ON NATURAL COALGEBRA DECOMPOSITIONS OF TENSOR ALGEBRAS AND LOOP SUSPENSIONS

PAUL SELICK AND JIE WU

ABSTRACT. We consider functorial decompositions of $\Omega \Sigma X$ in the case where $X$ is a $p$-torsion suspension. By means of a geometric realization theorem, we show that the problem can be reduced to the one obtained by applying homology: that of finding natural coalgebra decompositions of tensor algebras. We solve the algebraic problem and give properties of the piece $A_\text{min}(V)$ of the decomposition of $T(V)$ which contains $V$ itself, including verification of the Cohen conjecture that in characteristic $p$ the primitives of $A_\text{min}(V)$ are concentrated in degrees of the form $p^i$. The results tie in with the representation theory of the symmetric group and in particular produce a maximum projective submodule of the important $S_n$-module Lie($n$).

CONTENTS

1. Introduction 2
2. Natural coalgebra transformations of tensor algebras 7
3. Geometric Realizations and the Proof of Theorem 1.3 15
4. Existence of Minimal Natural Coalgebra Retracts of Tensor Algebras 19
5. Some Lemmas on Coalgebras 28
6. Functorial Version of the Poincaré-Birkhoff-Witt Theorem 32
7. Projective $k(S_n)$-Submodules of Lie($n$) 46
8. The Functor $A_{\text{min}}$ over a Field of Characteristic $p > 0$ 55
  8.1. An upper bound on the size of $A_{\text{min}}(V)$ 55
  8.2. Some general theorems on natural coalgebra retracts of $T(V)$ 62
  8.3. A coalgebra filtration on the functor $A_{\text{min}}$ 68
  8.4. A lower bound on the growth of $A_{\text{min}}(V)$ 71
9. Proof of Theorems 1.1 and 1.6 80
10. The Functor $L'_n$ and the Associated $k(\Sigma_n)$-Module Lie($n$) 84
11. Examples 97
  11.1. The functor $A_{\text{min}}^{n}$ for $n \leq p$ 97
  11.2. The functor $B_{\text{max}}$ 100

The authors are grateful to NSERC and the Fields Institute for support.
11.3. The symmetric group module $\text{Lie}^{\text{max}}_P(p)$ 102  
11.4. The symmetric group module $\text{Lie}^{\text{max}}_P(6)$ in characteristic 2 103  
11.5. Decompositions of $\Omega\Sigma^2 X$ for two-cell complexes $X$ 104  
11.6. The PBW map in characteristic 0 107  
References 108

1. Introduction

In classifying any mathematical structure, it is helpful to analyze the irreducible or indecomposable components. The geometric problem studied in this paper is that of finding natural homotopy decompositions of the loop-suspension $\Omega\Sigma X$ where $X$ is itself a $p$-torsion suspension for some prime $p$. The first main result is the construction of a functor $A(X)$ which gives the smallest natural homotopy retract of $\Omega\Sigma X$ whose mod $p$ homology contains $H_n(X; \mathbb{Z}/p\mathbb{Z})$. (See Theorem 1.1.) Among the properties of $A(X)$ which we show is that, as conjectured by Cohen, the primitives in the coalgebra $H_n(A(X); \mathbb{Z}/p\mathbb{Z})$ are concentrated in weights of the form $p^k$ within the tensor algebra $T(H(X; \mathbb{Z}/p\mathbb{Z}) \cong H_n(\Omega\Sigma X; \mathbb{Z}/p\mathbb{Z})$.

By means of the “Geometric Realization Theorem” (see Theorem 1.3) we show the above geometric problem to be equivalent to the algebraic problem of finding natural coalgebra decompositions of (ungraded) primitively generated tensor algebras over the field $\mathbb{F}_p$. With the problem thus reduced to algebra we consider the latter problem over an arbitrary field (not necessarily of characteristic $p$). The solution to this problem (and thus to the original geometric problem) is given in Theorem 1.5 and its generalization Theorem 6.5, which gives the more complete decomposition.

We refer to Theorem 6.5 as a functorial version of the Poincaré-Birkhoff-Witt Theorem. Recall that the Poincaré-Birkhoff-Witt Theorem gives a coalgebra isomorphism $U(L) \cong S(L)$ for a Lie algebra $L$, where $S(W)$ denotes the free commutative algebra on $W$, and the coalgebra structure on $S(W)$ is the one under which it becomes a primitively generated Hopf algebra. (That is, elements of $W$ are primitive.) Applying this in the case where $L = L(V)$, the free Lie algebra on the vector space $V$, gives a coalgebra decomposition $\text{PBW}_V : T(V) = U(L(V)) \cong S(L(V)) \cong S(\bigoplus_{n=1}^{\infty} L_n(V)) \cong \bigotimes_{n=1}^{\infty} S(L_n(V))$. However this isomorphism is not natural with respect to maps of the vector space $V$; it is natural with respect to maps of ordered bases for vector spaces. For example, if $V = \langle x, y \rangle$ with $x < y$ then $\text{PBW}_V(x \otimes y) = xy$ while $\text{PBW}_V(y \otimes x) = xy - [x, y]$ so $\text{PBW}_V$ does not commute with the map which interchanges $x$ and $y$. Without concerning ourselves for the moment as to the definition of the factors $A^{\text{min}}_n(V; L_n^{\text{max}})$ which appear in Theorem 6.5, we observe that if $\text{char}(k) = 0$ then according to Proposition 6.12, $A^{\text{min}}_n(V; L_n^{\text{max}}) \cong S(L_n(V))$, so
that the right hand side of Theorem 6.5 is identical to that of the Poincaré-Birkhoff-Witt decomposition in this case. However the isomorphism is different. For example, if \( \phi_V \) is the isomorphism of Theorem 6.5, then \( \phi_V(x \otimes y) = (xy + [x, y])/2 \) and \( \phi_V(y \otimes x) = (xy - [x, y])/2 \) and unlike PBW\(_V\), this is indeed natural with respect to maps of \( V \).

As the reader has no doubt already guessed upon seeing the division by 2 in the preceding formulas, Theorem 6.5 becomes substantially different when \( \text{char} \ k > 0 \). Since the maps are required to commute with interchange of variables, the representation theory over \( k \) of the symmetric groups \( S_n \) must be involved. In fact, any natural self-transformation of the tensor algebra functor \( T(\ ) \) is determined by a sequence of elements \( (\lambda_n) \in k(S_n) \) from the group algebras (see Lemma 2.1).

Let \( V = \langle x_1, x_2, \ldots, x_n \rangle \) be an \( n \)-dimensional vector space over \( k \). The \( k(S_n) \)-module \( \text{Lie}(n) \) is defined as the (vector) subspace of \( V^{\otimes n} \subset T(V) \) generated by the iterated commutators \( \{[\ldots[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}], \ldots, x_{\sigma(n)}]\}_{\sigma \in S_n} \), made into an \( k(S_n) \)-module by letting \( S_n \) act by permutation on the basis of \( V \). This intriguing module has arisen in recent work of Cohen, Dwyer-Hirschorn, and others, because it appears in the homology of various interesting spaces. In spite of much effort, representation theory of the symmetric groups in the modular case (when the characteristic of the field divides the order of the group) remains, for the most part, a mystery. In particular, not much is known about the algebraic properties of the module \( \text{Lie}(n) \).

An important aspect of our work is that it sheds some light on these properties as follows. Our natural decomposition of \( T(\ ) \) yields a decomposition of \( k(S_n) \) for each \( n \) resulting in the construction of an important projective \( k(S_n) \)-submodule of \( \text{Lie}(n) \), which we call \( \text{Lie}(n)^{\text{max}} \). We show (Theorem 7.4) that it is the maximum projective \( k(S_n) \)-submodule of \( \text{Lie}(n) \) in the sense that any projective \( k(S_n) \)-submodule of \( \text{Lie}(n) \) is a \( k(S_n) \)-retract of \( \text{Lie}(n)^{\text{max}} \). The interest here is in the modular case, since it is a well known consequence of the Dynkin-Specht-Weber relation \( \beta_n \circ \beta_n = n \beta_n \) that if \( \text{char} k \) does not divide \( n \) then \( \text{Lie}(n) \) is itself projective (and thus \( \text{Lie}(n)^{\text{max}} \) = \( \text{Lie}(n) \) for such \( n \)). Here \( \beta_n : V^{\otimes n} \to V^{\otimes n} \) is given by \( \beta_n(v_1 v_2 v_3 \cdots v_n) = [\ldots[v_1, v_2], v_3, \ldots, v_n] \).

The penultimate section of this paper considers the decomposition \( T(V) \cong A(V) \otimes B(V) \) of Theorem 1.5 and describes properties of the functor \( A(\ ) \) (denoted also as \( A^{\text{min}}(\ ) \)). This section contains the proof of the Cohen conjecture (Theorem 8.3) referred to earlier. Any decomposition of a Hopf algebra determines Hopf algebra structures on each of the factors, but in general these have no good Hopf algebra properties, and in particular the inclusion and projection maps onto the factors need not be Hopf algebra maps. Since part of the content of Theorem 1.4 is that this decomposition has the extra property that \( B(V) \) is a sub Hopf algebra of \( T(V) \), we are able to use our solution to the Cohen conjecture to give an upper bound on the size of \( A(V) \) (Corollary 8.5). Although we give (Corollary 8.27) a complete description
of $A_n^{\text{min}}(V)$ in the first interesting case ($n = p$), we are unable to give a complete description in general. It turns out that $A_n^{\text{min}}(V)$ agrees with the upper bound given by Corollary 8.5 when $n < p^2$; however we know (unpublished) that equality fails for $n = p^2$ when $p = 3$. This section also contains a coalgebra filtration on $A_n^{\text{min}}(V)$ which is used to give a lower bound on its growth.

While, by construction, there are no nontrivial natural retracts of $A(\ )$, it is possible that $A(X)$ decomposes further for particular spaces $X$. Indeed this is known to happen in the case where $X$ is the Moore space $P^n(p) = S^{n-1} \cup e^n$ for $p > 2$, according to the complete decomposition of these spaces given by Cohen-Moore-Neisendorfer ([5, 6, 7, 14]). There is no corresponding decomposition known however when $p = 2$, and one of the primary motivations for considering the problems discussed in this paper is the construction of the space $A(P^2(2))$. It is hoped that its study will lead to greater knowledge of the homotopy theory of the mod 2 Moore space and perhaps give information about mod 2 exponents for the homotopy groups of spheres.

