$ is a natural coalgebra retract of $A^{\min}(V;L^{\max}_{mk})$. By Corollary 6.4, we have

$$A^{\min}_j(V;\bar{L}_{mk}) = 0$$

for $0 < j < mk$, using Proposition 6.1 and Lemma 6.3. Thus there is a functorial inclusion $V \subseteq \overline{\overline{D}}(V)$ and so $A^{\min}(V)$ is a functorial retract of $\overline{\overline{D}}(V)$. Notice that

$$P(\bar{D}_{mk}(V)) = \bar{L}_{mk}(V) = P(A^{\min}_{mk}(V;\bar{L}_{mk})).$$

Thus $P(\overline{\overline{D}}_{mk}(V)) = 0$ and so

$$P(A^{\min}_{mk}(V)) = 0$$

for any **k**-module V. The induction is finished and the assertion follows. □

Recall that there is a functorial subHopf algebra $B^{\max}(V)$ such that there is a natural coalgebra decomposition $T(V) \cong B^{\max}(V) \otimes A^{\min}(V)$. (See Proposition 6.1.)

Corollary 8.4. *If n is not a power of p, then*

$$L_n(V) \subseteq B^{\max}(V).$$

Corollary 8.5. *$A^{\min}(V)$ is a natural coalgebra retract of the quotient of $T(V)$ modulo the right ideal generated by $\{L_n(V) \mid n$ not power of $p\}$.*

This gives an upper bound on the size of $A^{\min}$.

8.2. **Some general theorems on natural coalgebra retracts of $T(V)$.**

Theorem 8.6. *Let $B(V)$ be a functorial sub coalgebra of $T(V)$. Suppose that*

1) *There is a functorial multiplication $B(V) \otimes B(V) \to B(V)$ such that $B(V)$ is a quasi-Hopf algebra;*
2) *B is a retract of T as functors from* **k***-modules to* **k***modules.*

Then B is a retract of T as functors from **k***-modules to* **k***-coalgebras.*

Remark 8.7. *The inclusion $B(V)$ is only assumed to be a map of coalgebras. That is the inclusion might not be a map of algebras. Also we do not assume that the multiplication on $B(V)$ is associative.*

Proof. Let $B_m(V) = B(V) \cap T_m(V)$ and let $J_nB(V) = B(V) \cap J_n(V)$. Then $J_nB(V)$ is a functorial sub coalgebra of $J_n(V)$. Let $r\colon T(V) \to B(V)$ be a **k**-linear retraction. By Corollary 2.2, $r|_{J_n(V)}$ maps $J_n(V)$ onto $J_nB(V)$ and J_nB is a retract of J_n as functors from **k**-modules to **k**-modules. By Lemma 2.6, there is a functorial isomorphism of Hopf algebras $h_n\colon T(\bar{J}_n(V)) \to T(\bigoplus_{k=1}^n V^{\otimes k})$, where the Hopf algebra structure on $T(\bigoplus_{k=1}^n V^{\otimes k})$ is the one in which the elements in $\bigoplus_{k=1}^n V^{\otimes k}$ are primitive. Let $j_n\colon J_nB(V) \to J_n(V)$ be the inclusion and let $\bar{J}_nB(V) = J_nB(V)/J_0B(V)$. Let $\phi_n\colon T(\bar{J}_nB(V)) \to T(\bigoplus_{k=1}^n B_k(V))$ be the composite

$$T(\bar{J}_nB(V)) \xrightarrow{T(j_n)} T(\bar{J}_n(V)) \underset{\cong}{\xrightarrow{h_n}} T\Big(\bigoplus_{k=1}^{n} V^{\otimes k}\Big) \xrightarrow{T(\oplus_{k=1}^n r|_{T_k(V)})} T\Big(\bigoplus_{k=1}^{n} B_k(V)\Big).$$

Then ϕ_n is a map of Hopf algebras for each n. By considering the commutative diagram of Hopf algebras

$$\begin{array}{ccccccc}
T(B_n(V)) & \xrightarrow{T(j|_{B_n(V)})} & T(V^{\otimes n}) & = & T(V^{\otimes n}) & \xrightarrow{T(r|_{T_n(V)})} & T(B_n(V)) \\
\uparrow & & \uparrow & & \uparrow \text{proj} & & \text{proj} \uparrow \\
T(\bar{J}_nB(V)) & \longrightarrow & T(\bar{J}_n(V)) & \underset{\cong}{\xrightarrow{h_n}} & T\Big(\bigoplus_{k=1}^{n} V^{\otimes k}\Big) & \longrightarrow & T\Big(\bigoplus_{k=1}^{n} B_k(V)\Big) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
T(\bar{J}_{n-1}B(V)) & \longrightarrow & T(\bar{J}_{n-1}(V)) & \underset{\cong}{\xrightarrow{h_{n-1}}} & T\Big(\bigoplus_{k=1}^{n-1} V^{\otimes k}\Big) & \longrightarrow & T\Big(\bigoplus_{k=1}^{n-1} B_k(V)\Big)
\end{array}$$

we see that there is a commutative diagram of **k**-modules

$$\begin{array}{ccc}
B_n(V) & = & B_n(V) \\
\uparrow & & \uparrow \\
\bar{J}_nB(V) & \xrightarrow{Q(\phi_n)} & \bigoplus_{k=1}^{n} B_k(V) \\
\uparrow & & \uparrow \\
\bar{J}_{n-1}B(V) & \xrightarrow{Q(\phi_{n-1})} & \bigoplus_{k=1}^{n-1} B_k(V).
\end{array}$$

Notice that $Q(\phi_1)\colon \bar{J}_1(B(V)) \to B_1(V)$ is an isomorphism. Thus $Q(\phi_n)\colon \bar{J}_nB(V) \to \bigoplus_{k=1}^n B_k(V)$ is an isomorphism by induction on n. It follows that $\phi_n\colon T(\bar{J}_nB(V)) \to T(\bigoplus_{k=1}^n B_k(V))$ is an epimorphism for each V. Notice that $T(\bar{J}_nB(V))$ has the same Poincaré series as $T(\bigoplus_{k=1}^n B_k(V))$ if $\dim(V) < \infty$. Thus ϕ_n is an isomorphism if $\dim(V) < \infty$ and so ϕ_n is an isomorphism for any **k**-module V. Therefore the composite

$$\phi\colon T(\overline{B(V)}) \overset{T(j)}{\hookrightarrow} T(\overline{T(V)}) \xrightarrow[\cong]{h} T(\bigoplus_{n=1}^{\infty} V^{\otimes n}) \xrightarrow{T(r)} T(\bigoplus_{n=1}^{\infty} B_n(V))$$

is an isomorphism of Hopf algebras, where $\overline{B(V)} = B(V)/B_0(V)$ and $\overline{T(V)} = T(V)/T_0(V)$.