As noted above, the algebraic results of this paper are also valid in characteristic 0, however the resulting theorems are already well known in that case. Similarly, since a rational co-$H$-space is a wedge of spheres, the rational analogue of the geometric problem is not an interesting question. We also note that the condition that $X$ be a suspension can be weakened somewhat. The proofs go through without modification in the case where $X$ is a simply connected co-$H$-space, and all that is really needed is that $X$ be simply connected and conilpotent. (That is, that the reduced diagonal given by the composite $X \to X \times X - > X \wedge X$ be null.)

The remainder of this introduction describes the results in more detail.

Let $X$ be a $p$-torsion suspension and let $\Omega\Sigma X$ be the loop space of the suspension of $X$. The unreduced and reduced mod $p$ homology of $X$ will be denoted by $H_*(X)$ and $\tilde{H}_*(X)$, respectively. Recall that $H_*(\Omega\Sigma X)$ is isomorphic to the tensor algebra $T(V)$ as Hopf algebras, where $V = \tilde{H}_*(X)$ and $T(V)$ is given the Hopf algebra structure determined by making $V$ primitive. Let $L_n(V)$ be the set of homogeneous Lie elements of tensor length $n$ in the tensor algebra $T(V)$ for any (ungraded or connected graded) vector space $V$. In the following theorem, $V = \tilde{H}_*(X)$.

**Theorem 1.1.** Let $X$ be a $p$-torsion suspension. Then there is a natural homotopy decomposition

$$\Omega\Sigma X \simeq A(X) \times B(X)$$

such that

1) $V \subseteq H_*(A(X))$;

2) $B(X)$ is a loop suspension and the injection $B(X) \to \Omega\Sigma X$ is a loop map;

3) $L_n(V) \subseteq H_*(B(X))$ if $n$ is not a power of $p$.
Remark 1.2. The functor $A$ is the geometric realization of the algebraic functor $A\min$ described later which gives natural minimal coalgebra retracts of tensor algebras. The space $B(X)$ is the loop suspension of the wedge of certain functorial retracts of the self smash product $X^{\otimes n}$ for $n \geq 2$. So this theorem can be regarded as a functorial version (on general loop suspensions) of the classical theorems on the decomposition of $\Omega P^n(p^r)$ [5, Theorem 1.1].

If there is no possible confusion, we use the notation $V$ for either a graded module or an ungraded module. Otherwise, we will write $V^u$ for ungraded modules. Also we use the convention that an ungraded module $V^u$ can be regarded as a connected graded module by assigning dimension 2 to elements of $V^u$. [12].

The proof of this theorem will be given through the following steps. The first step is to reduce the problem of natural decompositions of loop spaces of double suspensions to the problem of natural coalgebra decomposition of tensor algebras.

Theorem 1.3 (Geometric Realization Theorem). Let $f_{V^u}: T(V^u) \to T(V^u)$ be a morphism of ungraded coalgebras over $\mathbb{Z}/p\mathbb{Z}$ such that $f_V$ is a natural transformation of the functor $T$. Then there is a functorial morphism of graded coalgebras $f_{\text{graded}}: T(V) \to T(V)$ for any connected graded module $V$ such that

1. $\varphi_{V^u} = f_{V^u}$ for any ungraded module $V^u$;
2. $f_{V^u}^\text{graded}: T(V) \to T(V)$ is a functorial morphism of bigraded coalgebras, where the bi-grading in $T(V)$ is given in the canonical way;
3. for any suspension $X$, then there exists a map $\phi_X: \Omega\Sigma X \to \Omega\Sigma X$, “functorial up to homotopy” in $X$, such that

$$\phi_X* = \varphi_{\Omega\Sigma X}^\text{graded}: H_*(\Omega\Sigma X) = T(\Omega\Sigma X) \to T(\Omega\Sigma X).$$

The map $\phi_X: \Omega\Sigma X \to \Omega\Sigma X$ is functorial up to homotopy means that for any suspensions $X$ and $Y$ and any map $f: X \to Y$, where we do not assume that $f$ is a suspension, the diagram

$$\begin{array}{ccc}
\Omega\Sigma X & \xrightarrow{\phi_X} & \Omega\Sigma X \\
\downarrow{\Omega\Sigma f} & & \downarrow{\Omega\Sigma f} \\
\Omega\Sigma Y & \xrightarrow{\phi_Y} & \Omega\Sigma Y
\end{array}$$

commutes up to homotopy. The idea of the proof of this theorem is to compare the Cohen progroup [3, 4] for natural transformations of loop spaces of double suspensions with the progroup introduced in Section 2.
The second step is to show the existence of functorial indecomposable coalgebra retracts of tensor algebras. In the following theorems, $V$ means an ungraded $k$-module and the ground ring $k$ is any field. Let $M$ be a functor from $k$-modules to $k$-modules such that $M(V)$ is a sub $k$-module of $T(V)$. A sub coalgebra $A(V)$ of $T(V)$ is called a natural minimal coalgebra retract over $M(V)$ if

1) $A$ is a functor and $M$ is a subfunctor of $A$;
2) $A$ is a retract of $T$ as functors from $k$-modules to coalgebras;
3) $A(V)$ satisfies the following minimality condition:
   if $C(V)$ is a sub coalgebra of $T(V)$ that satisfies conditions 1) and 2) above, then $A$ is a retract of $C$ as functors from $k$-modules to coalgebras.

**Theorem 1.4** (Existence of natural minimal coalgebra retracts). *For any functor $M$ from $k$-modules to $k$-modules such that $M(V)$ is a sub $k$-module of $T(V)$, there exists a unique, up to isomorphism, natural minimal coalgebra retract of $T(V)$ over $M(V)$."

The natural minimal coalgebra retract of $T(V)$ over $V$ will be denoted $A^{\min}(V)$. The proof of this theorem will be given by abstract argument instead of explicit determination.

The third step is to describe some properties of the primitive elements of $A^{\min}(V)$.

**Theorem 1.5.** *There is a natural sub Hopf algebra of $T(V)$, which is denoted by $B^{\max}(V)$, such that*

1) *there is a natural coalgebra decomposition
   \[ T(V) \cong B^{\max}(V) \otimes A^{\min}(V); \]
2) $L_n(V) \subseteq B^{\max}(V)$ if $n$ is not a power of $p$, where $p$ is the characteristic of $k$.

Note that while any coalgebra decomposition of a Hopf algebra determines a Hopf algebra structure on each of the factors, these structures are not in general compatible. In this case however, as noted in the theorem, the inclusion $B^{\max}(V) \hookrightarrow T(V)$ is multiplicative, although the inclusion of $A^{\min}(V)$ and projections onto the factors are not. Similarly, in the geometric realization, while each factor becomes an $H$-space, $B(X) \rightarrow \Omega \Sigma X$ is an $H$-map, but the other maps are not.

The functorial retract $A(X)$ in Theorem 1.1 is given by the geometric realization of the functor $A^{\min}$ as follows. Let $f: T(V) \rightarrow T(V)$ be the composite

\[ T(V) \xrightarrow{r^{A^{\min}}} A^{\min}(V) \subseteq T(V), \]

where $r^{A^{\min}}$ is a functorial coalgebra retraction. Given a suspension $X$, let $\phi_X: \Omega \Sigma X \rightarrow \Omega \Sigma X$ be the geometric realization of $f$ as in Theorem 1.3. Let $A^{\min}(X)$ be the homotopy colimit

\[ A^{\min}(X) = \text{hocolim}_{\phi_X} \Omega \Sigma X. \]
Then the functorial retract $A(X)$ is $A^\text{min}(X)$. Similarly, the complementary factor $B(X)$ is given by the geometric realization of the functor $B^\text{max}$.

The retract $A^\text{min}(X)$ catches the maximum exponent of $\pi_*(\Sigma X)$ in the following sense. Let $\exp(\pi_*(X)) \leq \infty$ denote the exponent of the homotopy groups of $X$ and let $X^{(n)}$ denote the $n$-fold self smash product of $X$.

**Theorem 1.6.** Let $X$ be a $p$-torsion suspension. Then there are inequalities

$$\exp(\pi_*(A^\text{min}(X))) \leq \exp(\pi_*(\Sigma X)) \leq \max\{\exp(\pi_*(A^\text{min}(X^{(n)}))) ; 1 \leq n < \infty\}.$$  

The article is organized as follows. In section 2, we study natural coalgebra transformations of tensor algebras. The proof of Theorem 1.3 is given in section 3, where Theorem 1.3 is Corollary 3.5. In section 4, we study the existence of natural minimal coalgebra retracts of tensor algebras. The proof of Theorem 1.4 is given in this section where Theorem 1.4 is Theorem 4.12. We give some lemmas on coalgebras in section 5. In section 6, we study the functorial version of the Poincaré-Birkhoff-Witt theorem. The fact that the complementary factor $B^\text{max}(V)$ is a sub Hopf algebra of $T(V)$ is given by Proposition 6.1. This proves the first part of Theorem 1.5. We give some lemmas on the $S_n$-module $\text{Lie}(n)$ in section 7. In section 8, we study the functors $A^\text{min}$ and $B^\text{max}$ over a field with non-zero characteristic. Sections 5, 6 and 7 are preparation for the proof of Theorem 8.3. The second part of Theorem 1.5 is Corollary 8.4. The proofs of Theorems 1.1 and 1.6 are given in section 9.

2. Natural coalgebra transformations of tensor algebras

In this section, the ground ring is a field $\mathbf{k}$ and $V$ is a connected graded $\mathbf{k}$-module.

**Lemma 2.1.** Let $\phi_V : V^\otimes n \rightarrow V^\otimes m$ be a functorial map of graded $\mathbf{k}$-modules and let $a_1, \ldots, a_n$ be $n$ homogeneous elements in $V$.

1) If $n = m$, then the element $\phi_V(a_1 \otimes \cdots \otimes a_n)$ lies in to the sub graded $\mathbf{k}$-module of $V^\otimes n$ spanned by the elements

$$a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

as $\sigma$ runs through all elements in $S_n$.

2) If $n \neq m$, then $\phi_V$ is the zero map.

**Proof.** Let $a_1 \cdots a_n$ denote $a_1 \otimes \cdots \otimes a_n$ in the tensor product. Let $\bar{V}$ be the graded $\mathbf{k}$-module with basis $\{x_1, \ldots, x_n\}$ where $|x_j| = |a_j|$ for $1 \leq j \leq n$. Let $f : \bar{V} \rightarrow V$ be the map of graded $\mathbf{k}$-modules given by $f(x_j) = a_j$. By hypothesis there is a
commutative diagram
\[
\begin{array}{ccc}
V_{\otimes}^{\ominus n} & f_{\otimes n} & V_{\otimes}^{\ominus n} \\
\phi_V & \Downarrow & \phi_V \\
V_{\otimes}^{\ominus m} & f_{\otimes m} & V_{\otimes}^{\ominus m}.
\end{array}
\]

It suffices to show that the assertions hold for the case where \( V = \tilde{V} \) and \( a_j = x_j \) for \( 1 \leq j \leq n \). Let \( d_j : \tilde{V} \to \tilde{V} \) be the map of graded \( k \)-modules defined by
\[
d_j(x_i) = \begin{cases} 
x_i & \text{if } i \neq j \\
0 & \text{if } i = j.
\end{cases}
\]

1) Let \( d_j \) denote \( d_j^{\otimes n} : V_{\otimes}^{\ominus n} \to V_{\otimes}^{\ominus n} \) for \( 1 \leq j \leq n \). Then we have
\[
d_j(x_{i_1} \cdots x_{i_n}) = \begin{cases} 
x_{i_1} \cdots x_{i_n} & \text{if } j \notin \{i_1, \ldots, i_n\} \\
0 & \text{if } j \in \{i_1, \ldots, i_n\}.
\end{cases}
\]

Let \( \gamma_n(x_1, \ldots, x_n) \) be the sub graded \( k \)-module of \( V_{\otimes}^{\ominus n} \) spanned by the elements
\[
x_{\sigma(1)} \cdots x_{\sigma(n)},
\]
as \( \sigma \) runs through all elements in \( S_n \). Then
\[
\gamma_n(x_1, \ldots, x_n) \subseteq \bigcap_{1 \leq j \leq n} \ker(d_j).
\]

Conversely, let
\[
\alpha = \sum_I k_I x_{i_1} \cdots x_{i_n} \in \bigcap_{1 \leq j \leq n} \ker(d_j),
\]
where \( I = (i_1, \ldots, i_n) \) with \( 1 \leq i_p \leq n \) for \( 1 \leq p \leq n \) and \( k_I \in R \). Let \( \alpha_1 = \sum_{i \notin I} k_I x_{i_1} \cdots x_{i_n} \) and let \( \alpha_2 = \sum_{i \in I} k_I x_{i_1} \cdots x_{i_n} \). Then
\[
\alpha = \alpha_1 + \alpha_2, \quad d_1(\alpha_1) = \alpha_1, \quad d_1(\alpha_2) = 0.
\]

Notice that \( d_1(\alpha) = 0 \). Thus
\[
\alpha_1 = d_1(\alpha_1) = d_1(\alpha) = 0
\]
and so
\[
\alpha = \alpha_2 = \sum_{i \in I} k_I x_{i_1} \cdots x_{i_n}.
Inductively, we have
\[ \alpha = \sum_{1, 2, \ldots, n \in I} k_I x_1 \cdots x_n \in \gamma_n(x_1, \ldots, x_n). \]

Thus
\[ \gamma_n(x_1, \ldots, x_n) = \bigcap_{1 \leq j \leq n} \text{Ker}(d_j). \]

Notice that
\[ d_j \circ \phi_V = \phi_V \circ d_j \]
for \( 1 \leq j \leq n \). Thus
\[ \phi_V(\gamma_n(x_1, \ldots, x_n)) = \phi_V\left( \bigcap_{1 \leq j \leq n} \text{Ker}(d_j) \right) \subseteq \bigcap_{1 \leq j \leq n} \text{Ker}(d_j) = \gamma_n(x_1, \ldots, x_n). \]

Assertion 1) follows.

2) We consider two cases.

Case I: \( m < n \).

From the commutative diagram
\[
\begin{array}{ccc}
\widetilde{V} \otimes^n & \xrightarrow{\phi_V} & \widetilde{V} \otimes^m \\
\downarrow d^\otimes_j & & \downarrow d^\otimes_j \\
\widetilde{V} \otimes^n & \xrightarrow{\phi_V} & \widetilde{V} \otimes^m 
\end{array}
\]
for \( 1 \leq j \leq n \), one gets
\[ \phi_V(x_1 \cdots x_n) \in \bigcap_{1 \leq j \leq n} \text{Ker}(d^\otimes_j: V^\otimes^m \to V^\otimes^m) = 0 \]
by the proof of assertion (1) and Case I of assertion 2) follows.

Case II: \( m > n \).

Let \( V \) be any connected graded \( k \)-module of finite type. Notice that the dual map
\[ \phi^*_V: (V^*)^\otimes^m \to (V^*)^\otimes^n \]
is natural transformation for \( k \)-modules \( V^* \) of finite type. By Case I of assertion 2), we have
\[ \phi^*_V = 0 \]
for any connected graded \( k \)-module of finite type and Case II of assertion 2) follows.

\[ \square \]
Let $V$ be a connected graded $k$-module. Then $T(V) = \bigoplus_{n \geq 0} V^\otimes n$ is a bigraded $k$-module.

**Corollary 2.2.** Let $\phi_V: T(V) \to T(V)$ be a functorial map of graded $k$-modules. Then $\phi_V$ is a functorial map of bi-graded $k$-modules.

**Definition 2.3.** The **James (coalgebra) filtration** of the tensor algebra $T(V)$ is defined by

$$J_n(V) = \bigoplus_{0 \leq j \leq n} T_j(V)$$

with the canonical comultiplication.

Let $C$ be a (graded) coalgebra and let $A$ be a (graded) algebra. Recall that the convolution product $f \ast g$ of $f, g: C \to A$ is defined by

$$C \xrightarrow{\psi} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A,$$

where $\psi: C \to C \otimes C$ is the comultiplication and $\mu: A \otimes A \to A$ is the multiplication. (See [12].) Let $\text{Hom}_{\text{coalg}}(J_n(-), T(-))$ and $\text{Hom}_{\text{coalg}}(T(-), T(-))$ denote the set of all of natural transformations of coalgebras from $J_n(V)$ and $T(V)$ to $T(V)$, respectively, with the multiplications given by the convolution product. The James filtration

$$J_0(V) \subseteq J_1(V) \subseteq \cdots \subseteq J_n(V) \subseteq \cdots \subseteq T(V)$$

induces a cofiltration of groups

$$\text{Hom}_{\text{coalg}}(T(-), T(-)) \to \cdots \to \text{Hom}_{\text{coalg}}(J_n(-), T(-)) \to \cdots \to \text{Hom}_{\text{coalg}}(J_0(-), T(-))$$

with

$$\text{Hom}_{\text{coalg}}(T(-), T(-)) = \lim_{\longrightarrow} \text{Hom}_{\text{coalg}}(J_n(-), T(-)).$$

Let $L_n(V)$ denote the set of homogeneous Lie elements of tensor length $n$ in the tensor algebra $T(V)$ and let $\text{Hom}_k((-)^\otimes n, L_n(-))$ denote the set of all of natural transformations of graded modules from the functor $(-)^\otimes n$ to the functor $L_n$.

**Proposition 2.4.** Let $\Gamma_n$ be the kernel of the homomorphism

$$\text{Hom}_{\text{coalg}}(J_n(-), T(-)) \to \text{Hom}_{\text{coalg}}(J_{n-1}(-), T(-)).$$

Then there is an isomorphism of groups

$$\Gamma_n \cong \text{Hom}_k((-)^\otimes n, L_n(-)).$$
Proof. Let $\phi_V \in \Gamma_n$ and let $\theta(\phi_V): V^\otimes n \to T(V)$ be the restriction of $\phi$ to the summand $V^\otimes n$ of $J_n(V)$. By Lemma 2.1, we have that $\theta(\phi_V)(V^\otimes n) \subseteq V^\otimes n$. By hypothesis, $\phi_V$ is a morphism of graded coalgebras. From the commutative diagram

$$
\begin{array}{ccc}
T(V) & \xrightarrow{\phi_V} & T(V) \\
\downarrow{\psi} & & \downarrow{\psi} \\
T(V) \otimes T(V) & \xrightarrow{\phi_V \otimes \phi_V} & T(V) \otimes T(V)
\end{array}
$$

together with the condition that $\phi_V$ belongs to $\Gamma_n$, we have

$$
\theta(\phi_V)(V^\otimes n) \subseteq P_n(V),
$$

where $P_n(V)$ is the set of primitive elements of tensor length $n$ in $T(V)$.

Let $a_1, \ldots, a_n$ be homogeneous elements in $V$ and let $\tilde{V}$ be the graded $k$-module with basis $\{x_1, \ldots, x_n\}$ where $|x_j| = |a_j|$ for $1 \leq j \leq n$. Let $\gamma_n$ be the graded submodule of $\tilde{V}^\otimes n$ generated by the homogeneous elements

$$x_{\sigma(1)} \cdots x_{\sigma(n)}$$

for $\sigma \in S_n$. By Lemma 2.1, we have that

$$\theta(\phi_{\tilde{V}})(x_1 \cdots x_n) \in \gamma_n \cap P_n(\tilde{V}) \subseteq L_n(\tilde{V}).$$

Let $f: \tilde{V} \to V$ be a $k$-linear map given by

$$f(x_j) = a_j$$

for $1 \leq j \leq n$. From the commutative diagram

$$
\begin{array}{ccc}
\tilde{V}^\otimes n & \xrightarrow{f^\otimes n} & V^\otimes n \\
\downarrow{\theta(\phi_{\tilde{V}})} & & \downarrow{\theta(\phi_V)} \\
T(\tilde{V}) & \xrightarrow{T(f)} & T(V),
\end{array}
$$

we have that $\theta(\phi_V)(a_1 \cdots a_n) \in L_n(V)$ for any $a_1, \ldots, a_n \in V$ and so one gets a function

$$\theta: \Gamma_n \to \text{Hom}_k((-)^\otimes n, L_n(-)).$$

It is easy to check that

1. $\theta(\phi_V \ast \phi'_V) = \theta(\phi_V) + \theta(\phi'_V)$ for $\phi_V, \phi'_V \in \Gamma_n$;
2. $\theta$ is a monomorphism.
Conversely, given a natural $k$-linear map $\lambda_V : V^\otimes n \to L_n(V)$, let $\phi_V : J_n(V) \to T(V)$ be the map defined by

1. $\phi_V : T_0(V) = k \to T_0(V) = k$ is the identity;
2. $\phi_V : T_j(V) \to T_j(V)$ is zero for $0 < j < n$;
3. $\phi_V = \lambda_V : T_n(V) = V^\otimes n \to L_n(V) \subseteq T_n(V)$.