By hypothesis $B(V)$ is a functorial quasi-Hopf algebra. Let $f_V\colon T(\overline{B(V)}) \to B(V)$ be defined by

$$f(x_1x_2\cdots x_n) = (\cdots(x_1 \cdot x_2)\cdots)\cdot x_n$$

for $x_j \in \overline{B(V)} = I(B(V))$. Then it is routine to check that f_V is a map of coalgebras. Let $E\colon B(V) \to T(\overline{B(V)})$ be the canonical inclusion. Notice that the composite

$$B(V) \overset{E}{\hookrightarrow} T(\overline{B(V)}) \xrightarrow[\cong]{\phi} T(\bigoplus_{n=1}^{\infty} B_n(V)) \xrightarrow[\cong]{\phi^{-1}} T(\overline{B(V)}) \xrightarrow{f_V} B(V)$$

is the identity map of $B(V)$ and the composite $\phi \circ E\colon B(V) \to T(\bigoplus_{n=1}^{\infty} B_n(V))$ factors through $T(V)$. Thus $B(V)$ is a functorial coalgebra retract of $T(V)$, which is the assertion. □

If $B(V)$ is a functorial sub Hopf algebra of $T(V)$, we will show by example that there is no **functorial** isomorphism of Hopf algebras $T(Q(B(V)) \to B(V)$ although $B(V)$

is isomorphic to $T(Q(B(V))$ as Hopf algebras for any **individual** vector space V. However, for some special functorial sub Hopf algebras of $T(V)$, this statement holds.

Theorem 8.8. *Let $B(V)$ be a functorial sub Hopf algebra of $T(V)$. Suppose that B is a retract of T as functors from $\mathbf{k}$-modules to $\mathbf{k}$-modules. Then there is a functorial isomorphism of Hopf agebras*

$$T(Q(B(V)) \cong B(V).$$

Furthermore, $B(V)$ is generated by some elements in $L(V) \cap B(V)$.

Proof. Let $Q_n(V)$ denote $Q(B(V))_n$, the set of indecomposable elements of tensor length n. Notice that $B(V)$ is primitively generated and is a tensor algebra (being a subalgebra of a tensor algebra). It suffices to show that there is a functorial cross-section from $Q(B(V))$ to $L(V) \cap B(V) \subseteq P(B(V))$. This is given in assertion 1) of the following statement.

Statement: 1) There is a functorial submodule $Q'_n(V) \subseteq L_n(V) \cap B(V)$ such that the composite $Q'_n(V) \to L_n(V) \cap B(V) \xrightarrow{p_n} Q_n(V)$ is an isomorphism, where p_n is the restriction of the canonical map from $B_n(V)$ to $Q_n(V)$; and 2) $Q_n(V)$ is a functorial retract of $V^{\otimes n}$.

The proof of this statement is given by induction on n. The statement holds obviously for $n = 1$. Suppose that the statement holds for $k < n$ with $n > 1$.

Let $j_k \colon Q'_k(V) \to L_k(V) \cap B(V)$ be the inclusion for $k < n$ and let $j \colon \bigoplus_{k=1}^{n-1} Q'_k(V) \to \bigoplus_{k=1}^{n-1} L_k(V) \cap B(V) \subseteq B(V)$ be the inclusion induced by j_k. Let $T(j) \colon T(\bigoplus_{k=1}^{n-1} Q'_k(V)) \to B(V) \subseteq T(V)$ be the map of Hopf algebras induced by the map j. Notice that $B(V)$ is a tensor algebra and $j \colon \oplus_{k=1}^{n-1} Q'_k(V) \to Q(B(V))$ is a monomorphism. Thus $T(j)$ is a monomorphism. Notice that $Q_k(V)$ is a functorial retract of $V^{\otimes k}$ for $k < n$ and $Q'_k(V)$ is functorially isomorphic to $Q_k(V)$. Thus $Q'_k(V)$ is a functorial retract of $V^{\otimes k}$. Let $q \geq 1$ and let $k_1, k_2, \ldots, k_q$ be positive integers with $k_j < n$. Let $l_1, l_2, \ldots, l_q$ be positive integers. Then

$$Q_{k_1}(V)^{\otimes l_1} \otimes Q_{k_2}(V)^{\otimes l_2} \otimes \cdots \otimes Q_{k_q}(V)^{\otimes l_q}$$

is a functorial retract of $V^{\otimes k_1 l_1 + k_1 l_2 + \cdots k_q l_q}$. Let $m = k_1 l_1 + k_1 l_2 + \cdots k_q l_q$ and let $\bar{V} = \langle x_1, x_2, \ldots, x_m \rangle$ be an m-dimensional vector space over $\mathbf{k}$. Let γ_m be the $\mathbf{k}$-submodule of $\bar{V}^{\otimes m}$ generated by the homogeneous elements $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}$ for $\sigma \in S_m$. Then

$$(Q_{k_1}(V)^{\otimes l_1} \otimes Q_{k_2}(V)^{\otimes l_2} \otimes \cdots \otimes Q_{k_q}(V)^{\otimes l_q}) \cap \gamma_m$$

is a projective $\mathbf{k}(S_m)$-module. Notice that

$$T(\bigoplus_{k=1}^{n-1} Q'_k(V))\cap T_m(V) = \bigoplus_{\substack{1\leq k_1,k_2,\ldots,k_q\leq n-1\\ k_1l_1+k_2l_2+\cdots+k_ql_q=m}} Q_{k_1}(V)^{\otimes l_1}\otimes Q_{k_2}(V)^{\otimes l_2}\otimes\cdots\otimes Q_{k_q}(V)^{\otimes l_q}$$

for $m\geq 1$. Thus

$$T(\bigoplus_{k=1}^{n-1} Q'_k(\bar V))\cap T_m(\bar V)\cap\gamma_m$$

is a projective $\mathbf{k}(S_m)$-module. Notice that

$$T(\bigoplus_{k=1}^{n-1} Q'_k(V))\cap T_m(V) = (T(\bigoplus_{k=1}^{n-1} Q'_k(\bar V))\cap T_m(\bar V)\cap\gamma_m)\otimes_{\mathbf{k}(S_m)} V^{\otimes m}$$

for any $\mathbf{k}$-module V. Thus

$$T(\bigoplus_{k=1}^{n-1} Q'_k(V))\cap T_m(V)$$

is a functorial retract of $\gamma_m\otimes_{\mathbf{k}(S_m)} V^{\otimes m}$ and so $T(\bigoplus_{k=1}^{n-1}Q'_k(-))$ is a retract of T as functors from modules to modules. By Theorem 8.6, $T(\bigoplus_{k=1}^{n-1}Q'_k(V))$ is a functorial coalgebra retract of $T(V)$ and so $T(\bigoplus_{k=1}^{n-1}Q'_k(V))$ is a functorial coalgebra retract of $B(V)$. Let $r\colon B(V)\to T(\bigoplus_{k=1}^{n-1}Q'_k(V))$ be a functorial coalgebra retraction and let