Then $\phi_V \in \Gamma_n$ with $\theta(\phi_V) = \lambda_V$. The assertion follows.

Let $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$ be an $n$-tuple of positive integers. The length $l(I)$ is defined by

$$l(I) = i_1 + i_2 + \cdots + i_n.$$ 

Let $\tilde{V}(I)$ be the graded $k$-module with basis $\{x_1, \ldots, x_n\}$ where $|x_j| = i_j$ for $1 \leq j \leq n$. Let $\text{Lie}^I(n)$ be the graded sub $k$-module of $\tilde{V}(I)^\otimes n$ generated by the $n$-fold graded commutators

$$[[x_{\sigma(1)}, x_{\sigma(2)}], \ldots, x_{\sigma(n)}]$$

for $\sigma \in S_n$.

**Lemma 2.5.** There is an isomorphism of groups

$$\text{Hom}_k((-)^\otimes n, L_n(-)) \cong \prod_{1 \leq q < \infty} \bigoplus_{l(I) = q} \text{Lie}^I(n).$$

**Proof.** For a connected graded $k$-module $V = \bigoplus_{q=1}^\infty V_q$, we have natural decomposition

$$V^\otimes n \cong \bigoplus_{q=1}^\infty \bigoplus_{i_1 + \cdots + i_n = q} V_{i_1} \otimes \cdots \otimes V_{i_n}$$

in terms of grading. Thus there is a natural decomposition

$$\text{Hom}_k(V^\otimes n, L_n(V)) \cong \bigoplus_{q=1}^\infty \bigoplus_{i_1 + \cdots + i_n = q} \text{Hom}_k(V_{i_1} \otimes \cdots \otimes V_{i_n}, L_n(V)).$$

The assertion follows from the proof of Proposition 2.4.

Let $\tilde{J}_n(V) = J_n(V)/J_0(V)$. Notice that there is a unique Hopf algebra structure on $T(\tilde{J}_n(V))$ such that the comultiplication on $T(\tilde{J}_n(V))$ is induced by the comultiplication on $J_n(V)$.

**Lemma 2.6.** There is a functorial isomorphism of Hopf algebras

$$h_n : T(\tilde{J}_n(V)) \longrightarrow T(\bigoplus_{k=1}^n V^\otimes k),$$
such that the diagram

\[
\begin{array}{ccc}
T(J_n(V)) & \xrightarrow{h_n} & T(\bigoplus_{k=1}^n V^\otimes k) \\
\downarrow & & \downarrow \\
T(J_{n-1}(V)) & \xrightarrow{h_{n-1}} & T(\bigoplus_{k=1}^{n-1} V^\otimes k)
\end{array}
\]

commutes, where the Hopf algebra structure on \(T(\bigoplus_{k=1}^n V^\otimes k)\) is the one in which the elements in \(\bigoplus_{k=1}^n V^\otimes k\) are primitive.

**Proof.** Let \(H_k : T(V) \to T(V^\otimes k)\) be the James-Hopf map. (See [8].) Let \(f_n : J_n(V) \to T(\bigoplus_{k=1}^n V^\otimes k)\) be the composite

\[
J_n(V) \xrightarrow{\text{comult}} \bigotimes_{k=1}^n J_n(V) \hookrightarrow \bigotimes_{k=1}^n T(V) \xrightarrow{\otimes_{k=1}^n H_k} \bigotimes_{k=1}^n T(V^\otimes k) \xrightarrow{(i_1, i_2, \ldots, i_n)} T(\bigoplus_{k=1}^n V^\otimes k),
\]

where \((i_1, i_2, \ldots, i_n) : \otimes_{k=1}^n T(V^\otimes k) \to T(\bigoplus_{k=1}^n V^\otimes k)\) is the composite

\[
\bigotimes_{k=1}^n T(V^\otimes k) \xrightarrow{\otimes_{k=1}^n T(i_k)} \bigotimes_{k=1}^n T(V^\otimes k) \xrightarrow{\text{mult}} T(\bigoplus_{k=1}^n V^\otimes k)
\]

and \(i_s : V^\otimes s \to \bigoplus_{k=1}^s V^\otimes k\) is the canonical inclusion. Observe that because \(T(V)\) is cocommutative, each of the maps in the composition is a coalgebra map. Let \(h_n : T(J_n(V)) \to T(\bigoplus_{k=1}^n V^\otimes k)\) be the map of Hopf algebras induced by \(f_n\). It is routine to check that \(h_n\) is an isomorphism such that the diagram in the lemma commutes. The assertion follows. \(\square\)

**Theorem 2.7.** The cofiltration

\[
\text{Hom}_{\text{coalg}}(T(-), T(-)) \to \cdots \to \text{Hom}_{\text{coalg}}(J_n(-), T(-)) \to \cdots \to \text{Hom}_{\text{coalg}}(J_0(-), T(-))
\]

is a progroup with the property that there is an isomorphism of groups

\[
\text{Ker}(\text{Hom}_{\text{coalg}}(J_n(-), T(-)) \to \text{Hom}_{\text{coalg}}(J_{n-1}(-), T(-))) \cong \prod_{q=1}^\infty \bigoplus_{\ell(\ell) = q} \text{Lie}^\ell(n).
\]

**Proof.** By Proposition 2.4 and Lemma 2.5, it suffices to show that the homomorphism

\[
\text{Hom}_{\text{coalg}}(J_n(-), T(-)) \to \text{Hom}_{\text{coalg}}(J_{n-1}(-), T(-))
\]

is an epimorphism. By Lemma 2.6, \(T(J_{n-1}(V))\) is a natural coalgebra retract of \(T(J_n(V))\). The assertion follows. \(\square\)
A natural coalgebra transformation $\phi$ of graded tensor algebras may not have a geometric realization in general. That is, given a suspension $X$, there may not exist a map $f : \Omega \Sigma X \to \Omega \Sigma X$ such that

$$f_\ast = \phi : H_\ast(\Omega \Sigma X) = T(\tilde{H}_\ast(X)) \to H_\ast(\Omega \Sigma X) = T(\tilde{H}_\ast(X)).$$

To be realizable by a map up to a certain dimension often reflects certain properties of $X$.

**Example 2.8.** Let $k = \mathbb{Z}/2$ and let $f^n_V : V \to V$ be the $k$-linear map defined by

$$f^n_V(x) = \begin{cases} x & \text{if } |x| = n \\ 0 & \text{if } |x| \neq n \end{cases}$$

and let $T(f^n_V) : T(V) \to T(V)$ be the map of Hopf algebras by the functor $T$ on $f^n_V$. Let $n \geq 2$ and let $g = T(f^n_V) \ast T(f^{n+1}_V) : T(V) \to T(V)$ be the convolution product. Let $X = P^{n+1}(2) = \Sigma^{-1} \mathbb{R}P^2$ and let $u, v$ be non-zero elements in $H_n(X)$ and $H_{n+1}$, respectively. Consider the map $g_{H_\ast(X)} : H_\ast(\Omega \Sigma X) = T(u, v) \to H_\ast(\Omega \Sigma X) = T(u, v)$. Up to dimension $2n + 1$, we have

$$g_{H_\ast(X)}(u) = u, \quad g_{H_\ast(X)}(v) = v, \quad g_{H_\ast(X)}(u^2) = u^2,$$

$$g_{H_\ast(X)}(uv) = g_{H_\ast(X)}(vu) = uv, \quad g_{H_\ast(X)}([u, v]) = 0.$$

Using this, one can check that there exists a self map $\phi$ of the $(2n + 1)$-skeleton of $\Omega \Sigma X$ such that $\phi_* = g_{H_\ast(X)}$ up to dimension $2n + 1$ if and only if the element $[u, v]$ is spherical. Notice that $[u, v]$ is spherical in $H_\ast(\Omega \Sigma X) = H_\ast(\Omega P^{n+2}(2))$ if and only if the tangent bundle of $S^{n+2}$ does not have a nowhere zero vector field if and only if $n$ is even. (See [9].)

Now we consider the ungraded case. Although Example 2.8 shows us that an arbitrary natural transformation of graded tensor algebras need not be geometrically realizable, we will see in the next section that graded natural transformations which come from ungraded ones (in the sense of Definition 3.1) are realizable. Let $V^n$ be an ungraded $k$-module and let $\text{Hom}_k^n(T(-), T(-))$ be the set of all of natural morphisms of coalgebras from ungraded tensor algebra $T(V^n)$ to itself. Let $V$ be an ungraded $k$-module with basis $\{x_1, \ldots, x_n\}$ Recall that $\text{Lie}(n)$ is the sub $k$-module of $V^{\otimes n}$ generated by the $n$-fold (ungraded) commutators

$$[[x_{\sigma(1)}, x_{\sigma(2)}], \cdots, x_{\sigma(n)}]$$

for $\sigma \in S_n$. Notice that $V$ can be regarded as a graded $k$-module by assigning dimension 2 to its elements. We have the following theorem for the ungraded case.
Corollary 2.9. The cofiltration
\[ \text{Hom}^u_{\text{coalg}}(T(-), T(-)) \to \cdots \to \text{Hom}^u_{\text{coalg}}(J_n(-), T(-)) \to \cdots \to \text{Hom}^u_{\text{coalg}}(J_0(-), T(-)) \]
is a progroup with the property that there is an isomorphism of groups
\[ \text{Ker}(\text{Hom}^u_{\text{coalg}}(J_n(-), T(-))) \to \text{Hom}^u_{\text{coalg}}(J_{n-1}(-), T(-)) \cong \text{Lie}(n). \]

3. Geometric Realizations and the Proof of Theorem 1.3

In this section, we give geometric realizations of elements in \( \text{Hom}^u_{\text{coalg}}(J_n(-), T(-)) \) for \( 1 \leq n \leq \infty \). Theorem 1.3 will follow from these geometric realizations.

Let \( V^u \) be an ungraded \( k \)-module and let \( \phi_{V^u} : T(V^u) \to T(V^u) \) be a natural \( k \)-linear map. By Lemma 2.1, there exist \( k^n \in k \) for \( n \geq 1 \) and \( \sigma \in S_n \) such that
\[ \phi_{V^u}(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma \in S_n} k^n_{\sigma} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \]
for \( a_1, \ldots, a_n \in V^u \). The set \( \{k^n_{\sigma}\}_{n, \sigma} \) will be called the coefficient set of \( \phi_{V^u} \).

Let \( V \) be a connected graded \( k \)-module. Recall that the graded \( S_n \)-action on \( V^\otimes n \) is determined by
\[ \tau_{ij} : (\cdots \otimes a_i \otimes \cdots \otimes a_j \otimes \cdots) = (-1)^{|\text{deg} a_i||\text{deg} a_j|} (\cdots \otimes a_j \otimes \cdots \otimes a_i \otimes \cdots) \]
for the generators \( \tau_{ij} = (i \ j) \in S_n \).

Definition 3.1. Let \( \phi_{V^u} : T(V^u) \to T(V^u) \) be a natural \( k \)-linear map of ungraded tensor algebras with the coefficient set \( \{k^n_{\sigma}\}_{n, \sigma} \) and let \( V \) be a connected graded \( k \)-module. The graded \( k \)-linear map \( \phi_{V^u}^{\text{grade}} : T(V) \to T(V) \) is defined by
\[ \phi_{V^u}^{\text{grade}}(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma \in S_n} k^n_{\sigma} \cdot (a_1 \otimes \cdots \otimes a_n) \]
for \( n \geq 1 \) and \( a_1 \otimes \cdots \otimes a_n \in V^\otimes n \).

Lemma 3.2. Let \( \phi_{V^u} \) and \( \phi'_{V^u} : T(V^u) \to T(V^u) \) be natural \( k \)-linear maps of ungraded tensor algebras. Then
1) \( \phi_{V^u}^{\text{grade}} : T(V) \to T(V) \) is a natural \( k \)-linear map of graded tensor algebras;
2) If \( \phi_{V^u} \) is a natural morphism of coalgebras, then \( \phi_{V^u}^{\text{grade}} \) is a natural morphism of graded coalgebras;
3) The convolution products are preserved. That is,
\[ (\phi \ast \phi')^{\text{grade}}_{V^u} = \phi^{\text{grade}}_{V^u} \ast \phi'^{\text{grade}}_{V^u}; \]
4) \( \phi_{V^u}^{\text{grade}} \) is trivial if and only if \( \phi_{V^u} \) is trivial.

The proof is immediate.
Corollary 3.3. The function \( \phi_{1^i} \to \phi_{1^i}^{\text{grade}} \) is a monomorphism of groups from \( \text{Hom}_{\text{calc}}^u(T(-), T(-)) \) to \( \text{Hom}_{\text{calc}}(T(-), T(-)) \) and so \( \text{Hom}_{\text{calc}}^u(T(-), T(-)) \) is identified with a subgroup of \( \text{Hom}_{\text{calc}}(T(-), T(-)) \).

We need to recall some terminology from [3, 4] before proceeding to prove Theorem 1.3. Let \( X \) be a suspension. The group \( K_n(X) \) is the subgroup of \( [X^n, J(X)] \) generated by the homotopy classes that are represented by the composites

\[
X^n \xrightarrow{p_i} X \xrightarrow{c} J(X)
\]

for \( 1 \leq i \leq n \), where \( J(X) \) is the James-construction of \( X \) and \( p_i \) is the \( i \)-th coordinate projection. Let \( q_n : X^n \to J_n(X) \) be the quotient map. Notice that

\[
q_n^* : [J_n(X), J(X)] \to [X^n, J(X)]
\]

is a monomorphism of groups. Let \( H(n)(X) \) denote the intersection \( H(n)(X) = [J_n(X), J(X)] \cap K_n(X) \). Then one gets the progroup

\[
H(\infty)(X) \longrightarrow \cdots \longrightarrow H(n)(X) \longrightarrow \cdots \longrightarrow H(0)(X).
\]

As in [3, 4], we define the Cohen group \( K_n(x_1, x_2, \ldots, x_n) \) to be the group generated by \( \{x_1, x_2, \ldots, x_n\} \) with relations

1. \([\cdots [x_{i_1}, x_{i_2}], x_{i_3}], \cdots, x_{i_r}] = 1 \]
   if \( i_s = i_t \) for some \( 1 \leq s, t \leq r \), where \([a, b] = a^{-1} b^{-1} ab\).

2. \([\cdots [x_{i_1}^{m_1}, x_{i_2}^{m_2}], x_{i_3}^{m_3}], \cdots, x_{i_r}^{m_r}] = [\cdots [x_{i_1}, x_{i_2}], x_{i_3}], \cdots, x_{i_r}]^{m_1 m_2 m_3 \cdots m_r}\]

Following Cohen ([3, 4]) we also let \( K_n(x_1, x_2, \ldots, x_n) \) be the Cohen group (see [3, 4]) let \( H(n) \) be the equalizer of the projections \( p_j : K_n(x_1, \ldots, x_n) \to K_{n-1}(x_1, \ldots, x_{n-1}) \) for \( 1 \leq j \leq n \), where the homomorphism \( p_j : K_n(x_1, \ldots, x_n) \to K_{n-1}(x_1, \ldots, x_{n-1}) \) is given by

\[
p_j(x_i) = \begin{cases} 
  x_i & \text{for } i < j \\
  1 & \text{for } i = j \\
  x_{i-1} & \text{for } i > j.
\end{cases}
\]

Let \( H^R(\infty) = \lim_n H^R(n) \). Then \( H(n) \) is the universal group of \( H(n)(X) \) in the following sense:

1. There is a homomorphism

\[
e_X : H(n) \to H(n)(X) \subseteq [J_n(X), J(X)]
\]

for any suspension \( X \).
(2). There exists a suspension $X$ such that 
\[ e_X : H(n) \to [J_n(X), J(X)] \]
is a monomorphism.

(3). Let $\alpha \in H(n)$ and let $f_X : J_n(X) \to J(X)$ be a representation the homotopy
class $e_X(\alpha)$. Then $f_X$ is functorial up to homotopy.

The progroup $H(\infty)$ is the tower
\[ H(\infty) \longrightarrow \cdots \longrightarrow H(n) \longrightarrow \cdots \longrightarrow H(0) \]
with $e_X : H(\infty) \to H(\infty)(X) \subseteq [J(X), J(X)]$. Let $\Lambda(n)$ denote the kernel of
$H(n) \to H(n - 1)$. Let $\mathcal{V}$ be the free $\mathbb{Z}$-module with basis \{$x_1, \ldots, x_n$\}. Recall
that $\text{Lie}^\mathbb{Z}(n)$ is the sub $\mathbb{Z}$-module of $\mathcal{V}^{\otimes n}$ generated by the $n$-fold commutators
\[[[x_{\sigma(1)}, x_{\sigma(2)}], \ldots, x_{\sigma(n)}]]$ for $\sigma \in S_n$. Recall that, for each $\alpha \in \text{Lie}^\mathbb{Z}(n)$, there are
unique integers $n_\tau$ ($\tau \in S_{n-1}$) such that
\[ \alpha = \sum_{\tau \in S_{n-1}} n_\tau [[x_1, x_{\tau(2)}], \ldots, x_{\tau(n)}], \]
where $S_{n-1}$ acts on \{2, $\ldots$, $n$\}. See [4, 16].

**Theorem 3.4** (Cohen [4]). The group $\Lambda(n)$ is isomorphic to $\text{Lie}^\mathbb{Z}(n)$.

**Corollary 3.5** (Theorem 1.3). Let $f_{V^u} : T(V^u) \to T(V^u)$ be a morphism of ungraded
coulebras over $\mathbb{Z}/p\mathbb{Z}$ such that $f_V$ is a natural transformation of the functor $T$. Then
there is a functorial morphism of graded coulebras $f_{\text{gra}:} T(V) \to T(V)$ for any
connected graded module $V$ such that

1. $f_{V^u} = f_{V^u}$ for any ungraded module $V^u$;
2. $f_{V^u} = f_{V^u}$ is a functorial morphism of bigraded coulebras, where
the bi-grading in $T(V)$ is given in the canonical way;
3. for any suspension $X$, there exists a map $\phi_X : \Omega \Sigma X \to \Omega \Sigma X$, functorial
up to homotopy in $X$, such that
\[ \phi_{X*} = f_{\text{gra}, X}^* : H_*(\Omega \Sigma X) = T(H_*(X)) \to T(H_*(X)). \]

**Proof.** By taking homology, the map
\[ H_* : [J_n(X), J(X)] \longrightarrow \text{Hom}_{\text{coalg}}(H_*(J_n(X)), H_*(J(X))) = \]
\[ \text{Hom}_{\text{coalg}}(J_n(\tilde{H}_*(X)), T(\tilde{H}_*(X))) \]
induces a homomorphism
\[ \theta : H(n) \to \text{Hom}_{\text{coalg}}(J_n(-), T(-)) \]
for \( 0 \leq n \leq \infty \) such that the diagram

\[
\begin{array}{ccc}
[J_n(X), J(X)] & \xrightarrow{H_*} & \text{Hom}_{\text{colg}}(J_n(\H_*(X)), T(\H_*(X))) \\
\downarrow e_X & & \downarrow \text{evaluation} \\
H(n) & \xrightarrow{\theta} & \text{Hom}_{\text{colg}}(J_n(-), T(-))
\end{array}
\]

commutes for any suspension \( X \).

Let \( \tau \) belong to \( S_{n-1} \), acting on \( \{2, \ldots, n\} \), and let \( \alpha = [(x_1, x_{\tau(2)}), \ldots, x_{\tau(n)}] \in \Lambda(n) \). By the definition of \( K_n(X) \), the homotopy class

\[
e_X(\alpha) \in H(n)(X) \subseteq [J_n(X), J(X)] \xrightarrow{q_n^*} [X^n, J(X)]
\]

is represented by the composite

\[
\begin{array}{ccc}
X^n & \xrightarrow{q_n} & J_n(X) \\
& & \xrightarrow{\text{pinch}} X^{(n)} \\
& & \xrightarrow{1 \wedge \tau} X^{(n)} \\
& & \xrightarrow{W_n} J(X),
\end{array}
\]

where \( W_n \) is the iterated Samelson product taken from left to right and

\[
(1 \wedge \tau)(x_1 \wedge \cdots \wedge x_n) = x_1 \wedge x_{\tau(2)} \wedge \cdots \wedge x_{\tau(n)}
\]

for \( x_1, \ldots, x_n \in X \). Let \( f_X(\alpha) \) be the composite

\[
\begin{array}{ccc}
J_n(X) & \xrightarrow{\text{pinch}} & X^{(n)} \\
& & \xrightarrow{1 \wedge \tau} X^{(n)} \\
& & \xrightarrow{W_n} J(X).
\end{array}
\]

Consider \( f_X(\alpha)_* : J_n(\H_*(X)) \to T(\H_*(X)) \). Then

1. \( f_X(\alpha)_*|_{H_{n-1}(\H_*(X))} : H_{n-1}(\H_*(X)) \to T(\H_*(X)) \) is trivial;
2. \( f_X(\alpha)_*(a_1 \otimes \cdots \otimes a_n) = \beta_n(a_1 \otimes \tau \cdot (a_2 \otimes \cdots \otimes a_n)) \) for \( a_1, \ldots, a_n \in \H_*(X) \),

where \( \beta_n(a_1 \otimes \cdots \otimes a_n) = [[a_1, a_2], \ldots, a_n] \), the \textbf{graded} \( n \)-fold commutator, and \( S_{n-1} \) acts on \( H_*(X)^{\otimes n-1} \) by permuting position with sign.