$$B'(V)=\mathbf{k}\,\square_{T(\bigoplus_{k=1}^{n-1}Q'_k(V))} B(V).$$

By Lemma 5.3, there is a functorial isomorphism of coalgebras

$$B(V)\cong T(\bigoplus_{k=1}^{n-1} Q'_k(V))\otimes B'(V).$$

Notice that $T(\bigoplus_{k=1}^{n-1}Q'_k(V))$ is a sub Hopf algebra of $B(V)$ generated by the elements $x\in B(V)$ with the tensor length $|x|<n$. Thus $B'(V)_j=0$ for $0<j<n$ and $B'(V)_n\cong Q_n(V)$. Let $Q'_n(V)=B'(V)_n$. Then $Q'_n(V)\subseteq P(B(V))=L^{\mathrm{res}}(V)\cap B(V)$. Let $i\colon Q'_n(V)\to V^{\otimes n}$ be the inclusion of $Q'_n(V)\subseteq L^{\mathrm{res}}_n(V)\cap B(V)\subseteq V^{\otimes n}$. Notice that $B'(V)$ is a functorial retract of $T(V)$. Thus there is a functorial map $r'\colon V^{\otimes n}\to Q'_n(V)$ such that $r'\circ i\colon Q'_n(V)\to Q'_n(V)$ is the identity map and $Q'_n(V)$ is given by the set of the fixed points of the idempotent map $\phi=i\circ r'\colon V^{\otimes n}\to V^{\otimes n}$. Notice that $\mathrm{Im}(\phi\colon\gamma_n\to\gamma_n)\subseteq L^{\mathrm{res}}_n(\bar V)\cap\gamma_n=\mathrm{Lie}(n)$. Thus $\phi(x_1\cdots x_n)\in L_n(\bar V)$ and so $\phi(a_1\cdots a_n)\in L_n(V)$ for any $\mathbf{k}$-module V and any elements $a_j\in V$. Therefore $Q'_n(V)\subseteq L_n(V)\cap B(V)$. The induction is finished and the assertion follows. □

Let $Q^{\max}_n(V)$ denote $Q(B^{\max}(V))$ the set of indecomposable elements of $B^{\max}(V)$ with tensor length n.

Corollary 8.9. *The following statements hold:*

1) $Q^{\max}_1(V)=0$;
2) $Q^{\max}_n$ *is a retract of* L_n *as functors from modules to modules for each* $n\geq 2$;
3) $Q^{\max}_n(V)$ *is a functorial retract of* $V^{\otimes n}$ *for each* $n\geq 2$;
4) *there is a functorial isomorphism of Hopf algebras*

$$T(\bigoplus_{n=2}^{\infty} Q^{\max}_n(V))\cong B^{\max}(V).$$

A relation between the functor $Q^{\max}$ and the functor $A^{\min}$ is a follows.

Proposition 8.10. *There is a functorial isomorphism of* $\mathbf{k}$*-modules*

$$A^{\min}_{n-1}(V)\otimes V\cong Q^{\max}_n(V)\oplus A^{\min}_n(V).$$

for each $n\geq 1$.

Proof. Let $\tilde{A}^{\min}(V)=\mathbf{k}\otimes_{B^{\max}(V)}T(V)$. Then $\tilde{A}^{\min}(V)$ is functorially isomorphic to $A^{\min}(V)$ as a coalgebra. Let $p\colon T(V)\to\tilde{A}^{\min}(V)$ be the projection and let $I^{\max}(V)$ be the kernel of p. Notice that $I^{\max}(V)$ is a right ideal of $T(V)$ and $\tilde{A}^{\min}(V)$ is a right $T(V)$-module. Let $I\tilde{A}^{\min}(V)$ be the kernel of the augmentation map $\epsilon\colon\tilde{A}^{\min}(V)\to\mathbf{k}$ and let $q\colon T(V)\otimes V\to I\tilde{A}^{\min}(V)$ be the $\mathbf{k}$-linear map given by

$$q(\alpha\otimes x)=p(\alpha x)=(\alpha)\cdot x$$

for $\alpha\in T(V)$ and $x\in V$, where $\cdot$ is the right action. Then

$$q(\alpha\otimes x)=0$$

if $\alpha\in I^{\max}(V)$. Consider the commutative diagram of exact sequences of $\mathbf{k}$-modules

$$\begin{array}{ccccc}
I^{\max}\otimes V & = & I^{\max}(V)\otimes V & & \\
\downarrow{\scriptstyle\text{mult}} & & \downarrow & & \\
I^{\max}(V) & \hookrightarrow & T(V)\otimes V & \xrightarrow{q} & I\tilde{A}^{\min}(V) \\
\downarrow & & \downarrow{\scriptstyle p\otimes\mathrm{id}} & & \| \\
K(V) & \hookrightarrow & \tilde{A}^{\min}(V)\otimes V & \xrightarrow{\bar{q}} & I\tilde{A}^{\min}(V).
\end{array}$$

Then we have

$$K(V)\cong I^{\max}(V)/(I^{\max}(V)\cdot V)\cong I^{\max}(V)/(I^{\max}(V)\cdot IT(V)).$$

Notice that the canonical map $\pi\colon Q^{\max}(V)=IB^{\max}(V)/(IB^{\max}(V)\cdot IB^{\max}(V))\to I^{\max}(V)/(I^{\max}(V)\cdot IT(V))$ is an epimorphism. The formula for $\chi(Q^{\max})$ in terms of $\chi(A^{\min})$ and $\chi(V)$ is determined by: 1) $T(Q^{\max})\cong B^{\max}$ and 2) $T(V)\cong B^{\max}\otimes A^{\min}$. By comparing Poincaré series, we see that the map π is an isomorphism if $\dim(V)<\infty$ and so π is an isomorphism for any $\mathbf{k}$-module V. Thus $Q^{\max}(V)\cong K(V)$ and there is a short exact sequence

$$Q^{\max}_n(V)\hookrightarrow \tilde{A}^{\min}_{n-1}(V)\otimes V\twoheadrightarrow A^{\min}_n(V).$$

Notice that the $\mathbf{k}$-linear map $q\colon T(V)\otimes V\cong IT(V)\to IA^{\min}(V)$ has a functorial cross-section. The assertion follows. □

Remark 8.11. *It is the fact that $A^{\min}(V)$ and $\mathbf{k}\otimes_{B^{\max}(V)}T(V)$ are only functorially isomorphic as coalgebras rather than equal that prevents the filtration on $A^{\min}$ from being a Hopf algebra filtration instead of just a coalgebra filtration.*

8.3. **A coalgebra filtration on the functor $A^{\min}$.** Now we give a coalgebra filtration on $A^{\min}(V)$ which can be used to give a lower bound on its growth. The filtration will be given by inductively defining a descending natural sequence of subcoalgebras of $A^{\min}(V)$.

Let $T^{(1)}(V)=T(V)$ and let $E^{(1)}(V)=\mathbf{k}[V]$ be the polynomial algebra generated by V. Let

$$T^{(2)}(V)=\mathbf{k}\,\square_{E^{(1)}(V)}\,T^{(1)}(V).$$

Notice that $T^{(2)}(V)$ is a normal subHopf algebra of $T(V)$ generated by $L_n(V)$ with $n\geq 2$. Thus $B^{\max}(V)\subseteq T^{(2)}$. Let $A^{(2)}(V)$ be defined by

$$A^{(2)}(V)=\mathbf{k}\otimes_{B^{\max}(V)}T^{(2)}(V).$$

Suppose that we have already defined $T^{(k)}(V)$ and $A^{(k)}(V)$ for $k\leq n$ and $E^{(k)}(V)$ for $k<n$ with the properties:

1) $T^{(k)}(V)$ is a subHopf algebra of $T(V)$ for $1\leq k\leq n$ and
$$B^{\max}(V)\subseteq T^{(n)}(V)\subseteq T^{(n-1)}(V)\subseteq\cdots\subseteq T^{(2)}(V)\subseteq T^{(1)}(V)=T(V);$$
2) $E^{(k)}(V)$ is a commutative Hopf algebra such that
$$T^{(k+1)}(V)=\mathbf{k}\,\square_{E^{(k)}}\,T^{(k)}$$
for $k<n$.
3) $A^{(k)}=\mathbf{k}\otimes_{B^{\max}(V)}T^{(k)}(V)$.

We need to define $T^{(n+1)}(V)$, $A^{(n+1)}(V)$ and $E^{(n)}(V)$ for $n \geq 2$. Notice that $T^{(n+1)}(V)$ and $A^{(n+1)}(V)$ will be defined once $E^{(n)}(V)$ is defined. Now let $E^{(n)}(V)$ be defined to be the quotient Hopf algebra of $T^{(n)}(V)$ modulo the two sided ideal generated by

1) $B^{\max}(V)$;
2) the commutators $[x, y]$ for $x, y \in IT^{(n)}(V)$, where $IT^{(n)}(V)$ is the augmentation ideal of $T^{(n)}(V)$.

This gives a coalgebra filtration

$$\cdots A^{(n)}(V) \subseteq A^{(n-1)}(V) \subseteq \cdots \subseteq A^{(2)}(V) \subseteq A^{(1)}(V) \cong A^{\min}(V).$$

Observe that $\cap_{n=1}^{\infty} A^{(n)}(V) = 0$.

Proposition 8.12. *There is an equality of Poincaré series*

$$\chi(A^{\min}(V)) = \prod_{n=1}^{\infty} \chi(E^{(n)}(V)).$$

Proof. Notice that

$$\chi(T^{(n)}(V)) = \chi(T^{(n+1)}(V)) \cdot \chi(E^{(n)}(V)) = \chi(B^{\max}(V)) \cdot \chi(A^{(n)}(V));$$

$$\chi(T^{(n+1)}(V)) = \chi(B^{\max}(V)) \cdot \chi(A^{(n+1)}(V)).$$

Thus $\chi(A^{(n)}(V)) = \chi(A^{(n+1)}(V)) \cdot \chi(E^{(n)}(V))$. The assertion follows. □

Let $L(B^{\max}(V))$ denote $L(V) \cap B^{\max}(V)$. Let $L^{(1)}(V) = L(V)$ and let

$$L^{(n+1)}(V) = [L^{(n)}(V), L^{(n)}(V)] + L(B^{\max}(V))$$

be a sub Lie algebra of $L(V)$ for $n \geq 1$. This gives a descending sequence of Lie algebras

$$\cdots \subseteq L^{(n+1)}(V) \subseteq L^{(n)}(V) \subseteq \cdots \subseteq L^{(1)}(V) = L(V).$$

Let UL denote the universal enveloping algebra of a Lie algebra L.

Proposition 8.13. *Let $T^{(n)}$,$E^{(n)}$ and $L^{(n)}$ be defined as above. Then*

1) *there is a functorial isomorphism of Hopf algebras*

$$UL^{(n)}(V) \cong T^{(n)}(V)$$

for each $n \geq 1$;

2) *$E^{(n)}(V)$ is functorially isomorphic to the polynomial algebra generated by*

$$L^{(n)}(V)/L^{(n+1)}(V) = L^{(n)}(V)/([L^{(n)}(V), L^{(n)}(V)] + L(B^{\max}(V)))$$

for each $n \geq 1$.

Proof. Let $j_n\colon L^{(n)}(V) \to T^{(n)}(V)$ be the canonical inclusion. Notice that the composite $L^{(n)}(V) \longrightarrow T^{(n)}(V) \longrightarrow E^{(n)}(V)$ factors through $L^{(n)}(V)/L^{(n+1)}(V)$. Let $\bar{\jmath}_n\colon L^{(n)}(V)/L^{(n+1)}(V) \longrightarrow E^{(n)}(V)$ be the induced map. We will show the following statements:

$\mathcal{P}_n$) The induced map of Hopf algebras

$$U(j_n)\colon UL^{(n)}(V) \to T^{(n)}(V)$$

is an isomorphism for each $n \geq 1$.

$\bar{\mathcal{P}}_n$) The induced map of Hopf algebras

$$U(\bar{\jmath}_n)\colon U(L^{(n)}(V)/L^{(n+1)}(V)) \to E^{(n)}(V)$$

is an isomorphism for each $n \geq 1$.

Assertion 1) follows from statement $\mathcal{P}_n$. Notice that $L^{(n)}(V)/L^{(n+1)}(V)$ is an abelian Lie algebra and so assertion 2) follows from statement $\bar{\mathcal{P}}_n$. The proof of statements $\mathcal{P}_n$ and $\bar{\mathcal{P}}_n$ are given as follows.

1) Statement $\mathcal{P}_1$ holds obviously.
2) If statement $\mathcal{P}_n$ holds, then statement $\bar{\mathcal{P}}_n$ holds.
3) If statements $\mathcal{P}_n$ and $\bar{\mathcal{P}}_n$ hold, then statement $\mathcal{P}_{n+1}$ holds.

We first show that statement $\bar{\mathcal{P}}_n$ holds by assuming statement $\mathcal{P}_n$. By Corollary 8.9, $B^{\max}(V)$ is the sub Hopf algebra of $T(V)$ generated by $L(B^{\max}(V))$. By statement $\mathcal{P}_n$, $T^{(n)}(V)$ is the sub Hopf algebra of $T(V)$ generated by $L^{(n)}(V)$. Thus $E^{(n)}(V)$ is the quotient algebra of $T^{(n)}(V)$ modulo the two sided ideal generated by $L^{(n+1)}(V) = [L^{(n)}(V), L^{(n)}(V)] + L(B^{\max}(V))$. The standard arguments show that statement $\bar{\mathcal{P}}_n$ holds.

Now we show that statement $\bar{\mathcal{P}}_{n+1}$ holds by assuming statements $\mathcal{P}_n$ and $\bar{\mathcal{P}}_n$. It suffices to show statement $\mathcal{P}_{n+1}$ for the case that $\dim(V) < \infty$. Consider the short exact sequence of Lie algebras

$$L^{(n+1)}(V) \hookrightarrow L^{(n)}(V) \twoheadrightarrow L^{(n)}(V)/L^{(n+1)}(V).$$

There is a short exact sequence of Hopf algebras

$$UL^{(n+1)}(V) \hookrightarrow UL^{(n)}(V) \twoheadrightarrow U(L^{(n)}(V)/L^{(n+1)}(V)).$$

(See [5, Proposition 3.7].) Statement $\mathcal{P}_{n+1}$ follows from the following commutative diagram of Hopf algebras

$$\begin{array}{ccccc} UL^{(n+1)}(V) & \hookrightarrow & UL^{(n)}(V) & \twoheadrightarrow & U(L^{(n)}(V)/L^{(n+1)}(V)). \\ \downarrow & & U(j_n)\downarrow\cong & & U(\bar{j}_n)\downarrow\cong \\ T^{(n+1)}(V) & \hookrightarrow & T^{(n)}(V) & \twoheadrightarrow & E^{(n)}(V), \end{array}$$

where the rows are short exact sequences of Hopf algebras. This completes the proof. □

Notice that $A^{(n)}(V) = \mathbf{k} \otimes_{B^{\max}(V)} T^{(n)}(V)$ is a right $T^{(n)}(V)$-module. Let $IA^{(n)}(V)$ be the kernel of the augmentation map $A^{(n)}(V) \to \mathbf{k}$. By induction and using Proposition 8.13 and Corollary 8.4, we have

Corollary 8.14. *$E^{(n)}(V)$ is a polynomial algebra with indecomposable elements*

$$Q(E^{(n)}(V)) \cong IA^{(n)}(V) \otimes_{T^{(n)}(V)} \mathbf{k}.$$

Furthermore $Q(E^{(n)}(V))_j = 0$ for $j < p^{n-1}$.