Thus \( \theta([[x_1, x_{\tau(2)}], \cdots, x_{\tau(n)}]) \in \text{Hom}_{\text{colg}}^n(J_n(-), T(-)) \) for any \( \tau \in S_{n-1} \) and so

\[
\theta(\Lambda(n)) \subseteq \text{Hom}_{\text{colg}}^n(J_n(-), T(-))
\]

for each \( 1 \leq n \leq \infty \). From the commutative diagram

\[
\begin{array}{ccc}
H(n) & \xrightarrow{\theta} & \text{Hom}_{\text{colg}}(J_n(-), T(-)) \\
\downarrow & & \downarrow \\
H(n-1) & \xrightarrow{\theta} & \text{Hom}_{\text{colg}}(J_{n-1}(-), T(-)),
\end{array}
\]
by induction, we have
\[ \theta(H(n)) \subseteq \text{Hom}_\text{coalg}^n(J_n(-), T(-)) \]
for \(0 \leq n \leq \infty\). By Corollary 2.9, the kernel \(\Gamma_n\) of \(\text{Hom}_\text{coalg}^n(J_n(-), T(-)) \to \text{Hom}_\text{coalg}^n(J_{n-1}(-), T(-))\) is isomorphic to \(\text{Lie}(n)\) over \(\mathbb{Z}/p\). We have that the map \(\theta: \Lambda(n) \to \Gamma_n\) is onto. By induction, we have
\[ \theta: H(n) \to \text{Hom}_\text{coalg}^n(J_n(-), T(-)) \]
is onto for \(0 \leq n \leq \infty\) and so
\[ \theta: H(\infty) \to \text{Hom}_\text{coalg}^n(T(-), T(-)) \]
is onto. The assertion follows. \(\square\)

4. Existence of Minimal Natural Coalgebra Retracts of Tensor Algebras

In this section, we give a proof of Theorem 1.4. The ground ring is a field \(k\) and 
\(V\) means an ungraded \(k\)-module. Let \(M(V)\) be a functorial submodule of \(T(V)\).
That is \(M\) is a sub functor of \(T\) from \(k\)-modules to \(k\)-modules.

**Definition 4.1.** A coalgebra \(B(V)\) together with natural transformations \(s_B, r_B, j_B\)
is called a natural coalgebra retract of \(T(V)\) over \(M(V)\) if:

1. \(B\) is a functor from \(k\)-modules to \(k\)-coalgebras;
2. \(s_B\) and \(r_B\) are natural transformations of coalgebras \(s_B: B(V) \to T(V)\) and
   \(r_B: T(V) \to B(V)\) such that \(r_B \circ s_B: B(V) \to B(V)\) is the identity map of \(B(V)\);
3. \(j_B\) is a natural transformation of \(k\)-modules \(j_B: M(V) \to B(V)\) such that the diagram
   \[
   \begin{align*}
   & B(V) \xrightarrow{s_B} T(V) \xrightarrow{r_B} B(V) \\
   & \downarrow j_B \quad \quad \quad \quad \downarrow j \quad \quad \quad \quad \downarrow j_B \\
   & M(V) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
Lemma 4.2. Let \( B(V) \) be a natural coalgebra retract of \( T(V) \) over \( M(V) \). Then

1. \( B(V) = \bigoplus_m B_m(V) \) is a connected graded coalgebra;
2. the transformations \( s_B : B(V) \to T(V) \) and \( r_B : T(V) \to B(V) \) are natural transformations of graded coalgebras.

Let \( B(V) \) be a natural coalgebra retract of \( T(V) \) over \( M(V) \). Let \( f_B : T(V) \to T(V) \) denote the composite

\[
T(V) \xrightarrow{r_B} B(V) \xrightarrow{s_B} T(V).
\]

Then \( f_B \) is idempotent and the diagram

\[
\begin{array}{ccc}
T(V) & \xrightarrow{f_B} & T(V) \\
\downarrow{j} & & \downarrow{j} \\
M(V) & = & M(V)
\end{array}
\]

commutes. Thus

\[
B(V) \cong \text{colim}_{f_B} T(V)
\]

as graded coalgebras. Conversely, let \( f : T(V) \to T(V) \) be a natural transformation of coalgebras such that the diagram

\[
\begin{array}{ccc}
T(V) & \xrightarrow{f} & T(V) \\
\downarrow{j} & & \downarrow{j} \\
M(V) & = & M(V)
\end{array}
\]

commutes. If the ground field \( k \) is finite, then \( \text{colim}_{f} T(V) \) together with the obvious retraction and inclusions forms a natural coalgebra retract of \( T(V) \) over \( M(V) \). To draw this conclusion for an arbitrary field \( k \), we need some lemmas.

Lemma 4.3. Let \( \phi_V : V^\otimes k \to V^\otimes k \) be a natural transformation of \( k \)-modules and let \( \phi_V^n : V^\otimes k \to V^\otimes k \) denote the \( n \)-fold self composite of \( \phi_V \). Then there exists a positive integer \( N \gg 0 \) such that

1. \( N \) is independent of \( V \);
2. the functorial epimorphism

\[
\phi_V : \text{Im}(\phi_V^n : V^\otimes k \to V^\otimes k) \twoheadrightarrow \text{Im}(\phi_V^{n+1} : V^\otimes k \to V^\otimes k)
\]
is a functorial isomorphism for $n \geq N$.

3) the functorial injection

$$\text{Im}(\phi_{V}^{n+1} : V \otimes k \to V \otimes k) \subset \text{Im}(\phi_{V}^{n} : V \otimes k \to V \otimes k)$$

is a functorial isomorphism for $n \geq N$.

Proof. Let $V$ be the $k$-module with basis $\{x_1, x_2, \ldots, x_k\}$. Notice that there is a decreasing sequence

$$\text{dim}_k \tilde{V} \otimes k = (\text{dim}_k \tilde{V})^k \geq \text{dim}_k \text{Im}(\phi_{\tilde{V}}) \geq \text{dim}_k \text{Im}(\phi_{\tilde{V}}^2) \geq \cdots .$$

Thus there exists $N \gg 0$ such that

$$\text{dim}_k \text{Im}(\phi_{\tilde{V}}^n) = \text{dim}_k \text{Im}(\phi_{\tilde{V}}^{n+1})$$

for $n \geq N$ and so

$$\text{Im}(\phi_{\tilde{V}}^{n+1} : \tilde{V} \otimes k \to \tilde{V} \otimes k) = \text{Im}(\phi_{\tilde{V}}^n : \tilde{V} \otimes k \to \tilde{V} \otimes k)$$

for $n \geq N$. In particular, there exists $y_n \in \tilde{V} \otimes k$ such that

$$\phi_{\tilde{V}}^{n+1}(y_n) = \phi_{\tilde{V}}^n(x_1 \otimes \cdots \otimes x_k)$$

for $n \geq N$. Let $V$ be any $k$-module and let $a_1, \ldots, a_k$ be $k$ elements in $V$. Let $f : \tilde{V} \to V$ be the $k$-linear map given by $f(x_j) = a_j$ for $1 \leq j \leq k$. From the commutative diagram

$$\begin{array}{ccc}
\tilde{V} \otimes k & \xrightarrow{f \otimes k} & V \otimes k \\
\downarrow \phi_{\tilde{V}} & & \downarrow \phi_V \\
\tilde{V} \otimes k & \xrightarrow{f \otimes k} & V \otimes k,
\end{array}$$

we have

$$\phi_{V}^n(a_1 \otimes \cdots \otimes a_k) = f \otimes k \phi_{\tilde{V}}^n(x_1 \otimes x_k) = f \otimes k \phi_{\tilde{V}}^{n+1}(y_n) = \phi_{V}^{n+1}(f \otimes k(y_n))$$

for $n \geq N$. Thus

$$\text{Im}(\phi_{V}^{n+1} : V \otimes k \to V \otimes k) = \text{Im}(\phi_{V}^n : V \otimes k \to V \otimes k)$$

for any $k$-module $V$; which is assertion 2). Assertion 1) follows from assertion 2). □

Corollary 4.4. Let $\phi_V : V \otimes k \to V \otimes k$ be a natural transformation of $k$-modules. Then there exists a positive integer $N \gg 0$ such that

1) $N$ is independent of $V$;
2) the canonical map
\[ \text{Im}(\phi^n_k : V^k \to V^k) \to \text{colim}_{\phi^n_k} V^k \]
is a functorial isomorphism for \( n \geq N \).

Let \( M(V) \) be a functorial \( k \)-submodule of \( T(V) \). We now generalize the preceding slightly to allow for automorphism of \( M(V) \) other than the identity. A natural transformation \( f_V : T(V) \to T(V) \) is called a **natural transformation over** \( M(V) \) if there is a functorial \( k \)-linear isomorphism \( \bar{f}_V : M(V) \to M(V) \) the diagram
\[
\begin{array}{ccc}
T(V) & \longrightarrow & T(V) \\
\downarrow j & & \downarrow j \\
M(V) & \xrightarrow{\cong} & M(V)
\end{array}
\]
commutes.

**Theorem 4.5.** Let \( M(V) \) be a functorial \( k \)-submodule of \( T(V) \) and let
\[ f_V : T(V) \to T(V) \]
be a natural transformation of coalgebras over \( M(V) \). Then \( \text{colim}_{f_V} T(V) \) together with the canonical retraction and inclusions forms a natural coalgebra retract of \( T(V) \) over \( M(V) \).

**Proof.** By Corollary 4.4, for each \( k \geq 1 \) there exists an integer \( N_k \gg 0 \) such that
\[ \text{Im}(f_V^k : T_k(V) \to T_k(V)) \to \text{colim}_{f_V} T_k(V) \]
is an isomorphism for \( n \geq N_k \). We may assume that
\[ N_1 < N_2 < N_3 < \cdots . \]
Consider the descending chain of subcoalgebras of \( T(V) \)
\[ T(V) \supseteq \text{Im}(f_V^{N_1} : T(V) \to T(V)) \supseteq \text{Im}(f_V^{N_2} : T(V) \to T(V)) \supseteq \cdots \ . \]
Let
\[ B(V) = \bigcap_{k=1}^{\infty} \text{Im}(f_V^{N_k} : T(V) \to T(V)) . \]
Then \( B(V) \) is a natural subcoalgebra of \( T(V) \) over \( M(V) \). Let \( \theta : B(V) \to \text{colim}_{f_V} T(V) \) be the composite
\[ B(V) \longrightarrow T(V) \longrightarrow \text{colim}_{f_V} T(V) . \]
Notice that for each $i \geq 1$,

$$B_i(V) = \bigcap_{k=1}^{\infty} \text{Im}(f_{V_N}^{N_k} : T_i(V) \to T_i(V)) = \text{Im}(f_{V_N}^{N_k} : T_i(V) \to T_i(V)),$$

by Lemma 4.3. Thus $\theta : B(V) \to \text{colim}_{f_{V_N}} T(V)$ is an isomorphism of coalgebras. Notice that $\theta$ maps the submodule $M(V)$ of $B(V)$ onto the submodule $\text{colim}_{f_{V_N}} M(V)$ of $\text{colim}_{f_{V_N}} T(V)$. The composite $T(V) \to \text{colim}_{f_{V_N}} T(V) \xrightarrow{\theta^{-1}} B(V)$ is a functorial retraction over $M(V)$. The assertion follows.  

Let $M(V)$ be a given functorial $k$-submodule of $T(V)$. Let $\mathcal{R}_M$ denote the class of all natural coalgebra retracts of $T(V)$ over $M(V)$. Let $B(V) \in \mathcal{R}_M$.

**Lemma 4.6.** Given $M(V)$, for each $k \geq 1$ there exists a natural coalgebra retract of $T(V)$ over $M(V)$, denoted by $B^{\text{min}}_k(V; M)$, such that the dimension satisfies

$$\dim_k B^{\text{min}}_k(V; M) \leq \dim_k B_k(V)$$

for any $B(V) \in \mathcal{R}_M$ and any $k$-module $V$ with $\dim_k V \leq k$.

**Remark 4.7.** We usually write $B^{\text{min}}_k(V)$ instead of $B^{\text{min}}_k(V; M)$ when $M(V)$ is clearly understood.

**Proof.** Notice that $B_k(V)$ is a functorial retract of $V^{\otimes k}$ for any $B(V) \in \mathcal{R}_M$ and that for given $B_k(V)$, $\dim_k(B_k(V))$ only depends on $\dim_k(V)$ if $\dim_k(V)$ is finite. Let $B^{(i)}(V) \subseteq \mathcal{R}_M$ for $1 \leq i \leq k$ such that

$$\dim_k B^{(i)}_k(V) = \min\{\dim_k B_k(V) \mid B(V) \in \mathcal{R}_M\}$$

if $\dim_k V = i$. Let $s_i : B^{(i)}(V) \to T(V)$ be the functorial injection and let $r_i : T(V) \to B^{(i)}(V)$ be the functorial retraction for $1 \leq i \leq k$. Let $f_i$ be the composite

$$T(V) \xrightarrow{r_i} B^{(i)}(V) \xrightarrow{s_i} T(V)$$

and let $f_V : T(V) \to T(V)$ be the composite

$$T(V) \xrightarrow{f_1} T(V) \xrightarrow{f_2} T(V) \longrightarrow \cdots \xrightarrow{f_k} T(V).$$

Notice that

1) $f_i \circ f_i = f_i$ for $1 \leq i \leq k$;

2) $\text{Im}(f_i) \cong B^{(i)}(V)$ for $1 \leq i \leq k$.

Let

$$B^{\text{min}}_k(V) = \text{colim}_{f_{V_N}} T(V).$$

Then $B^{\text{min}}_k(V) \in \mathcal{R}_M$ and $B^{\text{min}}_k(V)$ is a natural coalgebra retract of $B^{(i)}(V)$ for each $1 \leq i \leq k$. The assertion follows.  

Remark 4.8. $B_{\text{min}}^k(V)$ is natural although not canonical. That is, any particular choice of ordering of the sequence $f_1, \ldots, f_k$ gives a natural transformation with the desired properties. Using a different ordering may give a different (although isomorphic) $B_{\text{min}}^k(V)$.

Lemma 4.9. Let $f_V : T(V) \to T(V)$ be any natural transformation of coalgebras over $M(V)$ and suppose $k \geq 1$. Then the composite

$$B_{\text{min}}^k(V) \xrightarrow{s_{B_{\text{min}}^k}} T_k(V) \xrightarrow{f_V} T_k(V) \xrightarrow{r_{B_{\text{min}}^k}} B_{\text{min}}^k(V)$$

is a functorial isomorphism of $k$-modules for any $B_{\text{min}}^k(V)$ satisfying the conditions of Lemma 4.6.

Proof. Let $g_V : T(V) \to T(V)$ denote the composite

$$T(V) \xrightarrow{r_{B_{\text{min}}^k}} B_{\text{min}}^k(V) \xrightarrow{s_{B_{\text{min}}^k}} T_k(V) \xrightarrow{f_V} T_k(V) \xrightarrow{r_{B_{\text{min}}^k}} B_{\text{min}}^k(V) \xrightarrow{s_{B_{\text{min}}^k}} T(V).$$

Let

$$A(V) = \text{colim}_{g_V} T(V).$$

Then $A(V)$ is a natural coalgebra retract of $T(V)$ under $V$ with a commutative diagram

$$\begin{array}{ccc}
B_{\text{min}}^k(V) & \xrightarrow{s_{B_{\text{min}}^k}} & B_{\text{min}}^k(V) \\
\phi_1 & & \phi_2 \\
A_k(V) & = & A_k(V),
\end{array}$$

where $\phi_1, \phi_2 : B_{\text{min}}^k(V) \to A(V)$ are the canonical maps to the colimit.

Notice that $\dim_k B_{\text{min}}^k(V)$ is minimum among $k$-modules $V$ with $\dim_k V \leq k$. Thus both maps

$$\phi_1, \phi_2 : B_{\text{min}}^k(V) \to A_k(V)$$

are isomorphisms for any $k$-modules $V$ with $\dim_k V \leq k$ and so the composite

$$B_{\text{min}}^k(V) \xrightarrow{s_{B_{\text{min}}^k}} T_k(V) \xrightarrow{f_V} T_k(V) \xrightarrow{r_{B_{\text{min}}^k}} B_{\text{min}}^k(V)$$

is a natural isomorphism for any $k$-modules $V$ with $\dim_k V \leq k$.

Now let $V$ be any finite dimensional $k$-module. It suffices to show that the map

$$r_{B_{\text{min}}^k} \circ f_V \circ s_{B_{\text{min}}^k} : B_{\text{min}}^k(V) \to B_{\text{min}}^k(V)$$
is an epimorphism. Notice that $B_k^{\text{min}}(V)$ is generated, as a graded $k$-module, by the elements

$$r_{B^{\text{min}}_k}(a_1 \otimes \cdots \otimes a_k),$$

where $a_j$ runs through all elements in $V$. Let $a_1, \ldots, a_k$ be nonzero elements in $V$ and let $\tilde{V}$ be the $k$-module with basis $\{x_1, \ldots, x_k\}$. Let

$$\phi: \tilde{V} \to V$$

be the $k$-linear map determined by

$$\phi(x_j) = a_j$$

for $1 \leq j \leq k$. Then

$$r_{B^{\text{min}}_k}(a_1 \otimes \cdots \otimes a_k) \in \text{Im}(B_k^{\text{min}}(\phi): B_k^{\text{min}}(\tilde{V}) \to B_k^{\text{min}}(V)).$$

From the commutative diagram

$$
\begin{array}{ccc}
B_k^{\text{min}}(V) & \xrightarrow{r_{B^{\text{min}}_k} \circ f_V \circ s_{B^{\text{min}}_k}} & B_k^{\text{min}}(V) \\
\downarrow{B_k^{\text{min}}(\phi)} & & \downarrow{B_k^{\text{min}}(\phi)} \\
B_k^{\text{min}}(\tilde{V}) & \xrightarrow{r_{B^{\text{min}}_k} \circ f_V \circ s_{B^{\text{min}}_k}} & B_k^{\text{min}}(\tilde{V}), \\
\end{array}
$$

one gets

$$r_{B^{\text{min}}_k}(a_1 \otimes \cdots \otimes a_k) \in \text{Im}(r_{B^{\text{min}}_k} \circ f_V \circ s_{B^{\text{min}}_k}: B_k^{\text{min}}(V) \to B_k^{\text{min}}(V)).$$

Thus the composite

$$
B_k^{\text{min}}(V) \xrightarrow{s_{B^{\text{min}}_k}} T_k(V) \xrightarrow{f_V} T_k(V) \xrightarrow{r_{B^{\text{min}}_k}} B_k^{\text{min}}(V)
$$

is a functorial isomorphism for any finite dimensional $k$-module $V$ and so for any $k$-module, which is the assertion. 

**Lemma 4.10.** For each $n \geq 1$, there exists a natural coalgebra retract of $T(V)$ over $M(V)$, denoted by $A^{\text{min}}_k(V; M)$, such that for any natural transformation

$$f_V: T(V) \to T(V),$$

of coalgebras over $M(V)$, the composite

$$A_k^{\text{min}}(V) \xrightarrow{S_{A^{\text{min}}_k}} T_k(V) \xrightarrow{f_V} T_k(V) \xrightarrow{r_{A^{\text{min}}_k}} A_k^{\text{min}}(V)$$

is a functorial isomorphism for $k \leq n$.

**Remark 4.11.** As with $B$ we usually write simply $A^{\text{min}}_k(V)$ for $A^{\text{min}}_k(V; M)$.
Proof. Let $g_k : T(V) \to T(V)$ denote the idempotent map

$$T(V) \xrightarrow{g_k} B_{\text{min}}^k (V) \xrightarrow{B_{\text{min}}^k} T(V)$$

for some choice of $B_{\text{min}}^k (V)$. Let $g_V : T(V) \to T(V)$ denote the composite

$$T(V) \xrightarrow{g_1} T(V) \xrightarrow{g_2} T(V) \xrightarrow{g_3} \cdots \xrightarrow{g_n} T(V).$$

Let

$$A_{\text{min}}^k (V) = \text{colim}_{g_V} T(V).$$

Then $A_{\text{min}}^k (V)$ is a natural coalgebra retract of $B_{\text{min}}^k (V)$ over $M(V)$ for $1 \leq k \leq n$ and so

$$A_{\text{min}}^k (V) \cong B_{\text{min}}^k (V)$$

for $1 \leq k \leq n$. The assertion follows from Lemma 4.9.