Proposition 8.15. *If $\mathrm{char}(\mathbf{k}) > 2$, then $T^{(n+1)}(V)$ is the normal sub Hopf algebra of $T^{(n)}(V)$ generated by $B^{\max}(V)$.*

Proof. Let B' be the normal sub Hopf algebra of $T^{(n)}$ generated by $B^{\max}(V)$. Then $B' \subseteq T^{(n+1)}(V)$. By Proposition 8.13, it suffices to show that $L^{(n+1)}(V) \subseteq L(B') = B' \cap L(V)$. Notice that $L^{(n+1)}(V) = [L^{(n)}(V), L^{(n)}(V)] + L(B^{\max}(V))$. If q is not a power of $p = \mathrm{char}(\mathbf{k})$, then $L(B^{\max}(V))_q = L_q(V)$ by Corollary 8.4. Notice that $[L^{(n)}(V), L^{(n)}(V)]$ is a two sided Lie ideal generated by the elements $[\alpha, \beta]$ for $\alpha, \beta \in L^{(n)}(V)$. Also notice that $L(B')$ is a two sided Lie ideal of $L(T^{(n)}(V)) = T^{(n)}(V) \cap L(V)$. It follows that $[L^{(n)}(V), L^{(n)}(V)]_q \subseteq L(B')$ for any q. The assertion follows. □

8.4. **A lower bound on the growth of $A^{\min}(V)$.** Now we give a lower bound for the functor $A^{\min}$ by determining $E^{(2)}(V)$. We need some preliminaries to determine $E^{(2)}(V)$.

Let S_n act on $\mathbf{k}^{\oplus n}$ by permuting the basis $\{y_1, y_2, \ldots, y_n\}$ and let $I(\mathbf{k}^{\oplus n})$ be the kernel of the augmentation map $\epsilon\colon \mathbf{k}^{\oplus n} \to \mathbf{k}$ given by $\epsilon(y_j) = 1$ for $1 \le j \le n$. Then $I(\mathbf{k}^{\oplus n})$ is a sub $(\mathbf{S_n})$-module of $\mathbf{k}^{\oplus n}$. Let $Q(T^{(2)})_n$ be the set of indecomposable elements of $T^{(2)}(V)$ which have tensor length n and let S_n act on $V^{\otimes n}$ by permuting factors.

Proposition 8.16. *There is a functorial isomorphism of* $\mathbf{k}$*-modules*

$$Q(T^{(2)}(V))_n \cong I(\mathbf{k}^{\oplus n}) \otimes_{\mathbf{k}(S_n)} V^{\otimes n}$$

for any $\mathbf{k}$*-module* V.

Proof. By the proof of Lemma 3.13 in [5]. There is a functorial short exact sequence of $\mathbf{k}$-modules

$$Q(T^{(2)}) \hookrightarrow \mathbf{k}[V] \otimes V \xrightarrow{\text{mult}} I(\mathbf{k}[V]),$$

where $\mathbf{k}[V]$ is the polynomial algebra generated by V.

Let $\bar{V} = \langle x_1, \ldots, x_n \rangle$ be an n-dimensional vector space over $\mathbf{k}$ and let γ_n be as defined earlier. Let M_n be the quotient $\mathbf{k}$-module of γ_n modulo the sub module generated by

$$x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(i-1)} x_{\sigma(i+1)} \cdots x_{\sigma(n)} x_i - x_1 x_2 \cdots \hat{x}_i \cdots x_n x_i$$

for $1 \leq i \leq n$ and $\sigma \in S_{n-1}$ acting on $\{1, 2, \ldots, \hat{i}, \ldots, n\}$. Then M_n is a quotient $\mathbf{k}(S_n)$-module of γ_n. Let $q\colon \gamma_n \to M_n$ be the quotient map and let $y_i = q(x_1 \cdots \hat{x}_i \cdots x_n x_i) \in M_n$ for $1 \leq i \leq n$. Then $\{y_1, y_2, \ldots, y_n\}$ is a $\mathbf{k}$-basis for M_n and there is an isomorphism of $\mathbf{k}(S_n)$-modules

$$M_n \cong \mathbf{k}^{\oplus n}.$$

Let $\epsilon_n\colon V^{\otimes n} \to \mathbf{k}[V]_n$ be the canonical quotient map. Then there is a commutative diagram

$$\begin{array}{ccccc}
V^{\otimes n} & \xrightarrow{\epsilon_{n-1} \otimes \mathrm{id}} & \mathbf{k}[V]_{n-1} \otimes V & \xrightarrow{\text{mult}} & \mathbf{k}[V]_n \\
\uparrow \cong & & \uparrow & & \uparrow \cong \\
\gamma_n \otimes_{\mathbf{k}(S_n)} V^{\otimes n} & \longrightarrow & M_n \otimes_{\mathbf{k}(S_n)} V^{\otimes n} & \xrightarrow{\epsilon \otimes \mathrm{id}} & \mathbf{k} \otimes_{\mathbf{k}(S_n)} V^{\otimes n},
\end{array}$$

where $\epsilon\colon M_n \to \mathbf{k}$ is given by $\epsilon(y_j) = 1$. Thus $M_n \otimes_{\mathbf{k}(S_n)} V^{\otimes n} \to \mathbf{k}[V]_{n-1} \otimes V$ is an epimorphism. Notice that $\dim(M_n \otimes_{\mathbf{k}(S_n)} V^{\otimes n}) \leq \dim(\mathbf{k}[V]_{n-1} \otimes V)$. Thus $M_n \otimes_{\mathbf{k}(S_n)} V^{\otimes n} \to \mathbf{k}[V]_{n-1} \otimes V$ is a (functorial) isomorphism. Consider the commutative diagram

$$\begin{array}{ccccc}
Q(T^{(2)}(V))_n & \hookrightarrow & \mathbf{k}[V]_{n-1} \otimes V & \longrightarrow & \mathbf{k}[V]_n \\
\uparrow & & \uparrow \cong & & \uparrow \cong \\
I(M_n) \otimes_{\mathbf{k}(S_n)} V^{\otimes n} & \longrightarrow & M_n \otimes_{\mathbf{k}(S_n)} V^{\otimes n} & \twoheadrightarrow & \mathbf{k} \otimes_{\mathbf{k}(S_n)} V^{\otimes n},
\end{array}$$

where the top row is a short exact sequence.Thus $I(M_n)\otimes_{\mathbf{k}(S_n)} V^{\otimes n} \to Q(T^{(2)}(V))$ is an epimorphism. Notice that $\dim(I(M_n)\otimes_{\mathbf{k}(S_n)} V^{\otimes n}) \le \dim(Q(T^{(2)}(V))$. Thus $I(M_n)\otimes_{\mathbf{k}(S_n)} V^{\otimes n} \to Q(T^{(2)}(V))$ is a functorial isomorphism if $\dim(V) < \infty$ and it is an isomorphism for any V, which is the assertion. □

Let $f_V\colon T(V) \to T(V)$ be a functorial map of coalgebras. Let $\alpha(f)_n \in \mathbf{k}(S_n)$ be the sequence of elements such that $\alpha(f)_n = f_V|_{T_n(V)}\colon T_n(V) = V^{\otimes n} \to T_n(V) = V^{\otimes n}$. (See section 7.) Let $\langle\ ,\ \rangle$ denote the canonical inner product in $\mathbf{k}(S_n)$. Notice that

$$\alpha = \sum_{\sigma\in S_n} \langle \alpha, \sigma\rangle \sigma$$

for any $\alpha \in \mathbf{k}(S_n)$.