**Theorem 4.12.** Let $M(V)$ be a functorial $k$-submodule of $T(V)$. Then there exists a natural coalgebra retract of $T(V)$ over $M(V)$, denoted by $A_{\text{min}}^\alpha(V; M)$, such that $A_{\text{min}}^\alpha(V; M)$ is minimal in the following sense:

Let $f_V : T(V) \to T(V)$ be any natural transformation of coalgebras over $M(V)$. Then the composite

$$A_{\text{min}}^\alpha(V; M) \xrightarrow{S_{A_{\text{min}}^\alpha}} T(V) \xrightarrow{f_V} T(V) \xrightarrow{T_{A_{\text{min}}^\alpha}} A_{\text{min}}^\alpha(V; M)$$

is a functorial isomorphism of coalgebras.

Proof. For each $n \geq 1$ chose $A_{\text{min}}^n$ satisfying Lemma 4.10. By Lemma 4.10, the composites

$$A_{\text{min}}^n (V) \xrightarrow{S_{A_{\text{min}}^n+1}} T_j (V) \xrightarrow{T_{A_{\text{min}}^n+1}} A_{\text{min}}^n (V) \xrightarrow{S_{A_{\text{min}}^n}} T_j (V) \xrightarrow{T_{A_{\text{min}}^n+1}} A_{\text{min}}^n+1 (V),$$

$$A_{\text{min}}^n (V) \xrightarrow{S_{A_{\text{min}}^n}} T_j (V) \xrightarrow{T_{A_{\text{min}}^n+1}} A_{\text{min}}^n+1 (V) \xrightarrow{S_{A_{\text{min}}^n+1}} T_j (V) \xrightarrow{T_{A_{\text{min}}^n+1}} A_{\text{min}}^n (V)$$

are isomorphisms for $j \leq n$. Let $f_{n+1} : A_{\text{min}}^n+1 (V) \to A_{\text{min}}^n (V)$ denote the composite

$$A_{\text{min}}^n+1 (V) \xrightarrow{S_{A_{\text{min}}^n+1}} T(V) \xrightarrow{T_{A_{\text{min}}^n}} A_{\text{min}}^n (V).$$

Let $g_{n+1} : A_{\text{min}}^n \to A_{\text{min}}^n+1$ denote the composite

$$A_{\text{min}}^n \xrightarrow{S_{A_{\text{min}}^n}} T(V) \xrightarrow{T_{A_{\text{min}}^n+1}} A_{\text{min}}^n+1 (V).$$

Then the maps

$$f_{n+1} : A_{\text{min}}^n+1 (V) \to A_{\text{min}}^n (V),$$

$$g_{n+1} : A_{\text{min}}^n (V) \to A_{\text{min}}^n+1 (V).$$
are isomorphisms for \( j \leq n \).

Let \( E(V) \) denote the inverse limit of the sequence

\[
\ldots \xrightarrow{f_{n+2}} A_{\text{min}_{n+1}}(V) \xrightarrow{f_{n+1}} A_{\text{min}_n}(V) \xrightarrow{f_n} \ldots \xrightarrow{f_2} A_{\text{min}_1}(V) \xrightarrow{S_{A_{\text{min}}}} T(V).
\]

Let \( \phi_n : E(V) \to A_{\text{min}_n}(V) \) be the canonical map from the inverse limit to \( A_{\text{min}_n}(V) \).

Then

\[
\phi_n : E_j(V) \to A_{j_{\text{min}_n}}(V)
\]

is an isomorphism for \( j \leq n \).

Let \( F(V) \) denote the direct limit of the sequence

\[
T(V) \xrightarrow{T_{A_{\text{min}_1}}} A_{\text{min}_1}(V) \xrightarrow{g_2} \ldots \xrightarrow{g_n} A_{\text{min}_n}(V) \xrightarrow{g_{n+1}} A_{\text{min}_{n+1}}(V) \xrightarrow{g_{n+2}} \ldots.
\]

Let \( \phi'_n : A_{\text{min}_n}(V) \to F(V) \) denote the canonical map from \( A_{\text{min}_n}(V) \) to the colimit.

Then

\[
\phi'_n : A_{j_{\text{min}_n}}(V) \to F_j(V)
\]

is an isomorphism for \( j \leq n \).

Let \( \theta_n : A_{\text{min}_n}(V) \to A_{\text{min}_n}(V) \) denote the composite

\[
A_{\text{min}_n}(V) \xrightarrow{f_n} \ldots \xrightarrow{f_2} A_{\text{min}_1}(V) \xrightarrow{S_{A_{\text{min}_1}}} T(V) \xrightarrow{T_{A_{\text{min}_1}}} A_{\text{min}_1}(V)
\]

\[
\ldots \xrightarrow{g_n} A_{\text{min}_n}(V).
\]

By Lemma 4.10, we have

\[
\theta_n : A_{j_{\text{min}_n}}(V) \to A_{j_{\text{min}_n}}(V)
\]

is an isomorphism for \( j \leq n \). Thus the canonical map

\[
E(V) \to F(V)
\]

is a natural isomorphism of graded coalgebras over \( M(V) \). Let \( A_{\text{min}}(V; M) = E(V) \).

Then \( A_{\text{min}}(V; M) \) is a natural coalgebra retract of \( T(V) \) over \( M(V) \). Notice that the map \( E(V) \to F(V) \) factors through \( A_{\text{min}_n}(V) \) for each \( n \). Thus \( A_{\text{min}}(V; M) \) is a natural coalgebra retract of \( A_{\text{min}_n}(V) \) over \( M(V) \) for each \( n \).

By Lemma 4.10, the composite

\[
A_{j_{\text{min}_n}}(V) \xrightarrow{S_{A_{\text{min}_n}}} T_j(V) \xrightarrow{T_{A_{\text{min}_n}}} A_{j_{\text{min}_n}}(V) \xrightarrow{S_{A_{\text{min}_n}}} T_j(V) \xrightarrow{T_{A_{\text{min}_n}}} A_{j_{\text{min}_n}}(V)
\]

is an isomorphism for \( j \leq n \). Thus \( A_{j_{\text{min}_n}}(V) \) is a functorial retract of \( A_{j_{\text{min}_n}}(V) \) for \( j \leq n \) and so, by the previous paragraph, the composites

\[
A_{j_{\text{min}_n}}(V) \xrightarrow{S_{A_{\text{min}_n}}} T_j(V) \xrightarrow{T_{A_{\text{min}_n}}} A_{j_{\text{min}_n}}(V),
\]

\[
A_{j_{\text{min}_n}}(V) \xrightarrow{S_{A_{\text{min}_n}}} T_j(V) \xrightarrow{T_{A_{\text{min}_n}}} A_{j_{\text{min}_n}}(V; M),
\]

\[
A_{j_{\text{min}_n}}(V) \xrightarrow{S_{A_{\text{min}_n}}} T_j(V) \xrightarrow{T_{A_{\text{min}_n}}} A_{j_{\text{min}_n}}(V; M),
\]
\[ A_j^{\text{min}}(V; M) \xrightarrow{S_{A_j^{\text{min}}}} T_j(V) \xrightarrow{T_{A_j^{\text{min}}}} A_j^{\text{min}}(V) \]

are isomorphisms for \( j \leq n \) if \( V \) is a finite dimensional \( k \)-module and so they are isomorphisms for \( j \leq n \) for any \( k \)-module \( V \).

Let \( f_V : T(V) \to T(V) \) be a functorial coalgebra morphism over \( M(V) \). Notice that, for each \( n \geq 1 \), the composite

\[ f_V : T_j(V) \xrightarrow{T_{A_j^{\text{min}}}} A_j^{\text{min}}(V) \xrightarrow{S_{A_j^{\text{min}}}} T_j(V) \]

is an isomorphism for \( j \leq n \) by Lemma 4.10. Thus the composite

\[ A_j^{\text{min}}(V) \xrightarrow{S_{A_j^{\text{min}}}} T_j(V) \xrightarrow{f_V} T_j(V) \xrightarrow{T_{A_j^{\text{min}}}} A_j^{\text{min}}(V) \]

is an isomorphism for any \( 1 \leq j \leq n \). The assertion follows. \( \square \)

**Corollary 4.13.** Let \( M(V) \) be a functorial \( k \)-submodule of \( T(V) \) and let \( B(V) \) be a natural coalgebra retract of \( T(V) \) over \( M(V) \). Then \( A_j^{\text{min}}(V; M) \) is a natural coalgebra retract of \( B(V) \) over \( M(V) \).

Let \( M(V) \) be a functorial \( k \)-submodule of \( T(V) \). The coalgebra \( A_j^{\text{min}}(V; M) \) is called the **minimal natural coalgebra retract of \( T(V) \) over \( M(V) \)**. Notice that the minimal natural coalgebra retract of \( T(V) \) of \( M(V) \) is unique up to isomorphism over \( M(V) \) by the universal property in Corollary 4.13. In the special case where \( M(V) = V \), let \( A_j^{\text{min}}(V) \) denote the **minimal natural coalgebra retract of \( T(V) \) over \( V \)**.

5. SOME LEMMAS ON COALGEBRAS

In this section, the term quasi-Hopf algebra will be used as in [12] to refer to a Hopf algebra in which the multiplication need not be associative, although the comultiplication is assumed to be coassociative.

Let \( C \) be a connected graded \( k \)-coalgebra. Let \( P(C) \) denote the set of primitive elements in \( C \). By the proof of Proposition 3.9 in [12], one has the following lemma.

**Lemma 5.1.** Let \( f : A \to B \) be a morphism of connected graded coalgebras over a field. Then \( f : A_j \to B_j \) is a monomorphism for \( j \leq n \) if and only if \( P(f) : P(A_j) \to P(B_j) \) is a monomorphism for \( j \leq n \).

Let \( C \) be a connected graded cocommutative coalgebra and let \( B \) be a connected graded cocommutative quasi-Hopf algebra. Notice that the convolution product may not be associative in this setting. There exists a unique left conjugation

\[ \chi : B \to B \]