Lemma 8.17. *Let $f_V\colon T(V) \to T(V)$ be a functorial map of coalgebras such that $(f_V)|_V\colon V \to V$ is the identity map for any V. Then*

$$\sum_{\sigma\in S_n} \langle \alpha(f)_n, \sigma\rangle = 1$$

for each $n \ge 1$.

Proof. Let $\bar{V}$ be the n-dimensional $\mathbf{k}$-module generated by $\{x_1, \dots, x_n\}$. Let $q_1 : T(V) \to T(V)$ be the projection map defined by

$$q_1(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in V = T_1(V); \\ \\ 0 & \text{if } \alpha \in T_n(V), n \ne 1. \end{cases}$$

Then one has

$$q_1 \circ f_V = q_1\colon T(\bar{V}) \to T(\bar{V}).$$

By hypothesis f is a map of coalgebras. Thus one has

$$q_1^{*n} \circ f = q_1^{*n}\colon T(\bar{V}) \to T(\bar{V}),$$

where q_1^{*n} is the n-fold self convolution product of q_1. Notice that

$$q_1^{*n}(x_1 \cdots x_n) = \sum_{\sigma\in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

and

$$\begin{aligned} q_1^{*n} \circ f(x_1 \cdots x_n) &= \sum_{\sigma\in S_n} \langle \alpha(f)_n, \sigma\rangle q_1^{*n}(x_{\sigma(1)} \cdots x_{\sigma(n)}) \\ &= (\sum_{\sigma\in S_n} \langle \alpha(f)_n, \sigma\rangle) \sum_{\tau\in S_n} x_{\tau(1)} \cdots x_{\tau(n)}. \end{aligned}$$

The assertion follows. □

Corollary 8.18. *Let $f_V\colon T(V)\to T(V)$ be a natural map of coalgebras such that $(f_V)|_V\colon V\to V$ is the identity map for any V. Then*

(1). $f|_{T_n(V)}(x^n)=x^n$ *for any* $x\in V$ *and* $n\geq 1$*;*

(2).
$$f|_{T_n(V)}(\sum_{\sigma\in S_n}x_{\sigma(1)}\cdots x_{\sigma(n)})=\sum_{\sigma\in S_n}x_{\sigma(1)}\cdots x_{\sigma(n)}$$
for any $x_1,\dots x_n\in V$ *and* $n\geq 1$.

Lemma 8.19. *Let $f_V\colon T(V)\to T(V)$ be a natural map of coalgebras such that $(f_V)|_V\colon V\to V$ is the identity map for any V and let $x,y\in V$. Then*
$$f|_{T_{p^r}(V)}(ad^{p^r-1}(y)(x))=ad^{p^r-1}(y)(x)$$
for any $r\geq 1$.

Proof. We may assume that V is a two dimensional $\mathbf{k}$-module generated by x and y. The assertion follows from the formula:
$$(x+y)^{p^r}=y^{p^r}+ad^{p^r-1}(y)(x)+\Delta,$$
where Δ is a linear combination of the homogeneous elements in which x appears at least twice. (See [16].) $\square$

Let $M_1,\dots,M_k$ be submodules of $T(V)$. We write $\langle M_1,\dots,M_k\rangle$ for the subHopf algebra of $T(V)$ generated by $M_1,\dots,M_k$. Given $n>1$, let $D^n(V)$ denote
$$\mathbf{k}\otimes_{\langle B^{max}(V),L_2,\dots,L_{n-1}\rangle}T(V),$$
where $L_j=L_j(V)$. Let $\alpha\in T(V)$. We write $\{\alpha\}$ for the image of α in $D^n(V)$.

Lemma 8.20. *Given $t\geq 1$, let $\bar V$ be the p^t-dimensional $\mathbf{k}$-module generated by $\{x_1,\dots,x_{p^t}\}$. Then the elements*
$$y_j=\{x_1\cdots\hat{x}_j\cdots x_{p^t}x_j\}$$
with $1\leq j\leq p^t$ are linearly independent in $D^{p^t}(\bar V)$.

Proof. Suppose that $y_1,y_2,\dots,y_{p^t}$ are linearly dependent in $D^{p^t}(\bar V)$. There exists $1\leq i\leq p^t$ such that
$$y_i=\sum_{j\neq i}k_jy_j$$
for some $k_j\in\mathbf{k}$. Let $j\neq i$ with $1\leq j\leq p^t$ and let $T_{i,j}\colon\bar V\cong\bar V$ be the $\mathbf{k}$-linear map given by $T_{i,j}(x_i)=x_j$,$T_{i,j}(x_j)=x_i$ and $T_{i,j}(x_k)=x_k$ for $k\neq i,j$. Let $D^{p^t}(T_{i,j})\colon D^{p^t}(\bar V)\cong D^{p^t}(\bar V)$ be the induced isomorphism. Notice that
$$\{x_{i_1}\cdots x_{i_{p^t-1}}x_{i_{p^t}}\}=\{x_{i_{\tau(1)}}\cdots x_{i_{\tau(p^t-1)}}x_{i_{p^t}}\}$$

since $\langle B^{max}(V), L_2, \ldots, L_{n-1}\rangle$ contains all commutators of length less than n. For any $\tau \in S_{p^{t-1}}$, one gets

$$D^{p^t}(T_{i,j})(y_k) = \begin{cases} y_j & \text{if} \quad k = i; \\ y_i & \text{if} \quad k = j; \\ y_k & \text{otherwise.} \end{cases}$$

Thus $y_j = \sum_{l \neq i,j} k_l y_l + k_j y_i$ and so $(1 + k_j)(y_i - y_j) = 0$. Let $\tilde{V}$ be the 2-dimensional $\mathbf{k}$-module generated by x and y and let $\phi\colon \bar{V} \to \tilde{V}$ be the $\mathbf{k}$-linear map given by $\phi(x_i) = x$ and $\phi(x_k) = y$ for $k \neq i$. Then

$$D^{p^t}(\phi)(y_i - y_j) = \{y^{p^t-1}x\} - \{xy^{p^t-1}\} = -\{ad^{p^t-1}(y)(x)\},$$

since we are in characteristic p. Consider the commutative diagram

$$\begin{array}{ccccc}
 & & D^{p^t}(\tilde{V}) & = = = = = & D^{p^t}(\tilde{V}) \\
 & & \uparrow & & \uparrow \\
B^{\max}(\tilde{V}) & \hookrightarrow & T(\tilde{V}) & \overset{r}{\twoheadrightarrow} & \mathbf{k} \otimes_{B^{\max}(\tilde{V})} T(\tilde{V}) \\
\| & & \uparrow & & \uparrow \\
B^{\max}(\tilde{V}) & \hookrightarrow & \langle B^{\max}, L_2, \ldots, L_{p^t-1}\rangle & \twoheadrightarrow & \mathbf{k} \otimes_{B^{\max}(\tilde{V})} \langle B^{\max}(\tilde{V}), L_2, \ldots, L_{p^t-1}\rangle.
\end{array}$$

Notice that $B^{\max}(\tilde{V})$ is a coalgebra retract of $T(\tilde{V})$. Thus $B^{\max}(\tilde{V})$ is a coalgebra retract of $\langle B^{\max}, L_2, \ldots, L_{p^t-1}\rangle$ and so $\mathbf{k} \otimes_{B^{\max}(\tilde{V})} \langle B^{\max}, L_2, \ldots, L_{p^t-1}\rangle$ is a coalgebra retract of $\langle B^{\max}, L_2, \ldots, L_{p^t-1}\rangle$. Observe that $B^{\max}_1(V) = 0$, thus the composite

$$\langle L_2, \ldots, L_{p^t-1}\rangle \hookrightarrow \langle B^{\max}(\tilde{V}), L_2, \ldots, L_{p^t-1}\rangle \longrightarrow \mathbf{k} \otimes_{B^{\max}(\tilde{V})} \langle B^{\max}, L_2, \ldots, L_{p^t-1}\rangle$$

is onto up to dimension p^t. Notice that $r(ad^{p^t-1}(y)(x))$ is not zero by Lemma 8.19 and is not in the image of the composite $\langle L_2, \ldots, L_{p^t-1}\rangle \to T(\tilde{V}) \to \mathbf{k} \otimes_{B^{\max}(\tilde{V})} T(\tilde{V})$. Thus the primitive element $r(ad^{p^t-1}(y)(x))$ is not in the subcoalgebra $\mathbf{k} \otimes_{B^{\max}(\tilde{V})} \langle B^{\max}(\tilde{V}), L_2, \ldots, L_{p^t-1}\rangle$ and so

$$\{ad^{p^t-1}(y)(x)\} \neq 0$$

in $D^{p^t}(\tilde{V})$. Therefore $y_i - y_j \neq 0$ and so

$$k_j = -1$$

for each $j \neq i$. This shows that

$$y_1 + y_2 + \cdots + y_{p^t} = 0.$$

Notice that

$$D^{p^t}(\phi)(y_1 + \cdots + y_{p^t}) = \{y^{p^t-1}x\} + (p^t - 1)\{xy^{p^t-1}\} = -\{ad^{p^t-1}(y)(x)\} \neq 0.$$

One has a contradiction and the assertion follows. □

Let $q\colon T(V) \to D^{p^t}(V)$ be the quotient map and let $\gamma(D^{p^t}_{p^t})$ denote the image

$$\gamma(D^{p^t}_{p^t}) = \mathrm{Im}(q|_{\gamma_{p^t}} \colon \gamma_{p^t} \to D^{p^t}_{p^t}(\bar{V})).$$

Notice that $\gamma(D^{p^t}_{p^t})$ is a quotient $\mathbf{k}(S_{p^t})$-module of $\gamma_{p^t} \cong \mathbf{k}(S_{p^t})$. By Lemma 8.20, a representation of the $\mathbf{k}(S_n)$-module $\gamma(D^{p^t}_{p^t})$ is as follows.

Corollary 8.21. *There is an isomorphism of $\mathbf{k}(S_n)$-modules*

$$\gamma(D^{p^t}_{p^t}) \cong \mathbf{k}^{\oplus p^t},$$

where the $\mathbf{k}(S_{p^t})$-action on $\mathbf{k}^{\oplus p^t}$ is the standard representation.

Proposition 8.22. *Given $t \geq 1$ and let $\bar{V}$ be the p^t-dimensional $\mathbf{k}$-module generated by $\{x_1, \ldots x_{p^t}\}$. Then the elements*

$$z_j = \{[[x_1, x_j, x_2, \ldots, \hat{x_j}, \ldots, x_{p^t}]\}$$

with $2 \leq j \leq p^t$ are linearly independent in $D^{p^t}(\bar{V})$.

Proof. The assertion follows from the fact

$$z_j = y_j - y_1$$

in $D^{p^t}(\bar{V})$. □

Let $\mathrm{Lie}(D^{p^t}_{p^t})$ denote the image

$$\mathrm{Lie}(D^{p^t}_{p^t}) = \mathrm{Im}(q|_{\mathrm{Lie}(p^t)} \colon \mathrm{Lie}(p^t) \to D^{p^t}_{p^t}(\bar{V})).$$

Corollary 8.23. *Let $a_1, \ldots, a_{p^t}$ be the standard basis for $\mathbf{k}^{\oplus p^t}$. Then $\mathrm{Lie}(D^{p^t}_{p^t})$ is isomorphic to the $\mathbf{k}$-submodule of $\mathbf{k}^{\oplus p^t}$ spanned by $a_2 - a_1, a_3 - a_1, \ldots, a_{p^t} - a_1$ as $\mathbf{k}(S_{p^t})$-modules.*

Proposition 8.24. *The* $\mathbf{k}$*-module* $D^n_j(V)$ *for* $j \le n$ *is determined as follows:*

$$D^n_j(V) \cong \begin{cases} \mathbf{k}[V]_j & if \quad j < n \\ \mathbf{k}[V]_n & if \quad j = n \neq p^t \ for\ some\ t \\ \mathbf{k}[V]_{p^t-1} \otimes V & if \quad j = n = p^t \end{cases}$$

Proof. Notice that $B^{\max}(V)$ is generated by certain choices of $L^{\max}_k(V) \subseteq L_k(V)$ for $k \ge 2$. Thus $\langle B^{\max}(V), L_2, \dots, L_{n-1}\rangle$ is equal to $\langle L_2, \dots, L_{n-1}, L^{\max}_n\rangle$ up to dimension n and so the only case we need to show is that

$$D^{p^t}_{p^t}(V) \cong \mathbf{k}[V]_{p^t-1} \otimes V.$$

Notice that the quotient map $T(V) \to D^{p^t}(V)$ induces a natural epimorphism

$$\theta \colon \mathbf{k}[V]_{p^t-1} \otimes V \to D^{p^t}_{p^t}(V).$$

By Lemma 8.20, it is routine to check that the natural map θ is a monomorphism. The assertion follows. □

Theorem 8.25. $E^{(2)}(V)$ *is functorially isomorphic to the polynomial algebra generated by*

$$\bigoplus_{t=1}^{\infty} I(\mathbf{k}^{\oplus p^t}) \otimes_{\mathbf{k}(S_{p^t})} V^{\otimes p^t}.$$

Proof. Let $t \ge 1$. Notice that $\langle B^{\max}(V), L_2, \dots, L_{p^t-1}\rangle \subseteq T^{(2)}(V)$. Let $D'(V) = \mathbf{k} \otimes_{\langle B^{\max}(V), L_2, \dots, L_{p^t-1}\rangle} T^{(2)}(V)$. Consider the commutative diagram

$$\begin{array}{ccccc} \langle B^{\max}(V), L_2, \dots, L_{p^t-1}\rangle & \hookrightarrow & T^{(2)}(V) & \twoheadrightarrow & D'(V) \\ \| & & \downarrow & & \downarrow \\ \langle B^{\max}(V), L_2, \dots, L_{p^t-1}\rangle & \hookrightarrow & T(V) & \twoheadrightarrow & D^{p^t}(V) \\ & & \downarrow & & \downarrow \\ & & \mathbf{k}[V] & = & \mathbf{k}[V]. \end{array}$$

Notice that $D'(V)$ is (p^t-1)-connected. By Proposition 8.24, there is a functorial isomorphism

$$D'(V)_{p^t} \cong \mathrm{Ker}(D^{p^t}_{p^t}(V) \to \mathbf{k}[V]_{p^t}) \cong \mathrm{Ker}(\mathbf{k}[V]_{p^t-1} \otimes V \to \mathbf{k}[V]_{p^t}).$$

By Proposition 8.16, the canonical map

$$Q(T^{(2)}(V))_{p^t} = (IT^{(2)}(V) \otimes_{T^{(2)}(V)} \mathbf{k})_{p^t} \to (I(D'(V)) \otimes_{T^{(2)}(V)} \mathbf{k})_{p^t} = D'(V)_{p^t}$$

is an isomorphism. Notice that the map $T^{(2)}(V) \to D'(V)$ factors through $A^{(2)}(V) = \mathbf{k} \otimes_{B^{\max}(V)} T^{(2)}(V)$. Thus the canonical map $Q(T^{(2)}(V))_{p^t} \to (I(A^{(2)}(V)) \otimes_{T^{(2)}(V)} \mathbf{k})_{p^t}$ is a monomorphism and so it is an isomorphism. By Corollary 8.14, the canonical map $Q(T^{(2)}(V))_{p^t} \to Q(E^{(2)})_{p^t}$ is an isomorphism. By Corollary 8.4, $Q(B^{\max}(V))_n \to Q(T^{(2)}(V))_n$ is onto if n is not a power of p and so the epimorphism $Q(T^{(2)}(V))_n \to Q(E^{(2)}(V))_n$ is the trivial map if n is not a power of p. The assertion follows. □

By Proposition 8.12, we have a lower bound on the growth of $A^{\min}(V)$.

Corollary 8.26. *If* $\dim(V) \geq 2$*, then* $A^{\min}(V)$ *has at least sub-exponential growth.*

Corollary 8.27. *There is a natural isomorphism*

$$A^{\min}_p(V) \cong \mathbf{k}[V]_{p-1} \otimes V.$$

Example 8.28. Let V be a two dimensional $\mathbf{k}$-module over the Steenrod algebra generated by $\{x, y\}$ with $P^1_*(y) = x$. Then, as a retract of $V^{\otimes p}$, $A^{\min}_p(V)$ is a $2p$-dimensional $\mathbf{k}$-module over Steenrod algebra with a basis:

$$\{y^p, P^1_*(y^p), \ldots, P^p_*(y^p), z, P^1_*(z), \ldots, P^{p-2}_*(z)\},$$

where $P^{p-1}_*(z) = P^p_*(y^p) = x^p$.

To conclude this section, as an example, we show that $T^{(2)}(V)$, which is the coalgebra kernel of the quotient map from tensor algebras to polynomial algebras, is not **functorially** isomorphic to the tensor algebra generated by $Q(T^{(2)}(V))$ if $\mathrm{char}(\mathbf{k}) > 0$.

Proposition 8.29. *The canonical quotient map* $I(T^{(2)}(V)) \to Q(T^{(2)}(V))$ *does NOT have a functorial* $\mathbf{k}$*-linear cross-section map.*

Proof. We show that $q\colon I(T^{(2)}(V))_n \to Q(T^{(2)}(V))_n$ does not have a functorial cross-section if $n \geq p+2$. Suppose that there were such a cross-section. Let $s\colon Q(T^{(2)}(V))_n \to I(T^{(2)}(V))_n$ be a functorial $\mathbf{k}$-linear map such that $q \circ s\colon Q(T^{(2)}(V))_n \to Q(T^{(2)}(V))_n$ is the identity.

Let $\bar{V} = \langle x_1, x_2, \dots, x_n \rangle$ be an n-dimensional vector space over $\mathbf{k}$ and let γ_n be as defined earlier. Let $M_n \cong \mathbf{k}^{\oplus n}$ be as defined in the proof of Proposition 8.16. Then there is a commutative diagram of short exact sequences of $\mathbf{k}(S_n)$-modules

$$\begin{array}{ccccc} I(M_n) & \hookrightarrow & M_n & \xrightarrow{\epsilon} & \mathbf{k} \\ \uparrow{\scriptstyle p'} & & \uparrow{\scriptstyle p''} & & \| \\ I(\gamma_n) & \hookrightarrow & \gamma_n & \xrightarrow{\epsilon} & \mathbf{k}, \end{array}$$

where $\epsilon(x_{\sigma(1)} \cdots x_{\sigma(n)}) = 1$ for $\sigma \in S_n$.

Notice that $Q(T^{(2)}(V))_n$ is isomorphic functorially to $I(M_n) \otimes_{\mathbf{k}(S_n)} V^{\otimes n}$ by Proposition 8.16. Let $\phi\colon I(M_n) \to \bar{V}^{\otimes n}$ be the composite

$$I(M_n) = I(M_n) \otimes_{\mathbf{k}(S_n)} \gamma_n \subseteq I(M_n) \otimes_{\mathbf{k}(S_n)} \bar{V}^{\otimes n} \cong Q(T^{(2)}(\bar{V}))_n \xrightarrow{s} I(T^{(2)}(\bar{V}))_n \subseteq \bar{V}^{\otimes n}.$$

Then $\phi(I(M_n)) \subseteq I(T^{(2)}(\bar{V})) \cap \gamma_n \subseteq I(\gamma_n) \subseteq \gamma_n$ and the map $\phi\colon I(M_n) \to \gamma_n$ is a map of $\mathbf{k}(S_n)$ modules. Notice that γ_n is a free $\mathbf{k}(S_n)$-module. Therefore γ_n is an injective $\mathbf{k}(S_n)$-module (see [13]) and so there is a map of $\mathbf{k}(S_n)$-modules $\tilde{\phi}\colon M_n \to \gamma_n$ such that $\tilde{\phi}|_{I(M_n)} = \phi\colon I(M_n) \to \gamma_n$.

Let $f\colon \gamma_n \to \gamma_n$ be the composite

$$\gamma_n \xrightarrow{p''} M_n \xrightarrow{\tilde{\phi}} \gamma_n$$

and let $N = \operatorname{colim}_f \gamma_n$. Then N is a $\mathbf{k}(S_n)$-retract of γ_n and so N is a projective $\mathbf{k}(S_n)$-module. Recall that the composite $q \circ s\colon Q(T^{(2)}(V))_n \to Q(T^{(2)}(V))_n$ is the identity. It follows that the composite

$$I(M_n) \xrightarrow{\phi} I(T^{(2)}(\bar{V})) \cap \gamma_n \hookrightarrow I(\gamma_n) \xrightarrow{p'} I(M_n)$$

is the identity map and so

$$\dim(N) \geq \dim(I(M_n)) = n - 1.$$

On the other hand, N is a $\mathbf{k}(S_n)$-retract of M_n. Thus

$$\dim(N) \leq \dim(M_n) = n.$$

Now N is a projective $\mathbf{k}(S_n)$-module. Let P be the Sylow p-subgroup of S_n. Since projective modules over $\mathbf{k}(G)$ are free when G is a p-group, N is a free $\mathbf{k}(P)$-module and so $\dim(N)$ is divisible by the order of P. One gets a contradiction and the assertion follows. □